Parallel and Distributed Sparse Optimization

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outline

1. sparse optimization background

2. model and data distribution

3. parallel and distributed algorithms

   parallel/distributed prox-linear iteration

   GRock: greedy coordinate-block descent

   parallel/distributed ADMM

4. numerical experiments

   Rice cluster STIC

   Amazon EC2

5. conclusions

sparse optimization

processes "large" data sets for "structured" solutions

key points:

- finding the sparsest solution is generally NP-hard
- under some conditions (e.g., enough equations, dual certificate, incoherence, RIP, etc.), minimal ℓ_1 solution is the sparsest solution [1]
- most time, ℓ_1 minimization in $\mathbb{R}^n$ is quicker than solving an $n \times n$ dense system
- has many interesting generalizations in signal/image processing, machine learning, data mining, etc.

[1] Donoho [2006], Candès, Romberg, and Tao [2006]

current challenges

the size, complexity, and diversity of sparse optimization grow significantly

examples:

- video, 4D CT images, dynamical system, higher-dimensional tensors
- (the Internet) distributed data, streaming data, cloud computing

single-threaded approaches for sparse optimization

off-the-shelf: subgradient descent, LP/SOCP/SDP

smoothing: approximate ℓ_1 by

- $\ell_{1+\epsilon}$-norm
- $\sum_i \sqrt{x_i^2 + \epsilon}$
- Huber-norm

and apply gradient-based and quasi-Newton methods.

splitting: split smooth and nonsmooth terms into simple subproblems

- operator splitting: GPSR/FPC/SpaRSA/FISTA/SPGL1/...[2]
- dual operator splitting: Bregman/ADMM/split Bregman/... [3]

all operations have forms: $\mathbf{Ax}$ and $\mathbf{A}^T\mathbf{y}$

greedy approaches: GRock, OMP, CoSaMP ...[4]

[2] Hale, Yin, and Zhang [2008], Beck and Teboulle [2009], van den Berg and Friedlander [2008]

[3] Yin [2010], Yang and Zhang [2011]

[4] Tropp and Gilbert [2007], Needell and Tropp [2009]

parallel/distributed sparse optimization

should be capable of handling instances of very large scales

this set of slides covers

- parallel/distrbuted alternating direction method of multipliers (ADMM)
- parallel/distrbuted prox-linear approaches
- (coming ...) parallel implementation of linearize Bregman
- GRock: parallel greedy coordinate-block descent

going down the list, algorithms assume more structures but are also faster

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general model

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \;\; \mathcal{F}(\mathbf{x}) = \lambda \cdot \mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$$

- $\mathcal{R}(\mathbf{x})$ is a regularizer such as $\|\mathbf{x}\|_1$, $\|\Psi^T\mathbf{x}\|_1$, $\|\mathbf{x}\|_{2,1}$, etc.
- $\mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$ is a fitting measure such as squared loss, logistic loss, etc.

- we say $f(\mathbf{x})$ is separable if

$$f(\mathbf{x}) = \sum_{s=1}^{S} f_s(\mathbf{x}_s)$$

distributing big data $\mathbf{A}$

reasons:

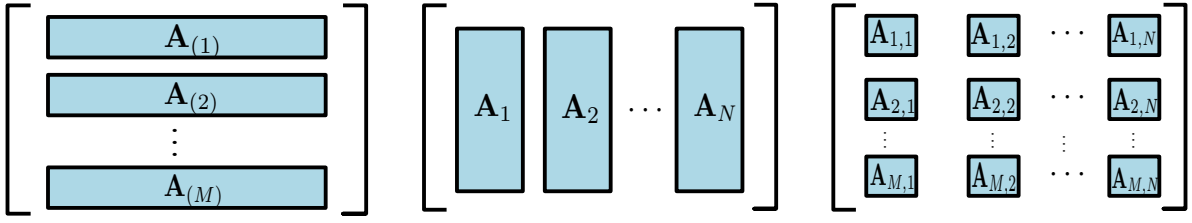
- $\mathbf{A}$ is too large to store centrally
- $\mathbf{A}$ is collected in a distributed manner

we consider three distribution schemes:

a. row blocks: stacking row-blocks $\mathbf{A}_{(1)}, \mathbf{A}_{(2)}, \ldots, \mathbf{A}_{(M)}$ forms $\mathbf{A}$

b. columns blocks $\mathbf{A} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \cdots \ \mathbf{A}_N]$

c. $\mathbf{A}$ is partitioned into MN sub-blocks. The (i,j)th block is $\mathbf{A}_{i,j}$.



a b c

assume each computing node stores a block

computing $\mathbf{A}\mathbf{x}$ and $\mathbf{A}^T\mathbf{y}$: broadcast then parallel

computing

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{A}_{(1)}\mathbf{x} \\ \mathbf{A}_{(2)}\mathbf{x} \\ \cdots \\ \mathbf{A}_{(M)}\mathbf{x} \end{bmatrix} \quad \text{in scenario a}$$

and

$$\mathbf{A}^T\mathbf{y} = \begin{bmatrix} \mathbf{A}_1^T\mathbf{y} \\ \mathbf{A}_2^T\mathbf{y} \\ \cdots \\ \mathbf{A}_N^T\mathbf{y} \end{bmatrix} \quad \text{in scenario b}$$

each takes two steps

1. $\mathbf{x}$ and $\mathbf{y}$ are *broadcasted to* or *synchronized among* all the nodes
2. every node *independently computes* $\mathbf{A}_{(i)}\mathbf{x}$ and $\mathbf{A}_j^T\mathbf{y}$

computing $\mathbf{Ax}$ and $\mathbf{A}^T\mathbf{y}$: parallel then reduce

computing

$$\mathbf{A}^T\mathbf{y} = \sum_{i=1}^{M} \mathbf{A}_{(i)}^T \mathbf{y}_i \quad \text{in scenario a}$$

and

$$\mathbf{Ax} = \sum_{i=1}^{N} \mathbf{A}_j \mathbf{x}_j \quad \text{in scenario b}$$

each takes two steps

1. every node computes $\mathbf{A}_{(i)}^T \mathbf{y}_i$ and $\mathbf{A}_j \mathbf{x}_j$ independently
2. a *reduce* operation computes the sum
   (*allreduce* also returns the sum to every node)

broadcast, parallel, then reduce

in scenario c, either one of $\mathbf{Ax}$ and $\mathbf{A}^T\mathbf{y}$ requires both broadcast and reduce

example of $\mathbf{Ax}$:

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{A}_{(1)}\mathbf{x} \\ \mathbf{A}_{(2)}\mathbf{x} \\ \cdots \\ \mathbf{A}_{(M)}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} \mathbf{A}_{1,j}\mathbf{x}_j \\ \sum_{j=1}^{N} \mathbf{A}_{2,j}\mathbf{x}_j \\ \cdots \\ \sum_{j=1}^{N} \mathbf{A}_{M,j}\mathbf{x}_j \end{bmatrix}$$

1. for every j, $\mathbf{x}_j$ is *broadcasted* to nodes $(1,j),(2,j),\ldots,(M,j)$
2. compute $\mathbf{A}_{i,j}\mathbf{x}_j$ in parallel over all i,j
3. for every i, *reduce* over nodes $(i,1),(i,2),\ldots,(i,N)$ to compute $\mathbf{A}_{(i)}\mathbf{x} = \sum_{j=1}^{N} \mathbf{A}_{i,j}\mathbf{x}_j$

node speed and block size

- most clusters have **identical nodes**
- but clouds can have **different nodes**
- most parallel algorithms **require synchronized computation**
  faster (or less busy) nodes wait for slower (or more busy) nodes
- equal block size + different nodes $\Longrightarrow$ faster nodes wait for slower ones

a better solution: larger blocks assigned to faster (or less busy) nodes

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background: prox-linear iteration

original model:

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\ \mathcal{F}(\mathbf{x}) = \lambda \cdot \mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$$

linearize $\mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$ at $\mathbf{x}^k$:

$$\mathbf{x}^{k+1} \leftarrow \arg\min_{\mathbf{x}} \lambda \cdot \mathcal{R}(\mathbf{x}) + \langle \mathbf{x}, \mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b}) \rangle + \frac{1}{2\delta_k} \|\mathbf{x} - \mathbf{x}^k\|_2^2,$$

advantages: well understood and its solution

$$\mathbf{x}^{k+1} = \mathbf{prox}_{\lambda\mathcal{R}}(\mathbf{x}^k - \delta_k \mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b})),$$

is often in **closed form**

example: if $\mathcal{R}(\mathbf{x}) = \|\mathbf{x}\|_1$, $\mathbf{prox}_{\lambda\mathcal{R}}$ is known as soft-thresholding

algorithms: FPC/SpaSRA/ISTA/FISTA ...

scenario a

assume separability $\mathcal{L}(\mathbf{z}) = \sum_i \mathcal{L}_i(\mathbf{z}_i)$

storage: node i keeps row block $\mathbf{A}_{(i)}$, subvector $\mathbf{b}_i$, and current $\mathbf{x}^k$

each iteration: compute

$$\mathbf{x}^{k+1} = \mathbf{prox}_{\lambda\mathcal{R}}(\mathbf{x}^k - \delta_k \mathbf{A}^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b})) \tag{1}$$

in three steps, noticing

$$\mathbf{A}^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b}) = \sum_{i=1}^{M} \mathbf{A}_{(i)}^T \nabla\mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}^k; \mathbf{b}_i)$$

1. parallel: every node i computes $\mathbf{A}_{(i)}^T \nabla\mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}^k, \mathbf{b}_i)$
2. *allreduce*: $\mathbf{A}^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b}) = \sum_{i=1}^{M} \mathbf{A}_{(i)}^T \nabla\mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}^k; \mathbf{b}_i)$ for all nodes
3. parallel: every node gets identical $\mathbf{x}^{k+1}$ following (1)

scenario b

assume separability $\mathcal{R}(\mathbf{x}) = \sum_j \mathcal{R}_j(\mathbf{x}_j)$

consequently:

storage: node i keeps column block $\mathbf{A}_j$, entire $\mathbf{b}$, and subvector $\mathbf{x}_j^k$

each iteration: computes

$$\mathbf{x}^{k+1} = \mathbf{prox}_{\lambda\mathcal{R}}(\mathbf{x}^k - \delta_k \mathbf{A}^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b}))$$

$$\Rightarrow \quad \text{for each } j, \quad \mathbf{x}_j^{k+1} = \mathbf{prox}_{\lambda\mathcal{R}_j}(\mathbf{x}_j^k - \delta_k \mathbf{A}_j^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b})) \qquad (2)$$

in three steps

- parallel: every node i computes $\mathbf{A}_j\mathbf{x}_j^k$
- *allreduce*: $\mathbf{A}\mathbf{x}^k = \sum_{j=1}^N \mathbf{A}_j\mathbf{x}_j^k$ for all nodes
- parallel: ever node computes $\mathbf{A}_j^T \nabla\mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b})$ and then $\mathbf{x}_j^{k+1}$ following (2)

scenario c

assume separability $\mathcal{R}(\mathbf{x}) = \sum_j \mathcal{R}(\mathbf{x}_j)$ **and** $\mathcal{L}(\mathbf{z}) = \sum_i \mathcal{L}(\mathbf{z}_i)$

two *allreduce* operations at each iteration, remaining operations are *parallel*

(details left to the reader)

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GRock: greedy coordinate-block descent

general messages about coordinate (block) descent:

- update one coordinate (block) at a time
- coordinate selection: cycle (Gauss-Seidel), random, greedy, mixtures
- advantage: update is easy
- disadvantage: (usually but not always) more iterations, may fail on non-smooth and non-convex problems (stuck at non-stationary points)

but **GRock converges after fewer iterations for sparse optimization with $\mathbf{A}$ formed of nearly orthogonal columns**

GRock iteration for scenario b

1. assign a merit value for each coordinate i

$$d_i = \arg\min_d \ \lambda \cdot r(x_i + d) + g_i d + \frac{1}{2} d^2$$

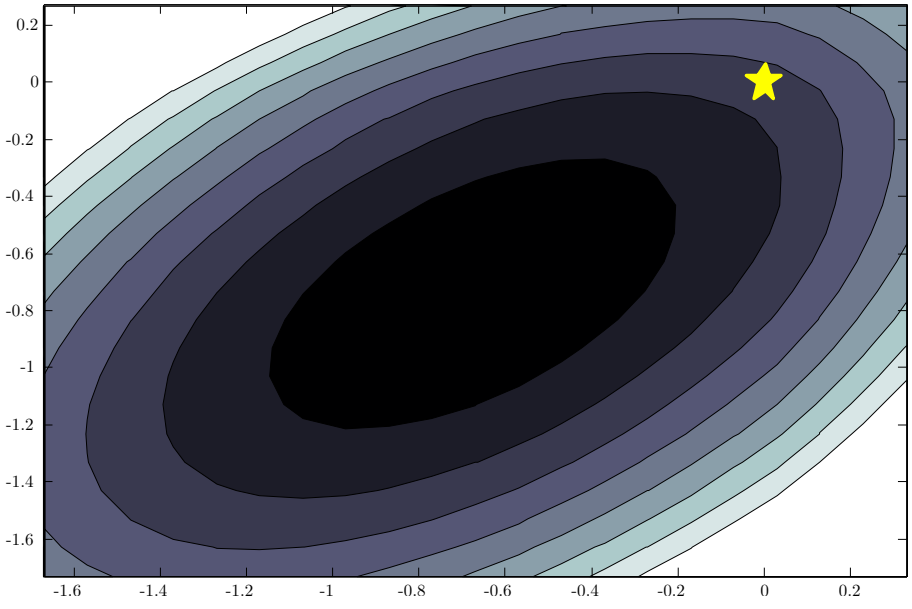
   - g_i is the ith component of $\mathbf{g}(\mathbf{x}) = \mathbf{A}_{(i)}^T \nabla \mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}; \mathbf{b}_i)$
   - larger d_i means x_i is more willing to reduce energy

2. assign a merit value for each block

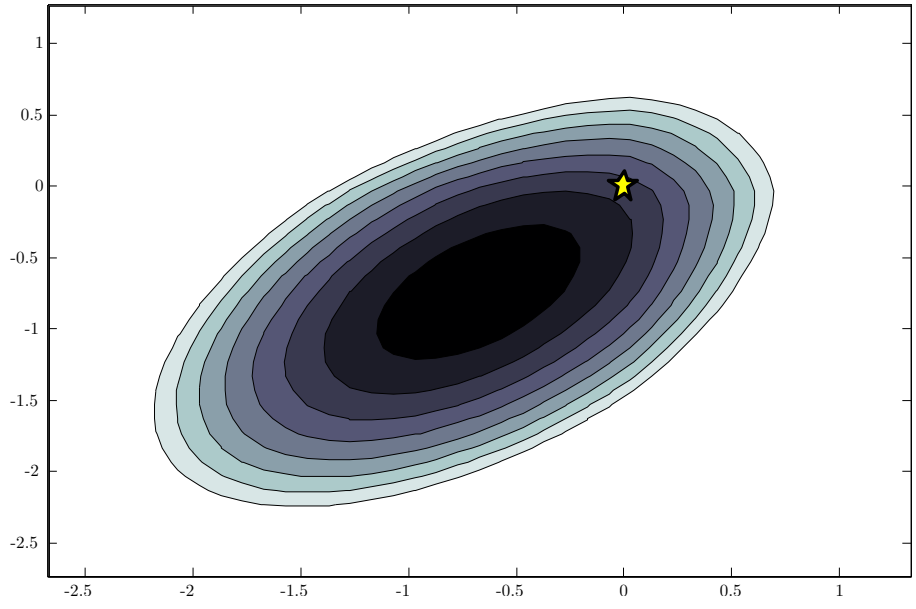
$$m_j = \max\{|d| : d \text{ is an element of } \mathbf{d}_j\}, \quad m_j = d_{s_j}$$

3. $\mathcal{P} \leftarrow$ select P blocks with largest m_j, $2 \leq P \leq N$
4. *parallel* update $x_{s_j} \leftarrow x_{s_j} + d_{s_j}$ for all $j \in \mathcal{P}$

how large P can be

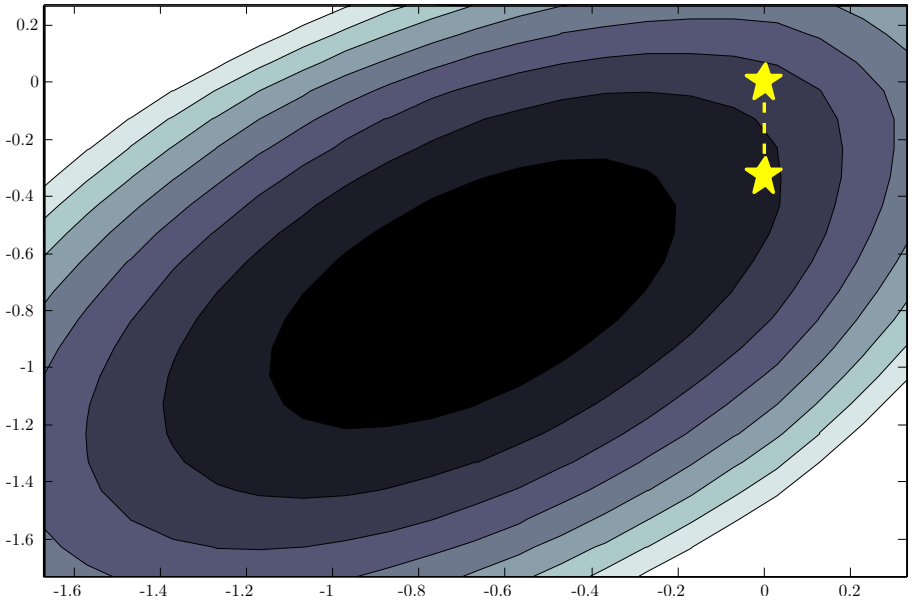


$P = 1$



$P = 3$

how large P can be



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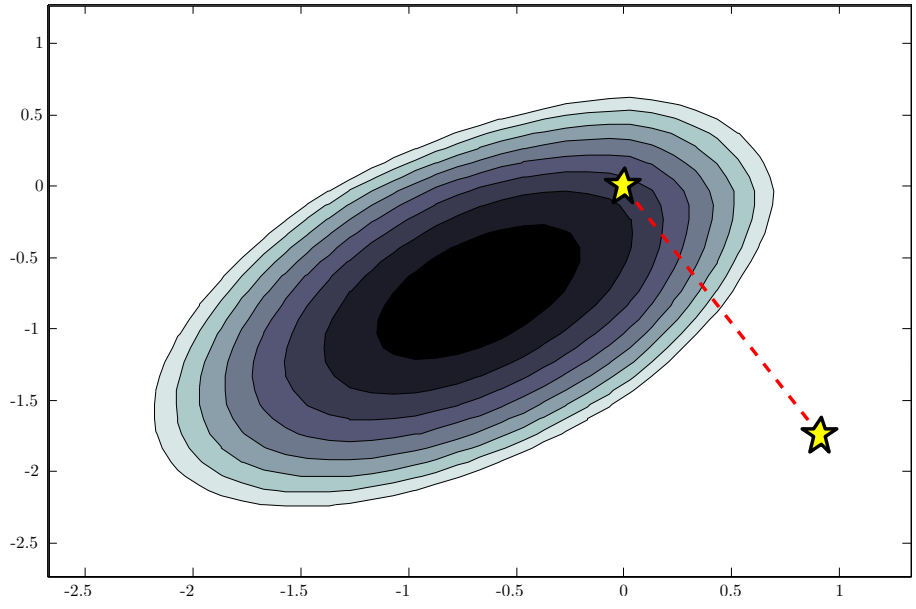
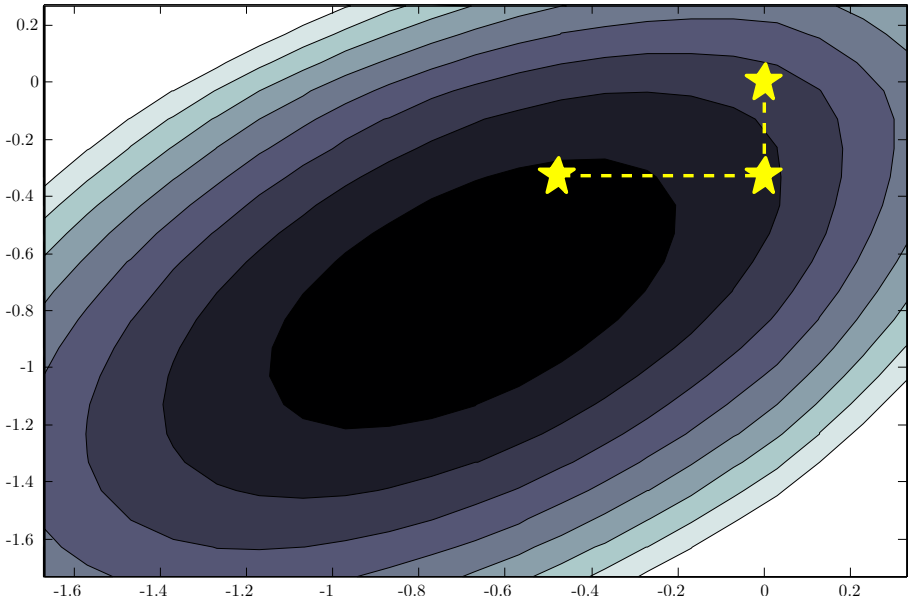
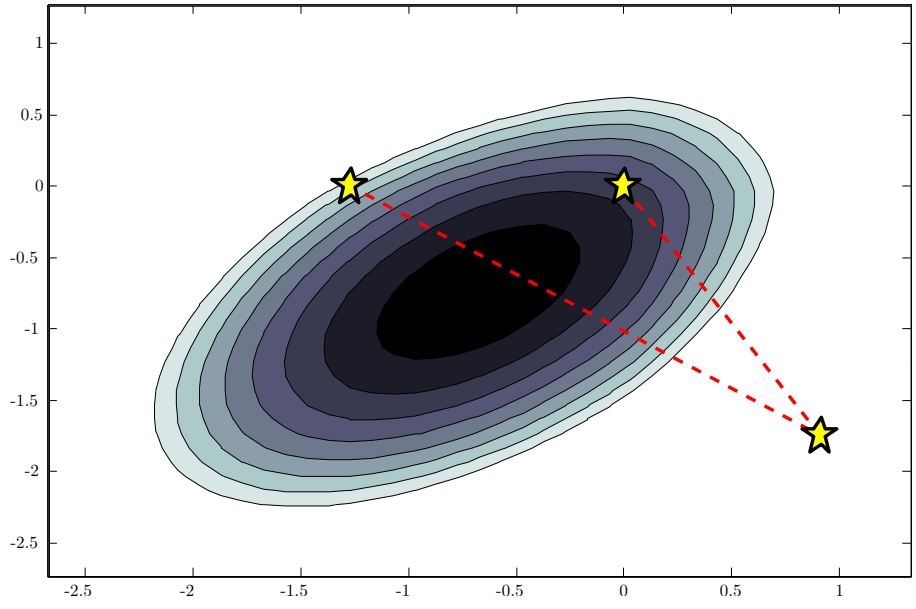


$P = 3$

how large P can be

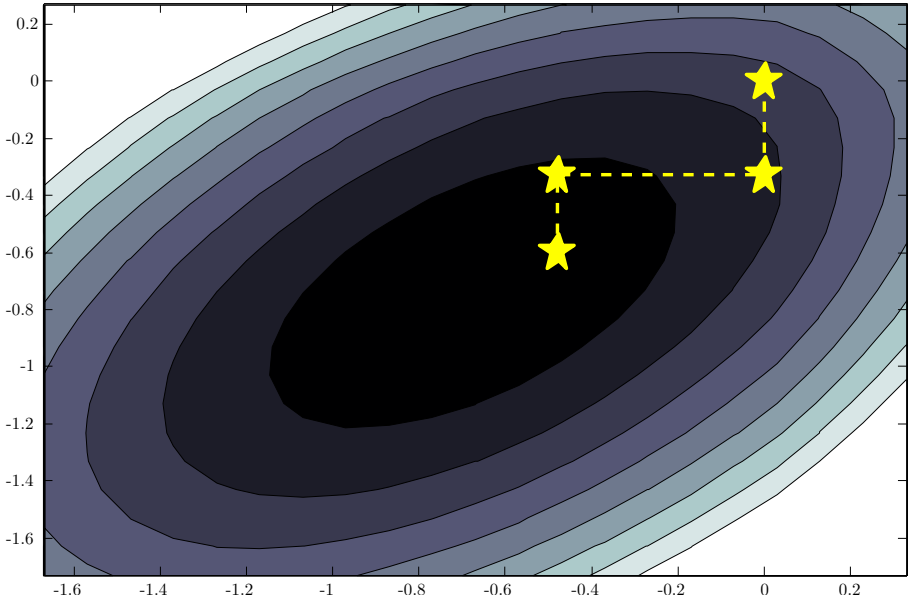


$P = 1$



$P = 3$

how large P can be



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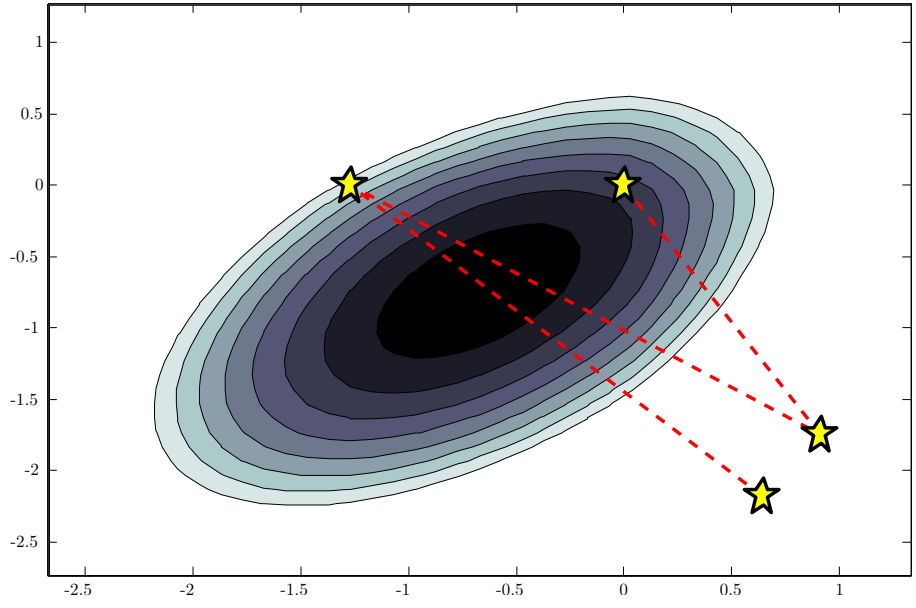
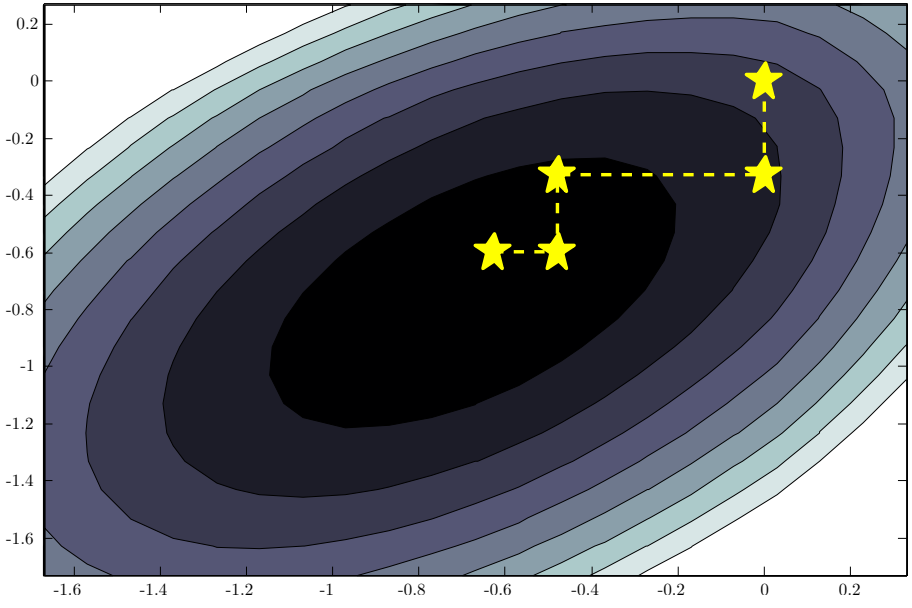
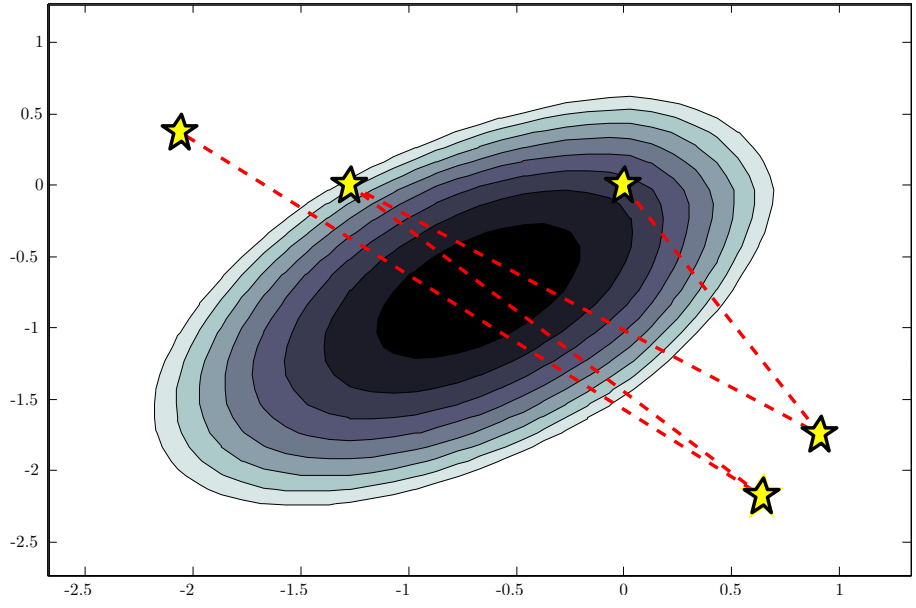


$P = 3$

how large P can be

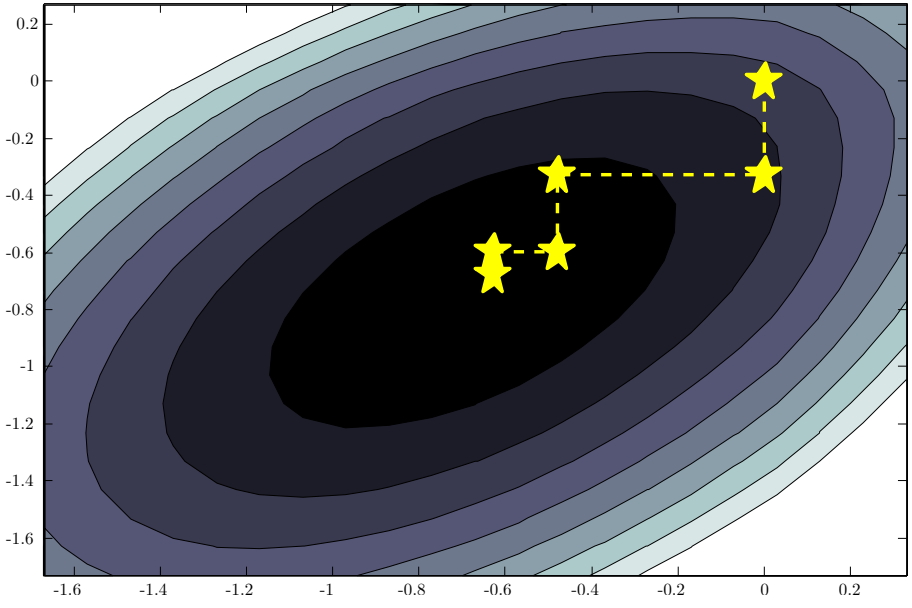


$P = 1$

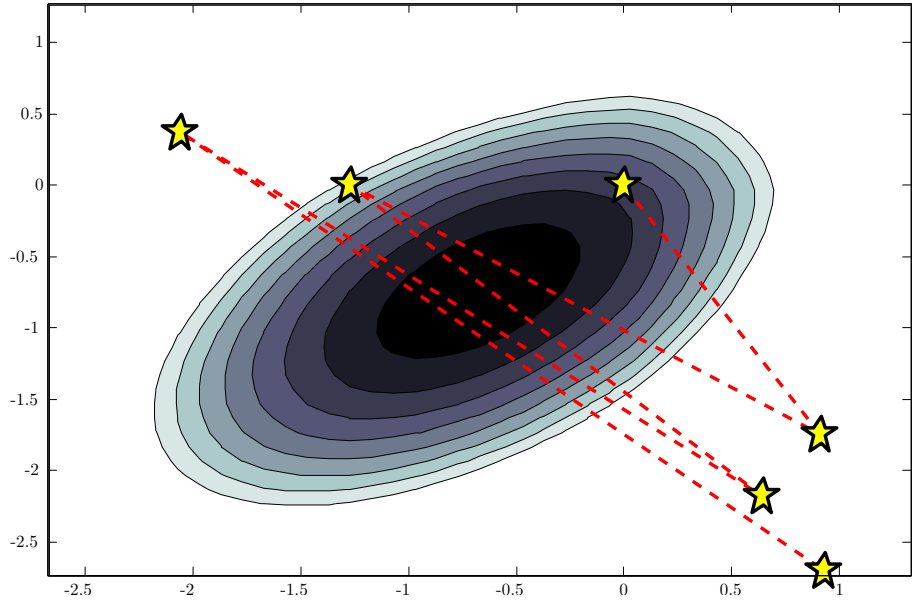


$P = 3$

how large P can be

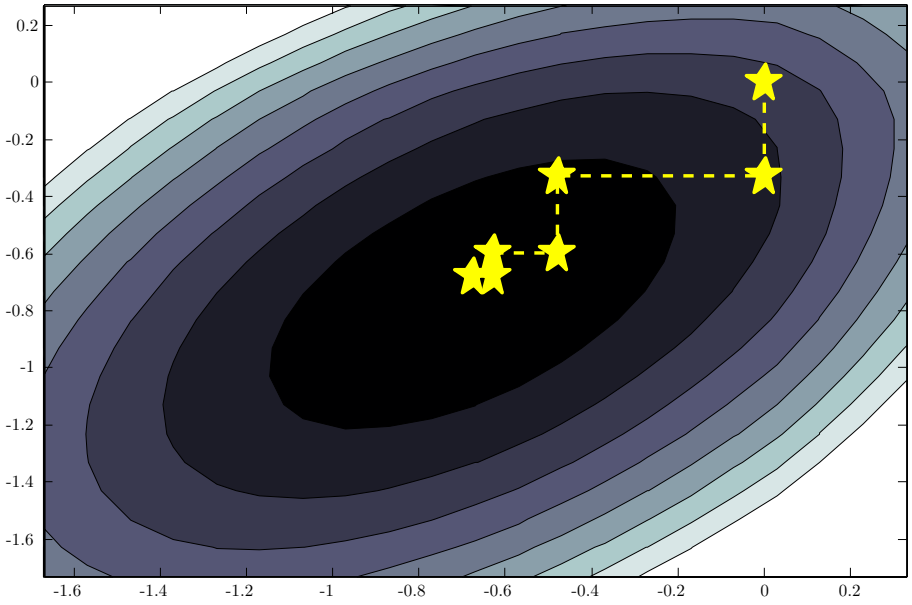


$P = 1$

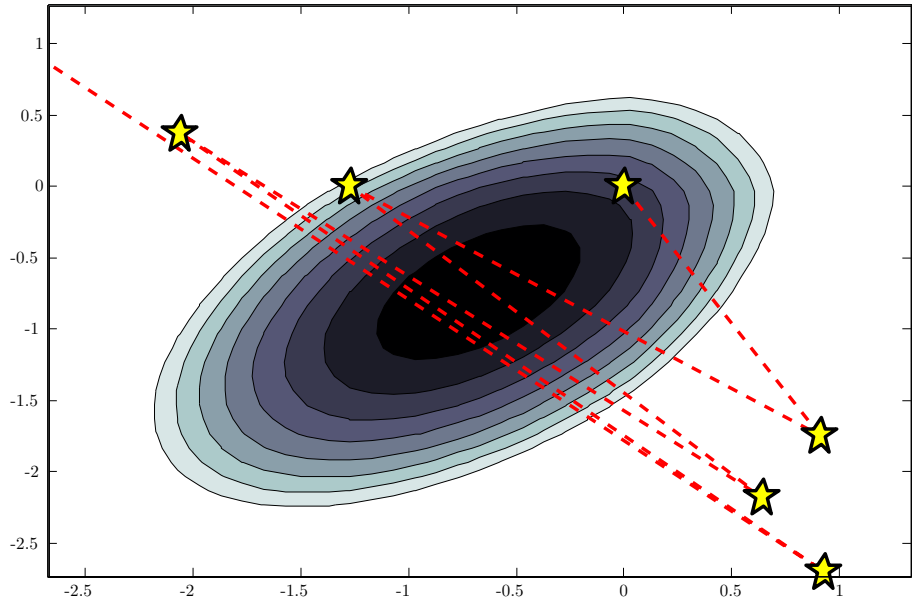


$P = 3$

how large P can be



$P = 1$ converge



$P = 3$ diverge

near orthogonality enables $P \geq 2$

assume each column of $\mathbf{A}$ has unit 2-norm

define block spectral radius

$$\rho_P = \max_{\mathbf{M} \in \mathcal{M}} \rho(\mathbf{M}),$$

where $\mathcal{M}$ is the set of all $P \times P$ submatrices that we can obtain from $\mathbf{A}^T\mathbf{A}$ corresponding to selecting exactly one column from each of the P blocks

Theorem

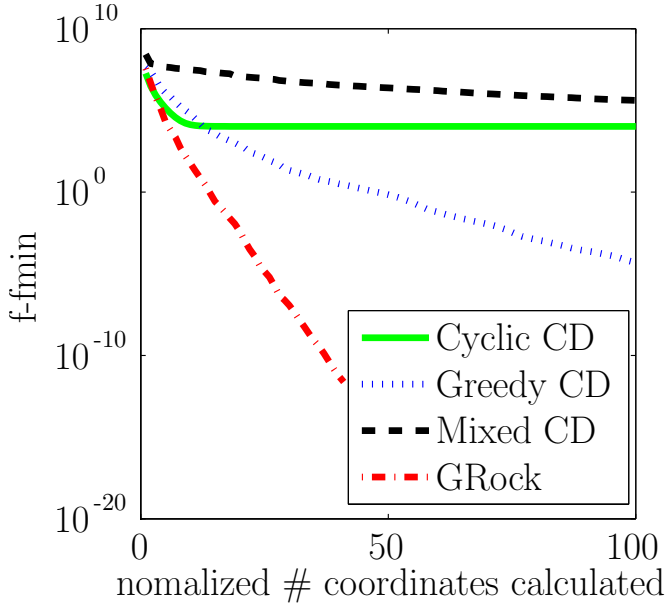
if $\rho_P < 2$, then GRock with P parallel updates per iteration gives

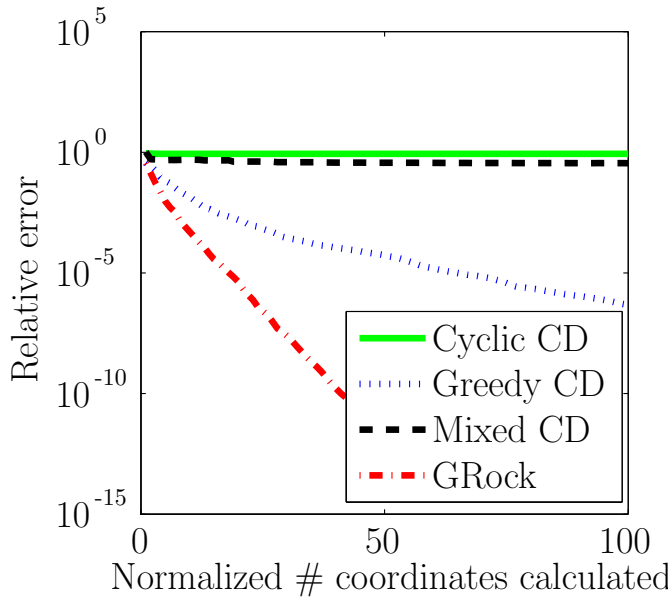
$$\mathcal{F}(\mathbf{x} + \mathbf{d}) - \mathcal{F}(\mathbf{x}) \leq \frac{\rho_P - 2}{2} \beta \|\mathbf{d}\|_2^2$$

moreover,

$$\mathcal{F}(\mathbf{x}^k) - \mathcal{F}(\mathbf{x}^*) \ \leq \ \frac{2C^2 \left(2L + \beta\sqrt{\frac{N}{P}}\right)^2}{(2 - \rho_P)\beta} \cdot \frac{1}{k}.$$

compare different block selection rules





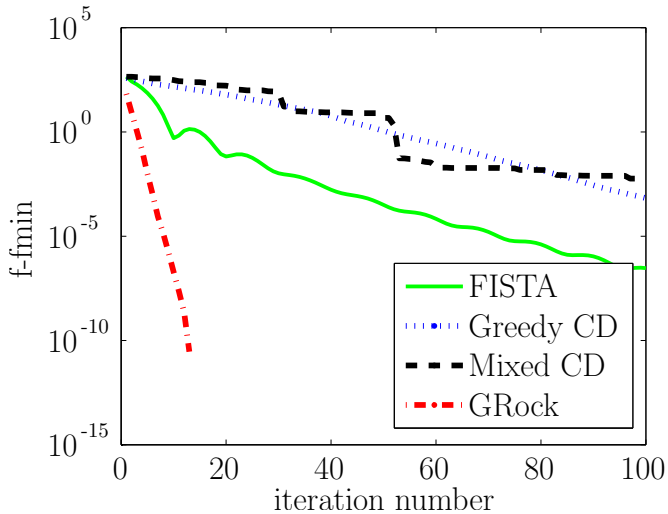
- LASSO test with $\mathbf{A} \in \mathbb{R}^{512\times 1024}$, $N = 64$ column blocks
- GRock uses $P = 8$ updates each iteration
- greedy CD [5] uses $P = 1$
- mixed CD [6] selects $P = 8$ random blocks and best coordinate in each iteration

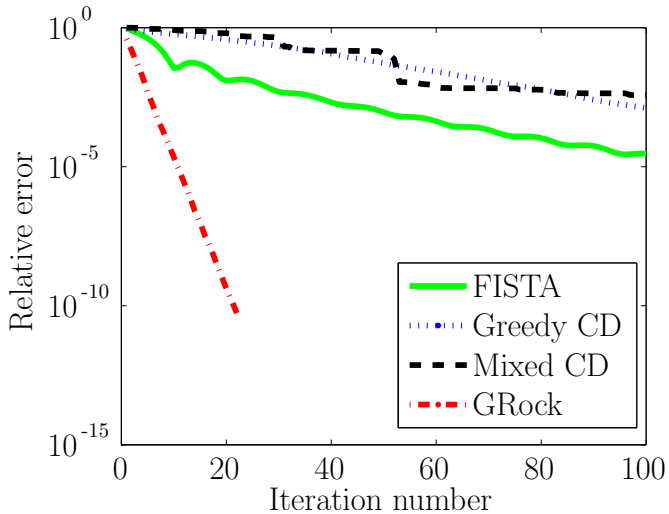
[5] Li and Osher [2009]

[6] Scherrer, Tewari, Halappanavar, and Haglin [2012]

real CMU data [7]

$\mathbf{A}$ has dimension [6205, 24820] with about 0.4% nonzero entries





GRock is often (but not always) quicker than FISTA

update of g in GRock is about 50% cheaper than in FISTA

GRock requires near orthogonality to work but FISTA does not

[7] Bradley, Kyrola, Bickson, Guestrin, and Guestrin [2011]

Why GRock rocks:

- coordinate selections often occur on those corresponding to nonzero entries in the solution
- such selection keeps most entries in $\mathbf{x}$ as zero throughout the iterations
- hence, the problem dimension is effectively reduced
- sparse update also makes gradient easier to update

prox-linear iterations typically have dense intermediate solutions at the beginning and can take many iterations before the intermediate solutions become finally sparse

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alternating direction method of multipliers (ADMM)

step 1: turn problem into the form of

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & f(\mathbf{x}) + g(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b}. \end{aligned}$$

f and g are **convex**, maybe **nonsmooth**, can **incorporate constraints**

step 2: iterate

1. $\mathbf{x}^{k+1} \leftarrow \min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{y}^k) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k\|_2^2$,
2. $\mathbf{y}^{k+1} \leftarrow \min_{\mathbf{y}} f(\mathbf{x}^{k+1}) + g(\mathbf{y}) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y} - \mathbf{b} - \mathbf{z}^k\|_2^2$,
3. $\mathbf{z}^{k+1} \leftarrow \mathbf{z}^k - (\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y}^{k+1} - \mathbf{b})$.

history: dates back to 60s, formalized 80s, parallel versions appeared late 90s, revived very recently, convergence results enriched recently

parallel/distributed ADMM[8]

idea: form *separable* $f = \sum_i f_i(\mathbf{x}_i)$, *block diagonal* $\mathbf{A}$, and *simple* g and $\mathbf{B}$

$$\begin{array}{ll} \min_{\mathbf{x},\mathbf{y}} & \sum_i f_i(\mathbf{x}_i) + g(\mathbf{y}) \\ \text{s.t.} & \begin{bmatrix} \mathbf{A}_1 & & \\ & \ddots & \\ & & \mathbf{A}_M \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_M \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_M \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_M \end{bmatrix} \end{array}$$

consequently, computing $\mathbf{x}^{k+1}$ reduces to parallel subproblems

$$\mathbf{x}_i^{k+1} \leftarrow \min_{\mathbf{x}} f_i(\mathbf{x}_i) + \frac{\beta}{2} \|\mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{y}^k - \mathbf{b}_i - \mathbf{z}_i^k\|_2^2, \quad i = 1, \ldots, M$$

[8] Boyd, Parikh, Chu, Peleato, and Eckstein [2011]

scenario a: primal ADMM

assume separability $\mathcal{L}(\mathbf{z}) = \sum_i \mathcal{L}_i(\mathbf{z}_i)$

reformulate

$$\min_{\mathbf{x}} \sum_i \mathcal{L}_i(\mathbf{A}_i\mathbf{x}, \mathbf{b}_i) + \mathcal{R}(\mathbf{x})$$

into

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \textstyle\sum_i \mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}_{(i)}, \mathbf{b}_i) \qquad + \mathcal{R}(\mathbf{x}) \\ \text{s.t.} \quad & \begin{bmatrix} I & & \\ & \ddots & \\ & & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(1)} \\ \vdots \\ \mathbf{x}_{(M)} \end{bmatrix} - \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

note: $\mathbf{x}_{(i)}$ are *not* subvectors of $\mathbf{x}$ but *local copies* of $\mathbf{x}$

scenario a: primal ADMM

storage: node i keeps row block $\mathbf{A}_{(i)}$, subvector $\mathbf{b}_i$, and current $\mathbf{x}_{(i)}^k$, $\mathbf{x}^k$, $\mathbf{z}^k$
each iteration:

1. *parallel*: $\mathbf{x}_{(i)}^{k+1} \leftarrow \min_{\mathbf{x}_{(i)}} \mathcal{L}_i(\mathbf{A}_i\mathbf{x}_{(i)}, \mathbf{b}_i) + \frac{\beta}{2}\|\mathbf{x}_{(i)} - \mathbf{x}^k - \mathbf{z}_i^k\|_2^2$, for every i
2. *allreduce*: $\frac{1}{M}\sum_i(\mathbf{x}_{(i)}^{k+1} - \mathbf{z}_i^k)$ for all nodes
3. *parallel*: $\mathbf{x}^{k+1} \leftarrow \mathcal{R}(\mathbf{x}) + \frac{\beta M}{2}\left\|\mathbf{x} - \frac{1}{M}\sum_i(\mathbf{x}_{(i)}^{k+1} - \mathbf{z}_i^k)\right\|_2^2$
4. *parallel*: $\mathbf{z}_i^{k+1} \leftarrow \mathbf{z}_i^k - (\mathbf{x}_{(i)}^{k+1} - \mathbf{x}^{k+1})$ for every i

note: steps 2 & 3 are due to

$$\frac{\beta}{2}\left\|\begin{bmatrix} I & & \\ & \ddots & \\ & & I \end{bmatrix}\begin{bmatrix} \mathbf{x}_{(1)} \\ \vdots \\ \mathbf{x}_{(M)} \end{bmatrix} - \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}\mathbf{x} - \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_M \end{bmatrix}\right\|_2^2 = \frac{\beta M}{2}\left\|\mathbf{x} - \frac{1}{M}\sum_i(\mathbf{x}_{(i)} - \mathbf{z}_i)\right\|_2^2 + C$$

where C is independent of $\mathbf{x}$

dual ADMM

reformulate

$$\min_{\mathbf{x}} \lambda\mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$$

into

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \lambda\mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{y}, \mathbf{b}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} - \mathbf{y} = 0 \end{aligned}$$

dualization:

$$\begin{aligned} \Rightarrow \quad & \min_{\mathbf{x},\mathbf{y}} \quad \max_{\mathbf{z}}\{\lambda\mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{y}, \mathbf{b}) + \mathbf{z}^T(\mathbf{A}\mathbf{x} - \mathbf{y})\} \\ \Rightarrow \quad & \max_{\mathbf{z}} \quad \min_{\mathbf{x},\mathbf{y}} \{\lambda\mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{y}, \mathbf{b}) + \mathbf{z}^T(\mathbf{A}\mathbf{x} - \mathbf{y})\} \\ \Rightarrow \quad & \max_{\mathbf{z}} \quad \min_{\mathbf{x}}\{\lambda\mathcal{R}(\mathbf{x}) + \mathbf{z}^T\mathbf{A}\mathbf{x}\} + \min_{\mathbf{y}}\{\mathcal{L}(\mathbf{y}, \mathbf{b}) - \mathbf{z}^T\mathbf{y}\} \\ \Rightarrow \quad & \max_{\mathbf{z}} \quad \mathcal{G}(\mathbf{A}^T\mathbf{z}) + \mathcal{H}(\mathbf{z}) \end{aligned}$$

where $\mathcal{G}(\mathbf{z}) := \min_{\mathbf{x}}\{\lambda\mathcal{R}(\mathbf{x}) + \mathbf{z}^T\mathbf{x}\}$ and $\mathcal{H}(\mathbf{z}) := \min_{\mathbf{y}}\{\mathcal{L}(\mathbf{y}, \mathbf{b}) - \mathbf{z}^T\mathbf{y}\}$

scenario b: dual ADMM

assume separability $\mathcal{R}(\mathbf{x}) = \sum_j \mathcal{R}_j(\mathbf{x}_j)$, then $\mathcal{G}$ is also separable

$$\mathcal{G}(\mathbf{A}^T\mathbf{z}) = \sum_j \mathcal{G}_j(\mathbf{A}_j^T\mathbf{z})$$

reformulate the dual problem as:

$$\min_{\mathbf{z}} \quad -\tfrac{1}{N}\textstyle\sum_j \mathcal{H}(\mathbf{z}_{(j)}) - \sum_j \mathcal{G}_j(\mathbf{y}_j)$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{A}_1^T & & \\ & \ddots & \\ & & \mathbf{A}_N^T \end{bmatrix} \begin{bmatrix} \mathbf{z}_{(1)} \\ \vdots \\ \mathbf{z}_{(N)} \end{bmatrix} - \begin{bmatrix} \mathbf{I} & & \\ & \ddots & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I} & & \\ & \ddots & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{(1)} \\ \vdots \\ \mathbf{z}_{(N)} \end{bmatrix} - \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \mathbf{z} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

note: $\mathbf{z}_{(j)}$ are *not* subvectors of $\mathbf{z}$ but *local copies* of $\mathbf{z}$

scenario b: dual ADMM

storage: node j keeps column block $\mathbf{A}_j$, vector $\mathbf{b}$, and current $\mathbf{y}_j^k$, $\mathbf{z}^k$, $\mathbf{z}_{(j)}^k$, and dual variables $\mathbf{x}_j^k$, $\mathbf{p}_i^k$ of the dual problem

each iteration:

1. *parallel*: $\mathbf{y}_i^{k+1} \leftarrow \arg\min_{\mathbf{y}_i} -\mathcal{G}_i(\mathbf{y}_i) + \frac{\alpha}{2}\|\mathbf{A}_i\mathbf{z}_{(i)}^k - \mathbf{y}_i - \mathbf{x}_i^k\|_2^2$, for every i
2. *parallel*: $\mathbf{z}_{(i)}^{k+1} \leftarrow \arg\min_{\mathbf{z}_{(i)}} -\frac{1}{N}\mathcal{H}(\mathbf{z}_{(i)}) + \frac{\alpha}{2}\|\mathbf{A}_i\mathbf{z}_{(i)} - \mathbf{y}_i^{k+1} - \mathbf{x}_i^k\|_2^2 + \frac{\beta}{2}\|\mathbf{z}_{(i)} - \mathbf{z}^k - \mathbf{p}_i^k\|_2^2$, for every i
3. *allreduce*: $\mathbf{z} \leftarrow \frac{1}{N}\sum_i^N(\mathbf{z}_{(i)}^{k+1} - \mathbf{p}_i^k)$ for all nodes
4. *parallel*: $\mathbf{x}_{(i)}^{k+1} \leftarrow \mathbf{x}_{(i)}^k - \gamma(\mathbf{A}_i^T\mathbf{z}_{(i)}^{k+1} - \mathbf{y}_i^{k+1})$ for every i
5. *parallel*: $\mathbf{p}_i^{k+1} \leftarrow \mathbf{p}_i^k - \gamma(\mathbf{z}_{(i)}^{k+1} - \mathbf{z}^{k+1})$ for every i

Outline

1. sparse optimization background
2. model and data distribution
3. parallel and distributed algorithms
   - parallel/distributed prox-linear iteration
   - GRock: greedy coordinate-block descent
   - parallel/distributed ADMM
4. numerical experiments
   - Rice cluster STIC
   - Amazon EC2
5. conclusions

test on STIC, a cluster at Rice university

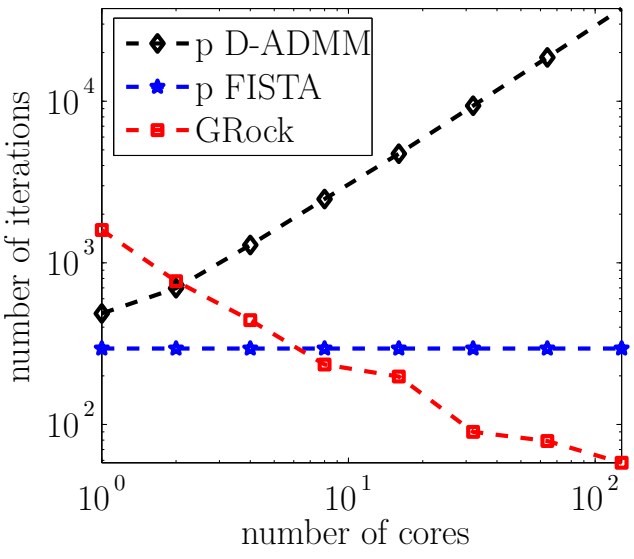
specs:

- 170 Appro Greenblade E5530 nodes each with two quad-core 2.4GHz Xeon (Nahalem) CPUs
- each node has 12GB of memory shared by all 8 cores
- # of processes used on each node = 8

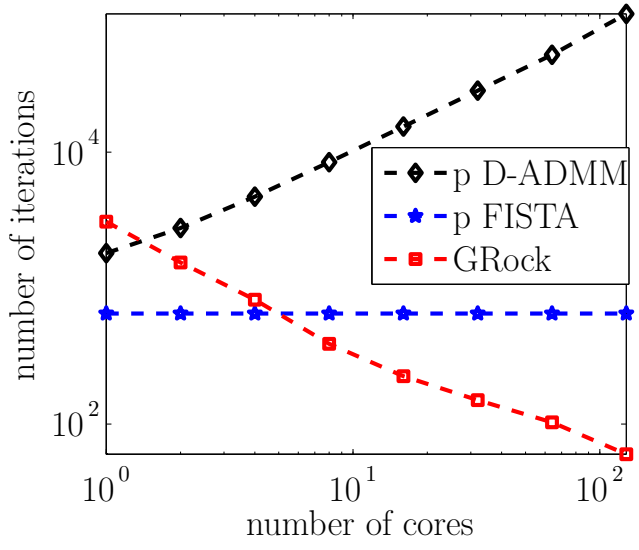
test dataset:

	$\mathbf{A}$ type	$\mathbf{A}$ size	λ	sparsity of $\mathbf{x}^*$
dataset I	Gaussian	1024×2048	0.1	100
dataset II	Gaussian	2048×4096	0.01	200

cores vs iterations

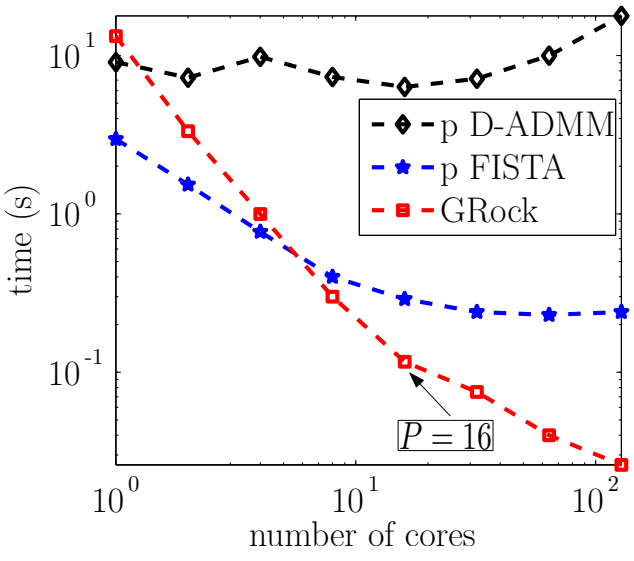


dataset I

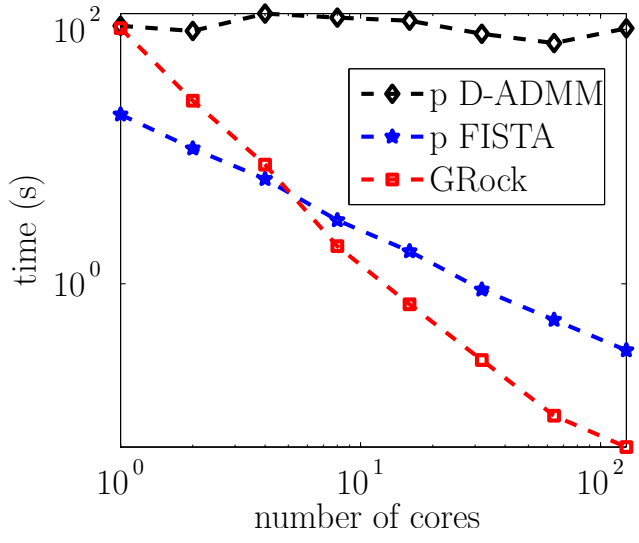


dataset II

cores vs time

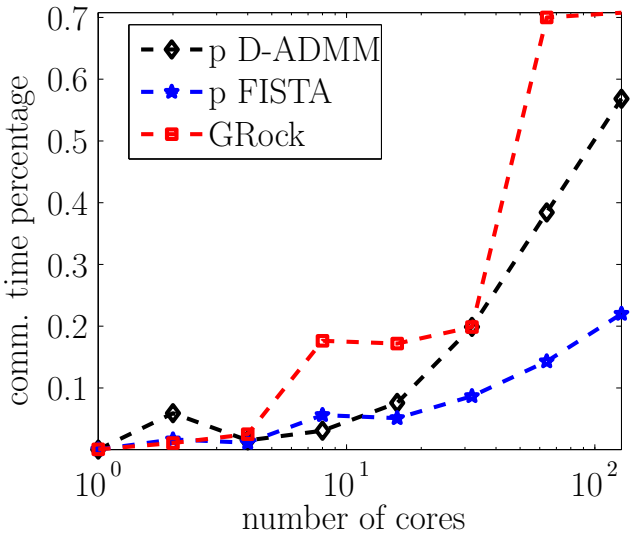


dataset I

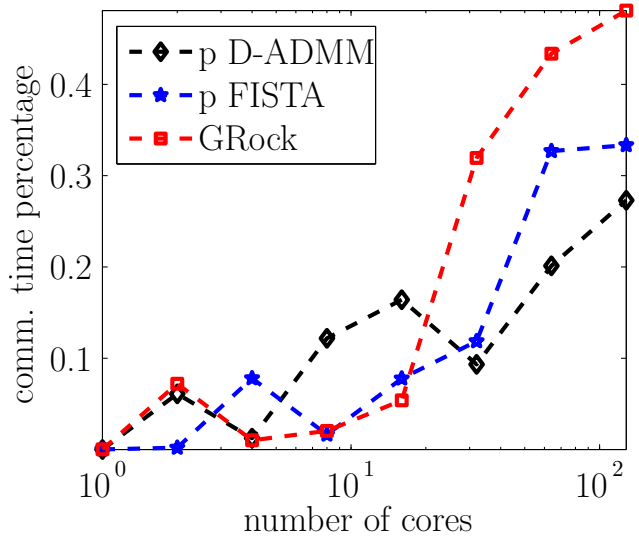


dataset II

cores vs (% communication/total time)



dataset I



dataset II

big data LASSO on Amazon EC2

Amazon EC2 is an elastic, pay-as-you-use cluster;

advantage: no hardware investment, everyone can have an account

test dataset

- $\mathbf{A}$ dense matrix, 20 billion entries, and 170GB size
- $\mathbf{x}$: 200K entries, 4K nonzeros, Gaussian values

requested system

- 20 "high-memory quadruple extra-large instances"
- each instance has 8 cores and 60GB memory

code

- written in C using GSL (for matrix-vector multiplication) and MPI

parallel dual ADMM vs GRock

	p D-ADMM	p FISTA	GRock
estimate stepsize (min.)	n/a	1.6	n/a
matrix factorization (min.)	51	n/a	n/a
iteration time (min.)	105	40	1.7
number of iterations	2500	2500	104
communication time (min.)	30.7	9.5	0.5
stopping relative error	**1E-1**	**1E-3**	**1E-5**
total time (min.)	**156**	**41.6**	**1.7**
Amazon charge	**$85**	**$22.6**	**$0.93**

- ADMM's performance depends on penalty parameter β
  we picked β as the best out of only a few trials (we cannot afford more)
- parallel dual ADMM stopped at 2500 iterations
- GRock stopped at relative error 1E-5

conclusions

summary of results

- ADMM is flexible but does not scale well; its iterations increase with distribution
- separable objectives enable parallel and distributed prox-linear iterations, which scale very well
- nearly orthogonal $\mathbf{A}$ further enables parallel GRock, which scales even better
- in general, model structures and data properties of sparse optimization enable very efficient parallel and distributed approaches

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open resources: report, slides, and codes available on our websites

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