Distributed Sparse Optimization

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STATOS 2013 – in memory of Alex Gershman

Zhimin Peng, Ming Yan, Wotao Yin. *Parallel and Distributed Sparse Optimization*, 2013

Outline

1. Background: Sparse optimization
2. Background: Parallel benefits, speedup, and overhead
3. Parallelize existing algorithms
4. Primal decomposition / dual decomposition
   - Parallel dual gradient ascent (linearized Bregman)
   - Parallel ADMM
5. Parallel greedy coordinate descent
6. Numerical results with big data

Coverage

This talk will not

- survey the existing sparse optimization models or algorithms
- cover decentralized algorithms due to time limit

This talk will

- illustrate how to apply parallel computing techniques on some large-scale sparse optimization problems
- discuss parallel speedup and overhead along the way

Sparse optimization

A tool that processes "large" data sets for "structured" solutions

Examples of structures: sparse, joint/group sparse, sparse graph, low-rank, smooth, low-dim manifold, their combinations, ...

Key points:

- finding sparsest solution is *generally NP-hard*[1]
- ℓ_1 minimization gives the sparsest solution, given sufficient measurements and other conditions[2]
- ℓ_1 minimization is tractable
- has many interesting applications and generalization (requiring smart designs of regularization and sensing matrix)

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[1] Natarajan [1995]

[2] Donoho and Elad [2003], Donoho [2006], Candès, Romberg, and Tao [2006]

Basic approach: ℓ_1 minimization

$$\underset{\mathbf{x}}{\text{minimize}} \, \|\Psi^T \mathbf{x}\|_1, \qquad \text{s.t.} \;\; \mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1a}$$

$$\underset{\mathbf{x}}{\text{minimize}} \, \lambda \|\Psi^T \mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{1b}$$

$$\underset{\mathbf{x}}{\text{minimize}} \, \|\Psi^T \mathbf{x}\|_1, \qquad \text{s.t.} \;\; \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \leq \delta. \tag{1c}$$

The settings:

- Ψ is the sparsifying transform; $\Psi^T \mathbf{x}$ is sparse
- $\mathbf{A}$ is a linear operator / sensing operator / feature matrix
- when $\mathbf{b}$ or $\Psi^T \mathbf{x}$ has noise, (1b) or (1c) is used

Other regularization: joint/group variants, nuclear norm, ...

Standard form of regularization: $\text{minimize} \sum_i (\text{atom}_i)$, good for parallelism

ℓ_1 optimization: single-threaded approaches

Off-the-shelf: subgradient descent, reduction to LP/SOCP/SDP

Smoothing: approximate ℓ_1 by $\ell_{1+\epsilon}$-norm, $\sum_i \sqrt{x_i^2 + \epsilon}$, Huber-norm, augmented ℓ_1

then, apply gradient-based and quasi-Newton methods.

Splitting: split smooth and nonsmooth terms into simple subproblems

- primal splitting: GPSR/FPC/SpaRSA/FISTA/SPGL1/...
- dual operator splitting: Bregman/ADMM/split Bregman/...

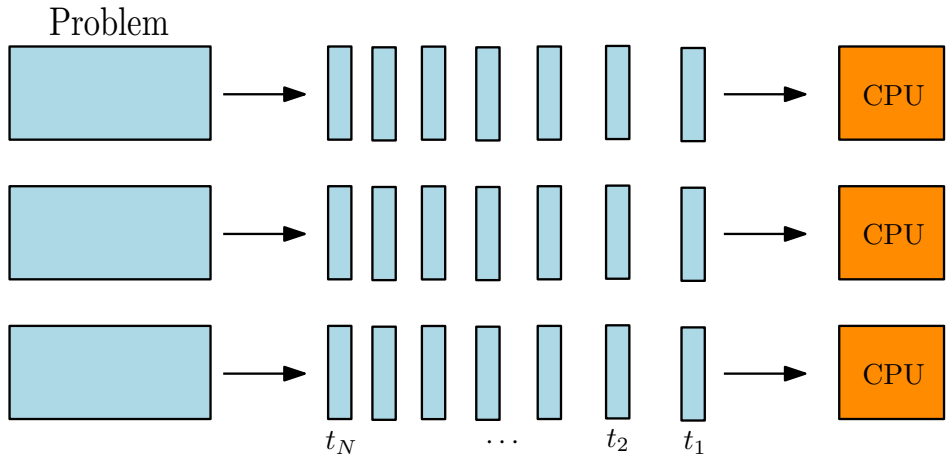
Greedy approaches: OMP and variants, greedy coordinate descent

Non-convex approaches

.....

Parallel computing

- a problem is broken into *concurrent* parts, executed *simultaneously*, coordinated by a controller
- uses multiple cores/CPUs/networked computers, or their *combinations*
- expect to solve a problem in *less time*, or a *larger problem*



Benefits of being parallel

- **break single-CPU limits**
- **save time**
- **process big data**
- **access non-local resource**
- **handle concurrent data streams / enable collaboration**

Parallel overhead

- **computing overhead:** *start up time, synchronization wait time, data communication time, termination (data collection time)*
- **I/O overhead:** slow read and write of *non-local* data
- **algorithm overhead:** *extra variables, data duplication*
- **coding overhead:** language, library, operating system, debug, maintenance

Parallel speedup

- **definition:**

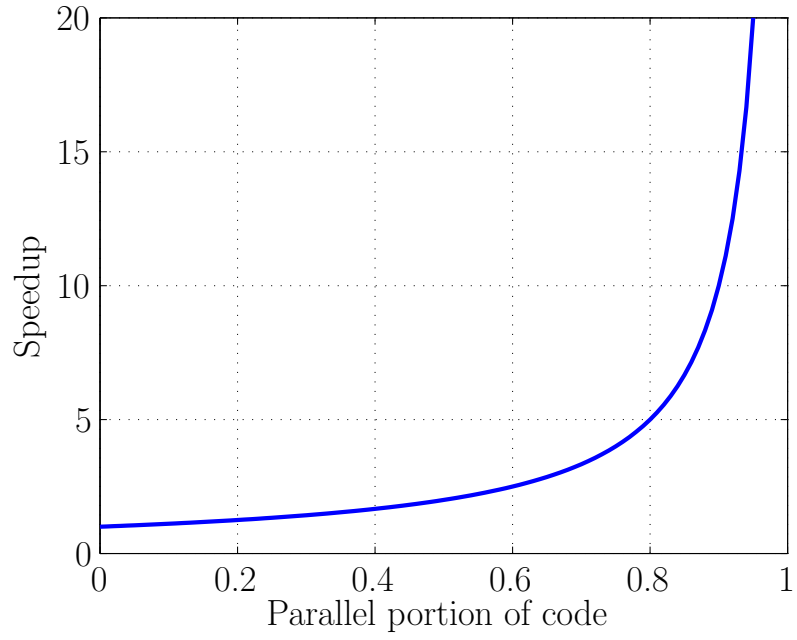
$$\text{speedup} = \frac{\text{serial time}}{\text{parallel time}}$$

  time is in the *wall-clock* sense

- **Amdahl's Law:** assume infinite processors and no overhead

$$\text{ideal max speedup} = \frac{1}{1-P}$$

  where $P =$ parallelized fraction of code

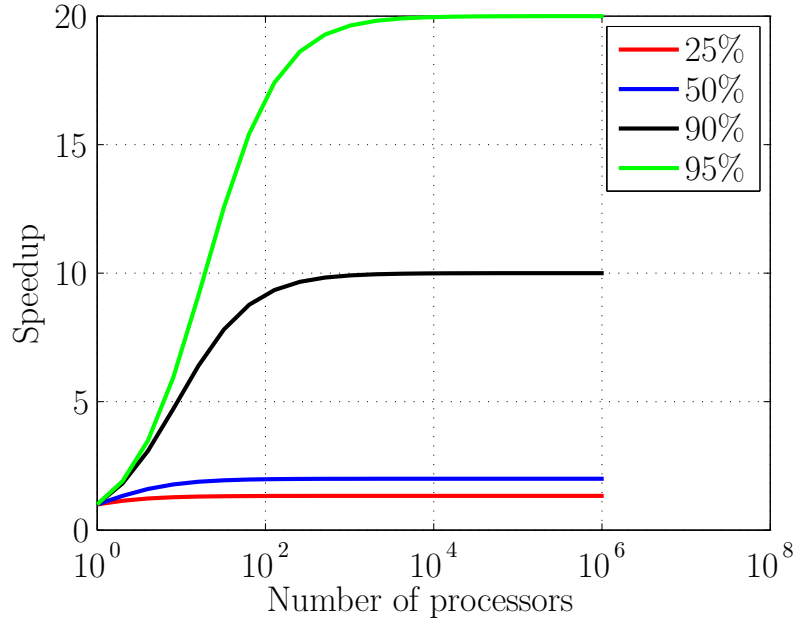


Parallel speedup

- assume N processors and no overhead

$$\text{ideal speedup} = \frac{1}{(P/N) + (1 - P)}$$

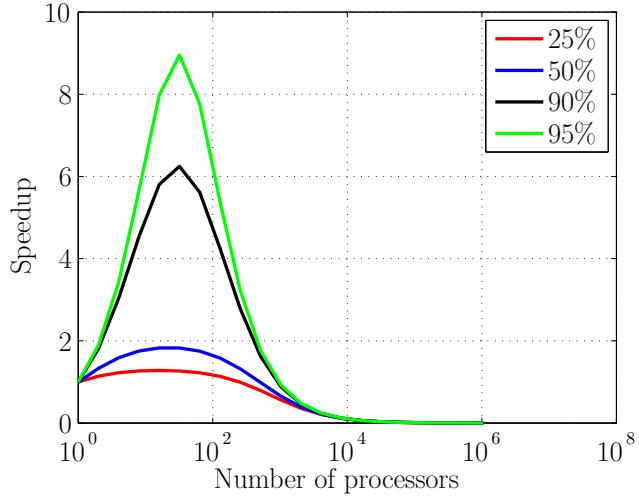
- **note:** P may depend on data size, which depends on input size and N



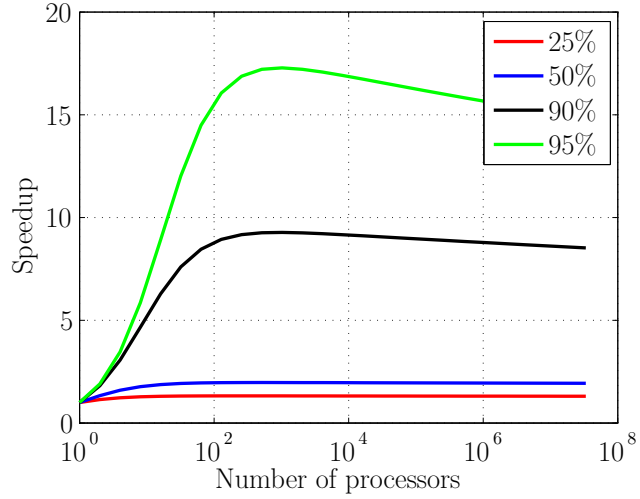
Parallel speedup

- $E :=$ parallel overhead
- **in the real world**

$$\text{actual speedup} = \frac{1}{(P/N) + (1-P) + E}$$

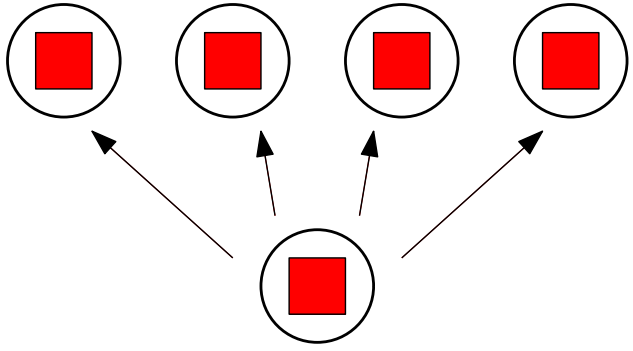


when $E = N$

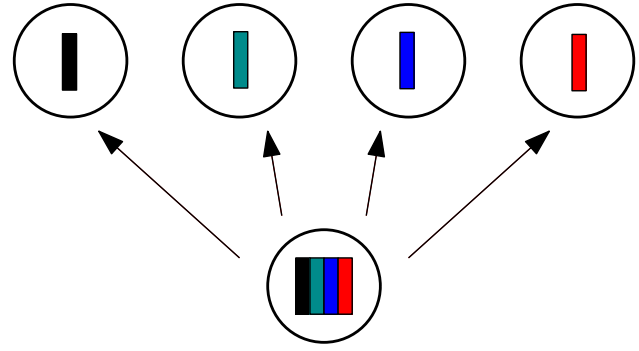


when $E = \log(N)$

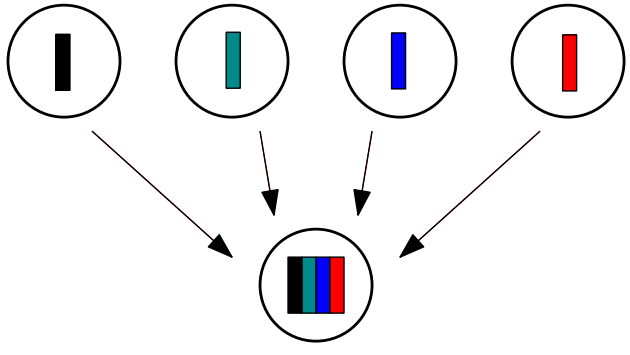
Types of communication



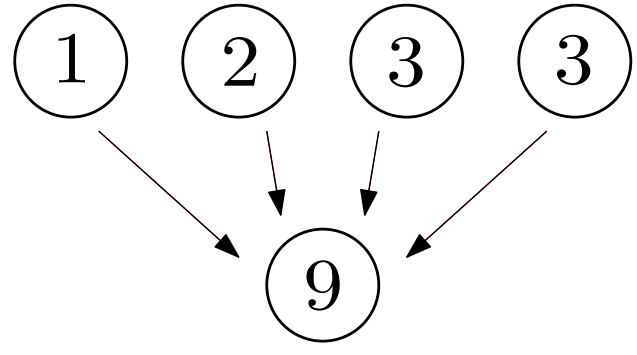
Broadcast



Scatter



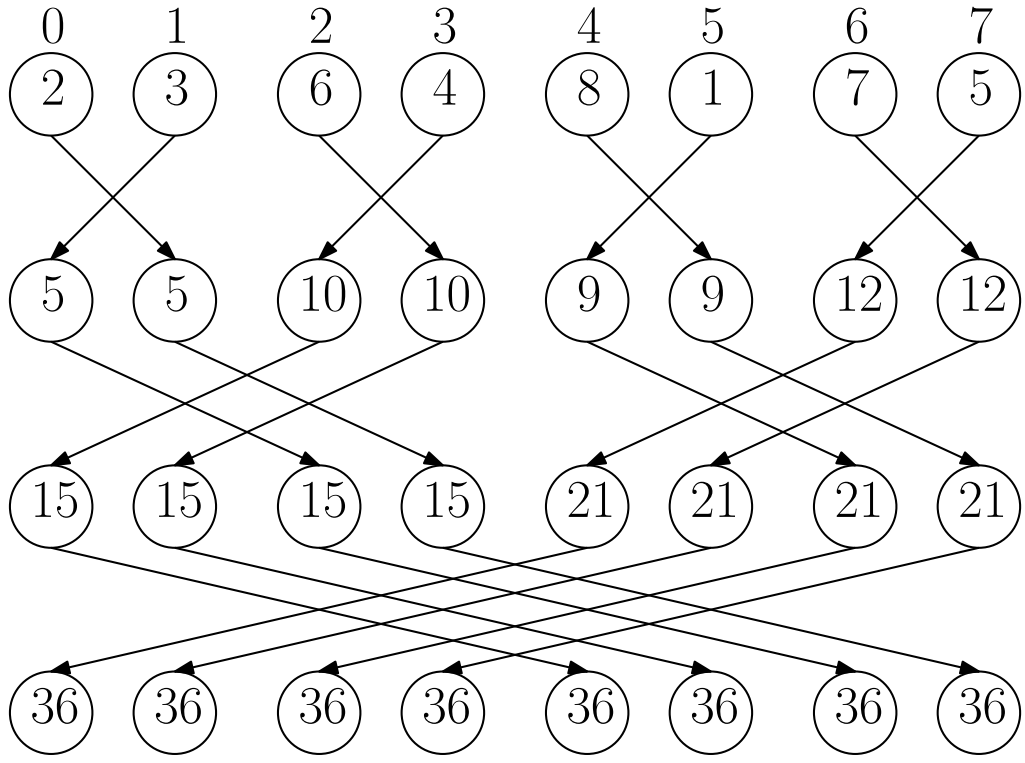
Gather



Reduce

Allreduce

Allreduce Sum via butterfly communication



$\log(N)$ layers, N parallel communications per layer

Decomposable objective and constraints enables parallelism

- variables $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]^T$
- embarrassingly parallel (not an interesting case): $\min \sum_i f_i(\mathbf{x}_i)$
- consensus minimization, no constraints

$$\min f(\mathbf{x}) = \sum_{i=1}^{N} f_i(\mathbf{x}) + r(\mathbf{x})$$

  example: $f(\mathbf{x}) = \frac{\lambda}{2} \sum_i \|\mathbf{A}_{(i)}\mathbf{x} - \mathbf{b}_i\|_2^2 + \|\mathbf{x}\|_1$

- coupling variable $\mathbf{z}$

$$\min_{\mathbf{x},\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{N} [f_i(\mathbf{x}_i, \mathbf{z}) + r_i(\mathbf{x}_i)] + r(\mathbf{z})$$

- coupling constraints

$$\min f(\mathbf{x}) = \sum_{i=1}^{N} f_i(\mathbf{x}_i) + r_i(\mathbf{x}_i)$$

  subject to
  - equality constraints: $\sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i = b$
  - inequality constraints: $\sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i \leq b$
  - graph-coupling constraints $\mathbf{A}\mathbf{x} = b$, where $\mathbf{A}$ is sparse

- combinations of independent variables, coupling variables and constraints
  example:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{w},\mathbf{z}} \quad & \textstyle\sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \\ \text{s.t.} \quad & \textstyle\sum_i \mathbf{A}_i \mathbf{x}_i \leq \mathbf{b} \\ & \mathbf{B}_i \mathbf{x}_i \leq \mathbf{w}, \ \forall i \\ & \mathbf{w} \in \Omega \end{aligned}$$

- variable coupling $\longleftrightarrow$ constraint coupling:

$$\min \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \longrightarrow \min \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}_i) \quad \text{s.t. } \mathbf{z}_i = \mathbf{z} \ \ \forall i$$

Approaches toward parallelism

- Keep an existing serial algorithm, parallelize its expensive part(s)
- Introduce a new parallel algorithm by
  - formulating an equivalent model
  - replacing the model by an approximate model that's easier to parallelize

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Example: parallel ISTA

$$\underset{\mathbf{x}}{\text{minimize}}\, \lambda\|\mathbf{x}\|_1 + f(\mathbf{x})$$

- Prox-linear approximation:

$$\arg\min_{\mathbf{x}} \lambda\|\mathbf{x}\|_1 + \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{x}^k \rangle + \frac{1}{2\delta_k}\|\mathbf{x} - \mathbf{x}^k\|_2^2$$

- Iterative soft-threshold algorithm:

$$\mathbf{x}^{k+1} = \text{shrink}\left(\mathbf{x}^k - \delta_k \nabla f(\mathbf{x}), \lambda\delta_k\right)$$

  - $\text{shrink}(\mathbf{y}, t)$ has a simple close form: `max(abs(y)-t,0).*sign(y)`
  - the dominating computation is $\nabla f(\mathbf{x})$

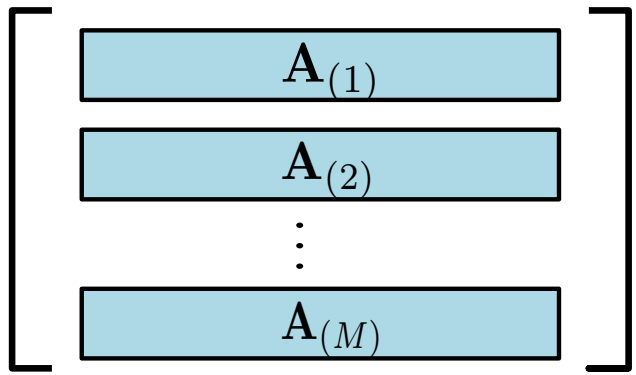
- examples: $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$, logistic loss of $\mathbf{A}\mathbf{x} - \mathbf{b}$, hinge loss ... where $\mathbf{A}$ is often dense and can go very large-scale

Parallel gradient computing of $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b})$: row partition

Let $\mathbf{A}$ be very big and dense, and $g(\mathbf{y}) = \sum g_i(\mathbf{y}_i)$.

Then $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b}) = \sum g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$

- let $\mathbf{A}_{(i)}$ and $\mathbf{b}_i$ be the ith row block of $\mathbf{A}$ and $\mathbf{b}$, respectively



- store $\mathbf{A}_{(i)}, \mathbf{b}_i$ on node i
- goal: to compute $\nabla f(\mathbf{x}) = \sum \mathbf{A}_{(i)}^T \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$, or similarly $\partial f(\mathbf{x})$

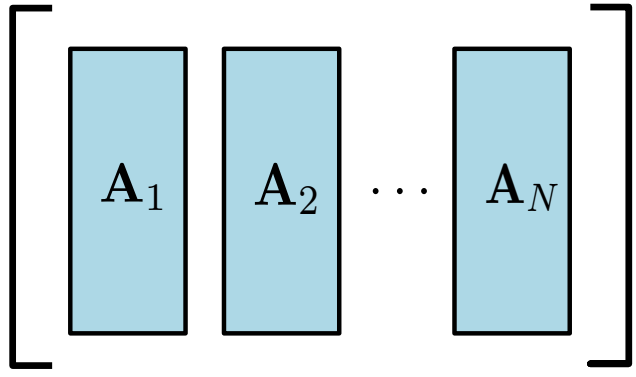
Parallel gradient computing of $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b})$: row partition

Steps

1. compute $\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i \ \forall i$ in parallel;
2. compute $\mathbf{g}_i = \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) \ \forall i$ in parallel;
3. compute $\mathbf{A}_{(i)}^T \mathbf{g}_i \ \forall i$ in parallel;
4. allreduce $\sum_{i=1}^{M} \mathbf{A}_{(i)}^T \mathbf{g}_i$.

- step 1, 2 and 3 are local computation and communication-free;
- step 4 requires a collective communication.

Parallel gradient computing of $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b})$: column partition



column block partition

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i \implies f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b}) = g(\sum \mathbf{A}_i \mathbf{x}_i + \mathbf{b})$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \mathbf{A}_1^T \nabla g(\mathbf{A}\mathbf{x} + \mathbf{b}) \\ \mathbf{A}_2^T \nabla g(\mathbf{A}\mathbf{x} + \mathbf{b}) \\ \vdots \\ \mathbf{A}_N^T \nabla g(\mathbf{A}\mathbf{x} + \mathbf{b}) \end{pmatrix}, \quad \text{similar for } \partial f(\mathbf{x})$$

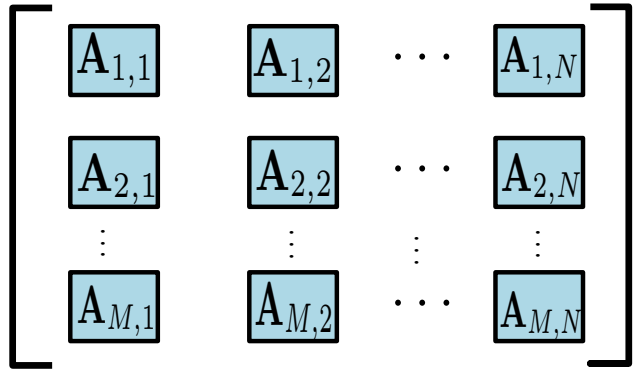
Parallel computing of $\nabla f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b})$: column partition

Steps

1. compute $\mathbf{A}_i\mathbf{x}_i + \frac{1}{N}\mathbf{b}\ \forall i$ in parallel
2. allreduce $\mathbf{A}\mathbf{x} + \mathbf{b} = \sum_{i=1}^{N} \mathbf{A}_i\mathbf{x}_i + \frac{1}{N}\mathbf{b}$;
3. evaluate $\nabla g(\mathbf{A}\mathbf{x} + \mathbf{b})\ \forall i$ in each process;
4. compute $\mathbf{A}_i^T \nabla g(\mathbf{A}\mathbf{x} + \mathbf{b})\ \forall i$ in parallel.

- each processor keeps $\mathbf{A}_i$ and $\mathbf{x}_i$, as well as a copy of $\mathbf{b}$;
- step 2 requires a collective communication;
- step 3 involves duplicated computation in each process.

Two-way partition



$$f(\mathbf{x}) = \sum g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) = \sum_i g_i(\sum_j \mathbf{A}_{i,j}\mathbf{x}_j + \mathbf{b}_i)$$

$$\nabla f(\mathbf{x}) = \sum_i \mathbf{A}_{(i)}^T \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) = \sum_i \mathbf{A}_{(i)}^T \nabla g_i(\sum_j \mathbf{A}_{i,j}\mathbf{x}_j + \mathbf{b}_i)$$

Two-way partition

Step

1. compute $\mathbf{A}_{i,j}\mathbf{x}_j + \frac{1}{N}\mathbf{b}_i \ \forall i$ in parallel;
2. allreduce $\sum_{j=1}^{N} \mathbf{A}_{i,j}\mathbf{x}_j + \mathbf{b}_i$ for every i;
3. compute $\mathbf{g}_i = \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) \ \forall i$ in parallel;
4. compute $\mathbf{A}_{j,i}^T \mathbf{g}_i \ \forall i$ in parallel;
5. allreduce $\sum_{i=1}^{N} \mathbf{A}_{j,i}^T \mathbf{g}_i$ for every j.

- processor (i, j) keeps $\mathbf{A}_{i,j}$, $\mathbf{x}_j$ and $\mathbf{b}_i$;
- step 2 and 5 require the collective communication;

Example: parallel ISTA for LASSO

LASSO

$$\min \ f(\mathbf{x}) = \lambda \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

algorithm

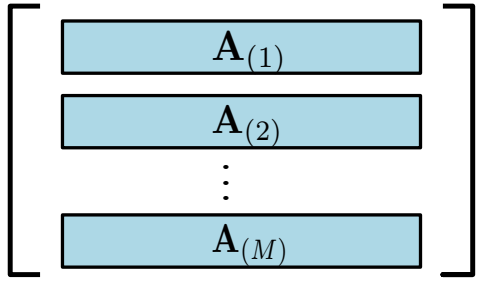
$$\mathbf{x}^{k+1} = \text{shrink}\left(\mathbf{x}^k - \delta_k \mathbf{A}^T\mathbf{A}\mathbf{x}^k + \delta_k \mathbf{A}^T\mathbf{b},\ \lambda\delta_k\right)$$

Serial pseudo-code

1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: pre-compute $\mathbf{A}^T\mathbf{b}$
3: **while** not converged **do**
4: $\quad \mathbf{g} = \mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{A}^T\mathbf{b}$
5: $\quad \mathbf{x} = \text{shrink}\,(\mathbf{x} - \delta\mathbf{g},\ \lambda\delta)$;
6: **end while**

Example: parallel ISTA for LASSO

distribute blocks of rows to M nodes

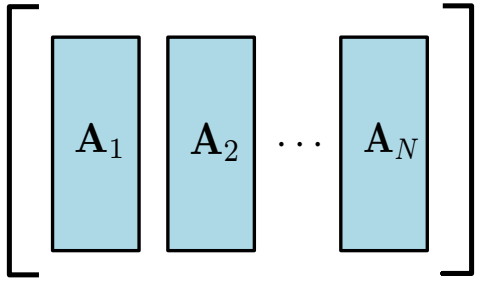


1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: $i =$ find_my_processor_id
3: processor i loads $\mathbf{A}_{(i)}$
4: pre-compute $\delta \mathbf{A}^T \mathbf{b}$
5: **while** not converged **do**
6: processor i computes $\mathbf{c}_i = \mathbf{A}_{(i)}^T \mathbf{A}_{(i)} \mathbf{x}$
7: $\mathbf{y} =$ allreduce($\mathbf{c}_i$, SUM)
8: $\mathbf{x} = \text{shrink}\left(\mathbf{x}^k - \delta \mathbf{c} + \delta \mathbf{A}^T \mathbf{b}, \lambda \delta\right)$;
9: **end while**

- assume $\mathbf{A} \in \mathbb{R}^{m \times n}$
- one allreduce per iteration
- speedup
  $\approx \frac{1}{P/M + (1-P) + O(\log(M))}$
- P is close to 1
- requires synchronization

Example: parallel ISTA for LASSO

distribute blocks of columns to N nodes



1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: i = find_my_processor_id
3: processor i loads $\mathbf{A}_i$
4: pre-compute $\delta \mathbf{A}_i^T \mathbf{b}$
5: **while** not converged **do**
6: processor i computes $\mathbf{y}_i = \mathbf{A}_i \mathbf{x}_i$
7: $\mathbf{y}$ = allreduce($\mathbf{y}_i$, SUM)
8: processor i computes $\mathbf{z}_i = \mathbf{A}_i^T \mathbf{y}$
9: $\mathbf{x}_i^{k+1} = \text{shrink}\left(\mathbf{x}_i^k - \delta \mathbf{z}_i + \delta \mathbf{A}_i^T \mathbf{b}, \lambda\delta\right)$;
10: **end while**

- one allreduce per iteration
- speedup $\approx \frac{1}{P/N + (1-P) + O(\frac{m}{n}\log(N))}$
- P is close to 1
- requires synchronization
- if $m \ll n$, this approach is faster

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[3] Dantzig and Wolfe [1960], Spingarn [1985], Bertsekas and Tsitsiklis [1997]

Unconstrained minimization with coupling variables

$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]^T$.

$\mathbf{z}$ is the **coupling/bridging/complicating** variable.

$$\min_{\mathbf{x},\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z})$$

equivalently to separable objective with **coupling constraints**

$$\min_{\mathbf{x},\mathbf{z}} \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}_i) \quad \text{s.t. } \mathbf{z}_i = \mathbf{z} \quad \forall i$$

Examples

- network utility maximization (NUM), extend to multiple layers[4]
- domain decomposition ($\mathbf{z}$ are overlapping boundary variables)
- system of equations: $\mathbf{A}$ is block-diagonal but with a few dense columns

[4] see tutorial: Palomar and Chiang [2006]

Primal decomposition (bi-level optimization)

Introduce **easier subproblems**

$$g_i(\mathbf{z}) = \min_{\mathbf{x}_i} f_i(\mathbf{x}_i, \mathbf{z}), \quad i = 1, \ldots, N$$

then

$$\min_{\mathbf{x},\mathbf{z}} \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \iff \min_{\mathbf{z}} \sum g_i(\mathbf{z})$$

the RHS is called the **master problem**

parallel computing: iterate

1. fix $\mathbf{z}$, update $\mathbf{x}_i$ and compute $g_i(\mathbf{z})$ *in parallel*
2. update $\mathbf{z}$ using a standard method (typically, subgradient descent)

- good for small-dimensional $\mathbf{z}$
- fast if the master problem converges fast

Dual decomposition

Consider

$$\begin{aligned} \min_{\mathbf{x},\mathbf{z}} \quad & \sum_i f_i(\mathbf{x}_i, \mathbf{z}_i) \\ \text{s.t.} \quad & \sum_i \mathbf{A}_i \mathbf{z}_i = \mathbf{b} \end{aligned}$$

Introduce

$$g_i(\mathbf{z}) = \min_{\mathbf{x}_i} f(\mathbf{x}_i, \mathbf{z})$$

$$g_i^*(\mathbf{y}) = \max_{\mathbf{z}} \mathbf{y}^T \mathbf{z} - g_i(\mathbf{z}) \qquad \text{//convex conjugate}$$

Dualization:

$$\Leftrightarrow \quad \min_{\mathbf{x},\mathbf{z}} \max_{\mathbf{y}} \sum_i f_i(\mathbf{x}_i, \mathbf{z}_i) + \mathbf{y}^T (\mathbf{b} - \sum_i \mathbf{A}_i \mathbf{z}_i)$$

(generalized minimax thm.) $$\Leftrightarrow \quad \max_{\mathbf{y}_i} \min_{\mathbf{x}_i, \mathbf{z}_i} \left\{ \sum_i \left(f_i(\mathbf{x}_i, \mathbf{z}_i) - \mathbf{y}^T \mathbf{A}_i \mathbf{z}_i \right) + \mathbf{y}^T \mathbf{b} \right\}$$

$$\Leftrightarrow \quad \max_{\mathbf{y}_i} \left\{ \min_{\mathbf{z}_i} \sum_i \left(g_i(\mathbf{z}_i) - \mathbf{y}^T \mathbf{A}_i \mathbf{z}_i \right) + \mathbf{y}^T \mathbf{b} \right\}$$

$$\Leftrightarrow \quad \min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T \mathbf{y}) - \mathbf{y}^T \mathbf{b}$$

Example: consensus and exchange problems

Consensus problem

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z})$$

Reformulate as

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z}_i) \quad \text{s.t. } \mathbf{z}_i = \mathbf{z} \ \ \forall i$$

Its dual problem is the *exchange problem*

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{y}_i) \quad \text{s.t. } \sum_i \mathbf{y}_i = 0$$

Primal-dual objective properties

Constrained separable problem

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z}) \quad \text{s.t. } \sum_i \mathbf{A}_i \mathbf{z}_i = \mathbf{b}$$

Dual problem

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T \mathbf{y}) - \mathbf{b}^T \mathbf{y}$$

- If g_i is proper, closed, convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is sub-differentiable
- If g_i is strictly convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is differentiable
- If g_i is strongly convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is differentiable and has Lipschitz gradient
- Easily obtaining $\mathbf{z}^*$ from $\mathbf{y}^*$ generally requires strict convexity of g_i or strict complementary slackness

Example:

- $g_i(\mathbf{z}) = |z_i|$ and $g_i^*(\mathbf{a}_i^T \mathbf{y}) = \iota\{-1 \leq \mathbf{a}_i^T \mathbf{y} \leq 1\}$
- $g_i(\mathbf{z}) = \frac{1}{2}|z_i|^2$ and $g_i^*(\mathbf{a}_i^T \mathbf{y}) = \frac{1}{2}|\mathbf{a}_i^T \mathbf{y}|^2$
- $g_i(\mathbf{z}) = \exp(z_i)$ and $g_i^*(\mathbf{a}_i^T \mathbf{y}) = (\mathbf{a}_i^T \mathbf{y}) \ln(\mathbf{a}_i^T \mathbf{y}) - \mathbf{a}_i^T \mathbf{y} + \iota\{\mathbf{a}_i^T \mathbf{y} \geq 0\}$

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[5] Yin, Osher, Goldfarb, and Darbon [2008]

Example: augmented ℓ_1

Change $|z_i|$ into strongly convex $g_i(z_i) = |z_i| + \frac{1}{2\alpha}|z_i|^2$

Dual problem

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T\mathbf{y}) - \mathbf{b}^T\mathbf{y} = \sum_i \underbrace{\frac{\alpha}{2}|\operatorname{shrink}(\mathbf{A}_i^T\mathbf{y})|^2}_{h_i(\mathbf{y})} - \mathbf{b}^T\mathbf{y}$$

Dual gradient iteration (equivalent to linearized Bregman):

$$\mathbf{y}^{k+1} = \mathbf{y}^k - t^k\left(\sum_i \nabla h_i(\mathbf{y}^k) - \mathbf{b}\right), \quad \text{where } \nabla h_i(\mathbf{y}) = \alpha\mathbf{A}_i \operatorname{shrink}(\mathbf{A}_i^T\mathbf{y}).$$

- **Dual is C^1 and unconstrained.** One can apply (accelerated) gradient descent, line search, quasi-Newton method, etc.
- **Global linear convergence.**[6]
- **Easy to parallelize**. One collective communication per iteration.
- Recover $\mathbf{x}^* = \alpha \operatorname{shrink}(\mathbf{A}_i^T\mathbf{y}^*)$.

[6] Lai and Yin [2011]

Augmented ℓ_1: exact regularization[7]

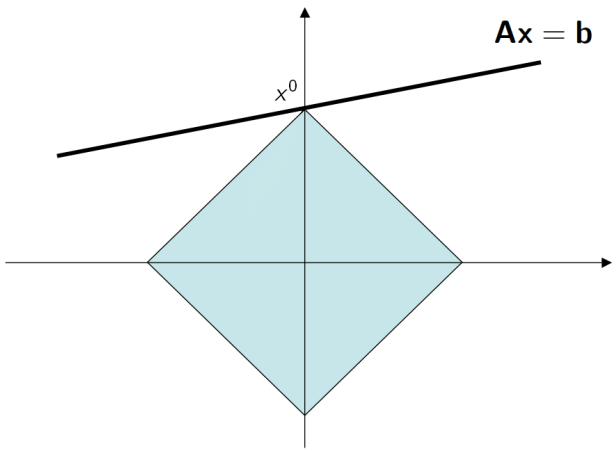
Theorem

There exists a finite $\alpha^0 > 0$ such that whenever $\alpha > \alpha^0$, the solution to

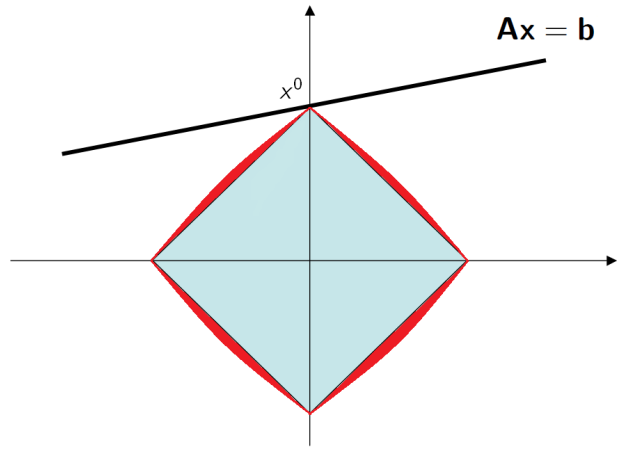
$$\textit{(L1+LS)} \qquad \min \|\mathbf{x}\|_1 + \frac{1}{2\alpha}\|\mathbf{x}\|_2^2 \quad \textit{s.t. } \mathbf{Ax} = \mathbf{b}$$

is also a solution to

$$\textit{(L1)} \qquad \min \|\mathbf{x}\|_1 \quad \textit{s.t. } \mathbf{Ax} = \mathbf{b}.$$



L1



L1+LS

[7] Friedlander and Tseng [2007], Yin [2010]

Augmented ℓ_1: CS recovery guarantees

- if $\min \|\mathbf{x}\|_1$ gives exact or stable recovery provided

$$\#\text{measurements} \geq C \cdot f(\text{signal dim}, \text{signal sparsity}).$$

- adding $\frac{1}{2\alpha}\|\mathbf{x}\|_2^2$ changes the condition to

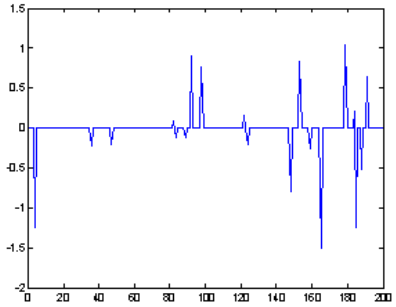
$$\#\text{measurements} \geq (C + O(\frac{1}{2\alpha})) \cdot f(\text{signal dim}, \text{signal sparsity}).$$

- in theory and practice, $\alpha \geq 10\|\mathbf{x}^o\|_\infty$ suffices[8]

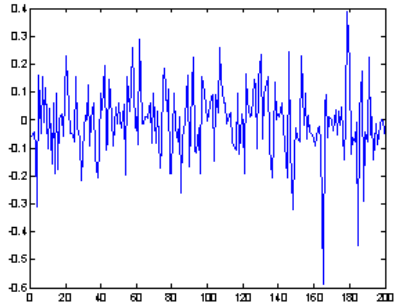
[8] Lai and Yin [2011]

Augmented ℓ_1: simple test

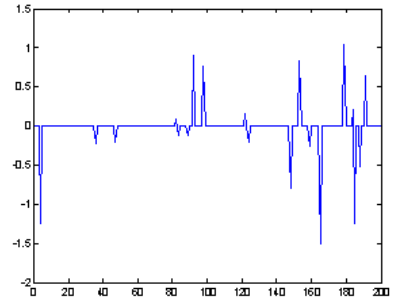
- Gaussian random $\mathbf{A}$
- $\mathbf{x}^o$ had 10% nonzero entries
- recover $\mathbf{x}^o$ from under-sampled $\mathbf{b} = \mathbf{A}\mathbf{x}^o$



$\min\{\|\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$



$\min\{\|\mathbf{x}\|_2^2 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$



$\min\{\|\mathbf{x}\|_1 + \frac{1}{25}\|\mathbf{x}\|_2^2 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$

exactly the same as ℓ_1 solution

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[9] Bertsekas and Tsitsiklis [1997]

Alternating direction method of multipliers (ADMM)

step 1: turn problem into the form of

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & f(\mathbf{x}) + g(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b}. \end{aligned}$$

f and g are **convex**, maybe **nonsmooth**, can **incorporate constraints**

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f and g are **convex**, maybe **nonsmooth**, can **incorporate constraints**

step 2: iterate

1. $\mathbf{x}^{k+1} \leftarrow \min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{y}^k) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k\|_2^2$,
2. $\mathbf{y}^{k+1} \leftarrow \min_{\mathbf{y}} f(\mathbf{x}^{k+1}) + g(\mathbf{y}) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y} - \mathbf{b} - \mathbf{z}^k\|_2^2$,
3. $\mathbf{z}^{k+1} \leftarrow \mathbf{z}^k - (\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y}^{k+1} - \mathbf{b})$.

history: dates back to 60s, formalized 80s, parallel versions appeared late 80s, revived very recently, new convergence results recently

ADMM and parallel/distributed computing

Two approaches:

- parallelize the serial algorithm(s) for ADMM subproblem(s),
- turn problem into a parallel-ready form[10]

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \sum_i f_i(\mathbf{x}_i) + g(\mathbf{y}) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{A}_1 & & \\ & \ddots & \\ & & \mathbf{A}_M \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_M \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_M \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_M \end{bmatrix} \end{aligned}$$

consequently, computing $\mathbf{x}^{k+1}$ reduces to parallel subproblems

$$\mathbf{x}_i^{k+1} \leftarrow \min_{\mathbf{x}} f_i(\mathbf{x}_i) + \frac{\beta}{2}\|\mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{y}^k - \mathbf{b}_i - \mathbf{z}_i^k\|_2^2, \quad i = 1, \ldots, M$$

Issue: to have *block diagonal* $\mathbf{A}$ and *simple* $\mathbf{B}$, additional variables are often needed, thus increasing the problem size and parallel overhead.

---

[10] a recent survey Boyd, Parikh, Chu, Peleato, and Eckstein [2011]

Parallel Linearized Bregman vs ADMM on basis pursuit

Basis pursuit:

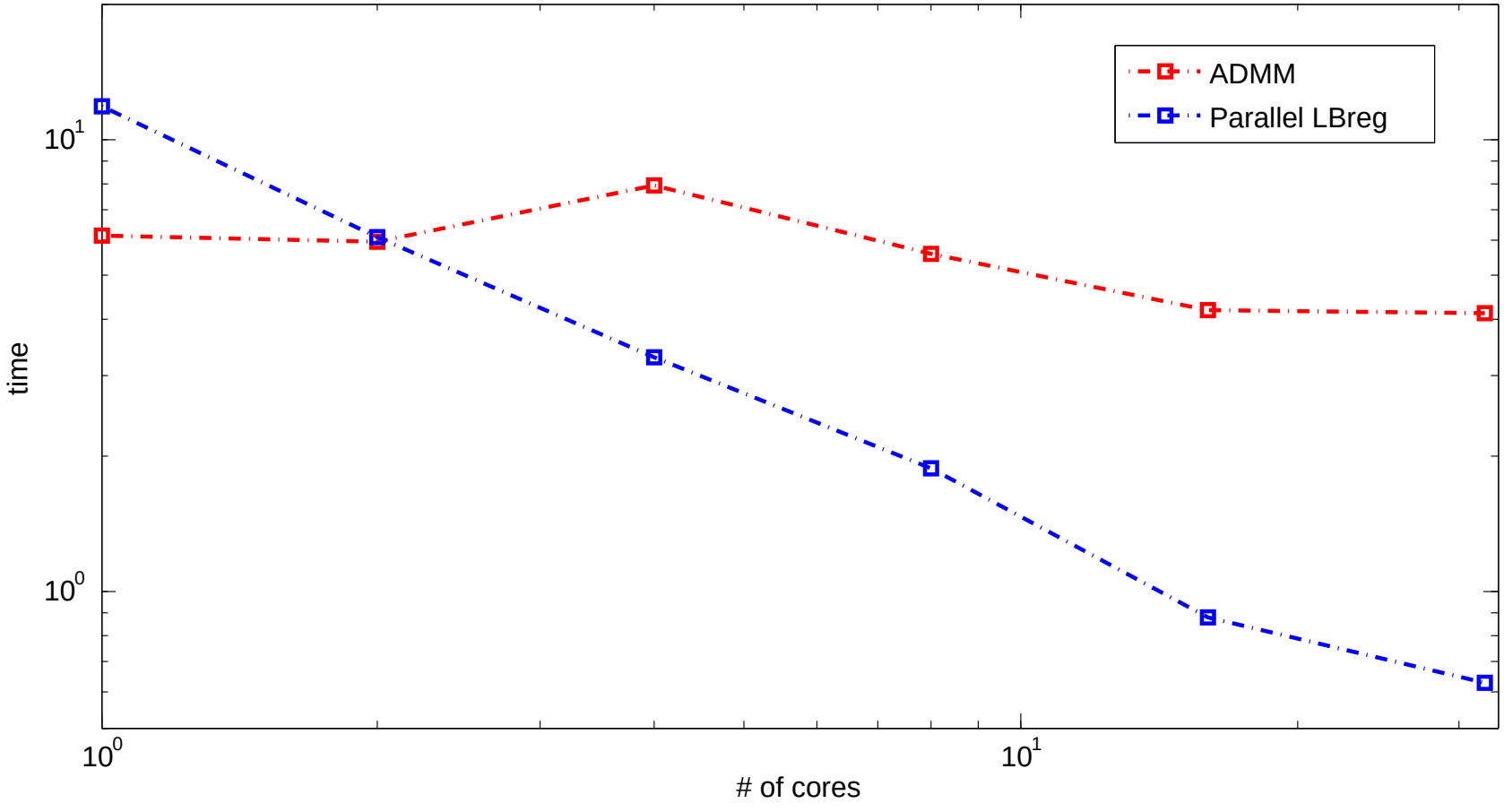
$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}.$$

- example with **dense** $\mathbf{A} \in \mathbb{R}^{1024\times 2048}$;
- distribute $\mathbf{A}$ to N computing nodes

$$\mathbf{A} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \cdots \ \mathbf{A}_N];$$

- compare
  1. parallel ADMM in the survey paper Boyd, Parikh, Chu, Peleato, and Eckstein [2011];
  2. parallel linearized Bregman (dual gradient descent), un-accelerated.
- tested $N = 1, 2, 4, \ldots, 32$ computing nodes;

parallel ADMM v.s. parallel linearized Bregman



parallel linearized Bregman scales much better

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(Block)-coordinate descent/update

Definition: update one block of variables each time, keeping others fixed

Advantage: update is simple

Disadvantages:

- more iterations (with exceptions)
- may stuck at non-stationary points if problem is non-convex or non-smooth (with exceptions)

Block selection: cycle (Gauss-Seidel), parallel (Jacobi), random, greedy

Specific for sparse optimization: greedy approach[11] can be exceptionally effective (because most time correct variables are selected to update; most zero variables are never touched; "reducing" the problem to a much smaller one.)

[11] Li and Osher [2009]

GRock: greedy coordinate-block descent

Example:

$$\min \lambda\|\mathbf{x}\|_1 + f(\mathbf{A}\mathbf{x} - \mathbf{b})$$

- decompose $\mathbf{A}\mathbf{x} = \sum_j \mathbf{A}_i\mathbf{x}_j$; block $\mathbf{A}_j$ and $\mathbf{x}_j$ are kept on node j

Parallel GRock:

1. (*parallel*) compute a merit value for each coordinate i

$$d_i = \arg\min_d \ \lambda \cdot r(x_i + d) + g_i d + \frac{1}{2}d^2, \quad \text{where } g_i = \mathbf{a}_{(i)}^T \nabla \mathbf{f}(\mathbf{A}\mathbf{x} - \mathbf{b})$$

2. (*parallel*) compute a merit value for each block j

$$m_j = \max\{|d| : d \text{ is an element of } \mathbf{d}_j\}$$

   let s_j be the index of the maximal coordinate within block j

3. (allreduce) $\mathcal{P} \leftarrow$ select P blocks with largest m_j, $2 \le P \le N$
4. (*parallel*) update $x_{s_j} \leftarrow x_{s_j} + d_{s_j}$ for all $j \in \mathcal{P}$
5. (allreduce) update $\mathbf{A}\mathbf{x}$

How large P can be?

P depends on the **block spectral radius**

$$\rho_P = \max_{\mathbf{M} \in \mathcal{M}} \rho(\mathbf{M}),$$

where $\mathcal{M}$ is the set of all $P \times P$ submatrices that we can obtain from $\mathbf{A}^T\mathbf{A}$ corresponding to selecting exactly one column from each of the P blocks

Theorem

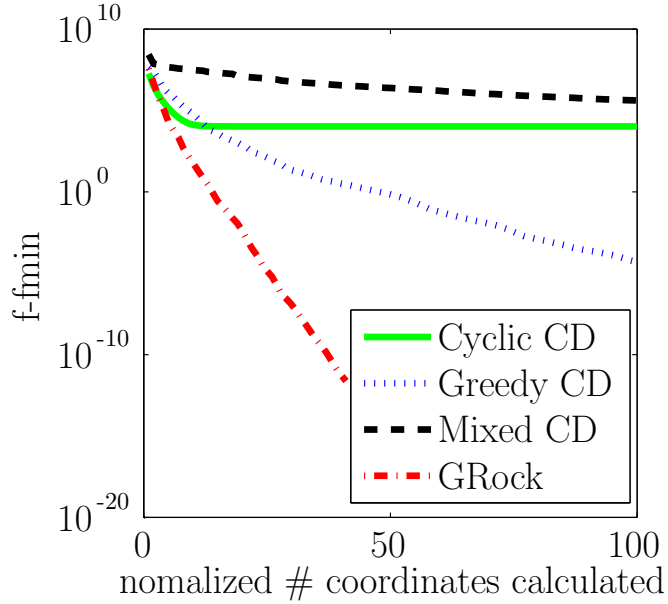
Assume each column of $\mathbf{A}$ has unit 2-norm. If $\rho_P < 2$, GRock with P parallel updates per iteration gives

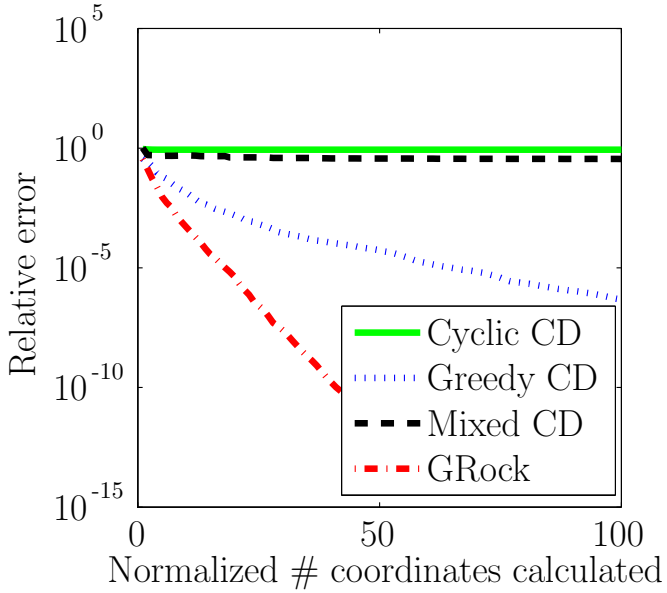
$$\mathcal{F}(\mathbf{x} + \mathbf{d}) - \mathcal{F}(\mathbf{x}) \leq \frac{\rho_P - 2}{2} \beta \|\mathbf{d}\|_2^2$$

moreover,

$$\mathcal{F}(\mathbf{x}^k) - \mathcal{F}(\mathbf{x}^*) \ \leq \ \frac{2C^2 \left(2L + \beta\sqrt{\frac{N}{P}}\right)^2}{(2 - \rho_P)\beta} \cdot \frac{1}{k}.$$

Compare different block selection rules





- LASSO test with $\mathbf{A} \in \mathbb{R}^{512\times1024}$, $N = 64$ column blocks
- GRock uses $P = 8$ updates each iteration
- greedy CD[12] uses $P = 1$
- mixed CD[13] selects $P = 8$ random blocks and best coordinate in each iteration

[12] Li and Osher [2009]

[13] Scherrer, Tewari, Halappanavar, and Haglin [2012]

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Test on Rice cluster STIC

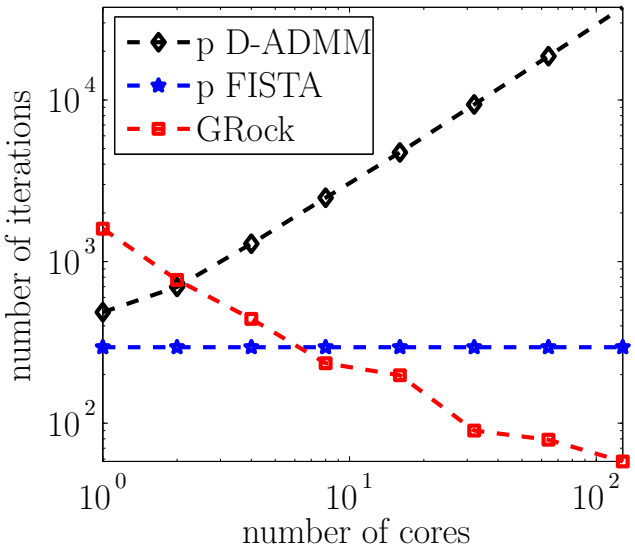
specs:

- 170 Appro Greenblade E5530 nodes each with two quad-core 2.4GHz Xeon (Nahalem) CPUs
- each node has 12GB of memory shared by all 8 cores
- # of processes used on each node = 8

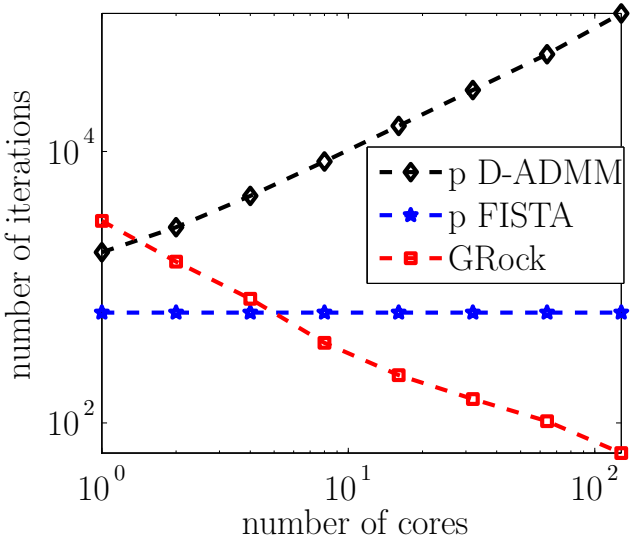
test dataset:

	$\mathbf{A}$ type	$\mathbf{A}$ size	λ	sparsity of $\mathbf{x}^*$
dataset I	Gaussian	1024×2048	0.1	100
dataset II	Gaussian	2048×4096	0.01	200

iterations vs cores



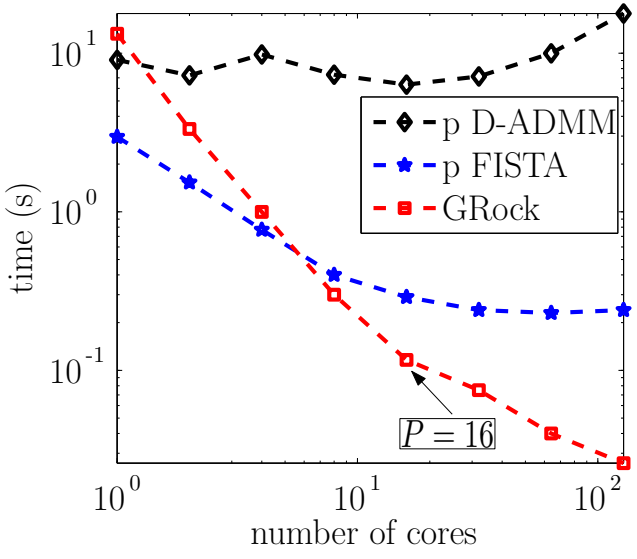
dataset I



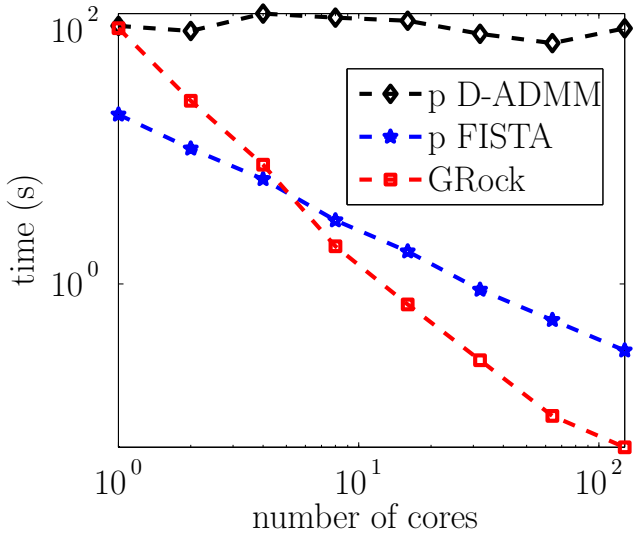
dataset II

Note: we set $P =$ number of cores

time vs cores



dataset I



dataset II

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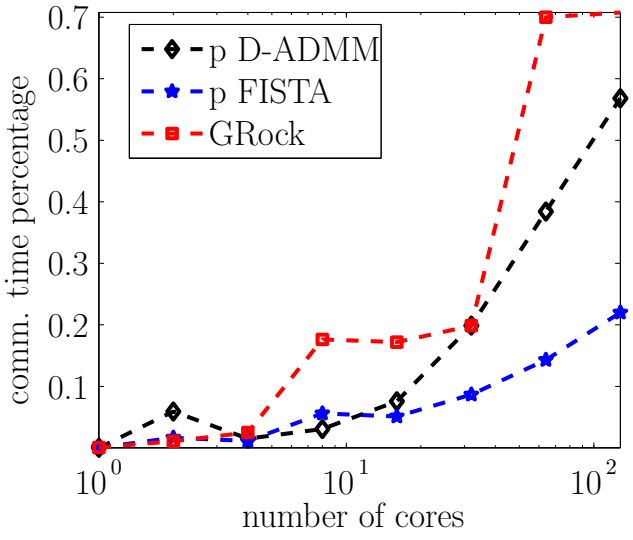
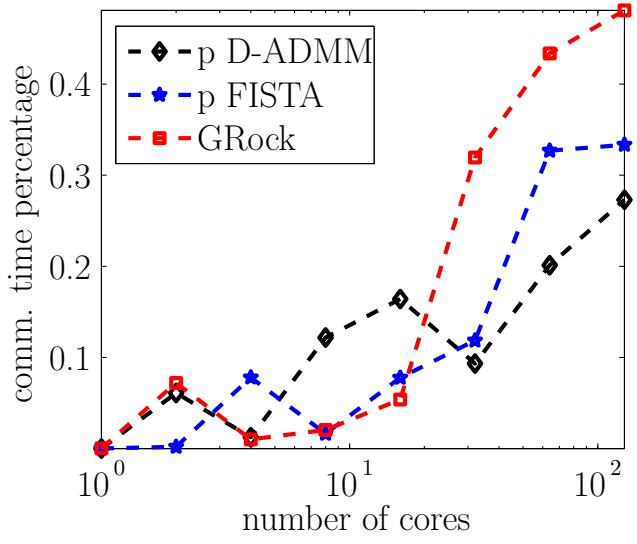
(% communication time) vs cores



dataset I



dataset II

Note: we set $P =$ number of cores

big data LASSO on Amazon EC2

Amazon EC2 is an elastic, pay-as-you-use cluster;

advantage: no hardware investment, everyone can have an account

test dataset

- $\mathbf{A}$ dense matrix, 20 billion entries, and **170GB** size
- $\mathbf{x}$: 200K entries, 4K nonzeros, Gaussian values

requested system

- 20 "high-memory quadruple extra-large instances"
- each instance has 8 cores and 60GB memory

code

- written in C using GSL (for matrix-vector multiplication) and MPI

parallel Dual-ADMM vs FISTA[14] vs GRock

	p D-ADMM	p FISTA	GRock
estimate stepsize (min.)	n/a	1.6	n/a
matrix factorization (min.)	51	n/a	n/a
iteration time (min.)	105	40	1.7
number of iterations	2500	2500	104
communication time (min.)	30.7	9.5	0.5
stopping relative error	**1E-1**	**1E-3**	**1E-5**
total time (min.)	**156**	**41.6**	**1.7**
Amazon charge	**$85**	**$22.6**	**$0.93**

- ADMM's performance depends on penalty parameter β
  we picked β as the best out of only a few trials (we cannot afford more)
- parallel Dual ADMM and FISTA were capped at 2500 iterations
- GRock used adaptive P and stopped at relative error 1E-5

[14] Beck and Teboulle [2009]

Software codes can be found in the author's website.

Funding agencies: US NSF, DoD

Acknowledgements: Qing Ling (USTC) and Zaiwen Wen (SJTU)

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