Math 273a: Optimization
Linear programming

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Fall 2015

some material taken from the textbook Chong-Zak, 4th Ed.

History

- The word “programming” used traditionally by planners to describe the process of operations planning and resource allocation.
- In 1930s–40s, this process could often be aided by solving LPs. Kantorovich: solutions to problems in production and transportation.
- The initial impetus came in the aftermath of World War II.
- In 1947, George Dantzig proposed the Simplex Method (poorly named great method[1]). Made the solution of LPs practical. But, it has exponential worst-case complexity.
- Advance in computer technology expand the applications of LP. Bringing people to study and apply LP extensively.

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[1] One of the 10 algorithms with the greatest influence on the development and practice of science and engineering in the twentieth century

The Best of the 20th Century: Top 10 Algorithms

by Barry A. Cipra

- 1946, von Neumann, Ulam, and Metropolis: Monte Carlo method
- 1947, Dantzig: the Simplex method
- 1950, Hestenes, Stiefel, and Lanczoz: Krylov subspace iteration methods
- 1951, Householder: decompositional approach to matrix computations
- 1957, Backus: Fortran optimizing compiler
- 1959–61: J.G.F. Francis of Ferranti Ltd.: QR algorithm
- 1962: Hoare: Quicksort
- 1965: Cooley and Tukey: the fast Fourier transform
- 1977, Ferguson and Forcade: integer relation detection algorithm
- 1987, Greengard and Rokhlin: fast multipole algorithm

Modern period

- 1950s –, Applications
- 1960s, Large-scale optimization
- 1970s, Complexity theory
- Khachyan, 1979, the ellipsoid algorithm, first polynomial-time algorithm, but impractical
- Karmakar, 1984, interior-point algorithms, lead to later interior-point methods.
- CPLEX 1.0, 1988, research shifts to commercial
- CPLEX acquired by ILOG, which was later acquired by IBM
- Guroi, 2008
- Today: huge-scale, distributed, streaming LP

Example: the diet problem

- n different foods, jth food sells at price c_j per unit
- m basic nutrients; for balanced diet, receive at least b_i units of ith nutrient
- each unit of food j contains a_{ij} units of ith nutrient
- variable x_j: # units of food j in diet
- total cost: $c_1 x_1 + \cdots + c_n x_n$
- nutritional constraints: $a_{i1} x_1 + \cdots + a_{in} x_n \geq b_i$, $i = 1, \ldots, m$.

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

Graphic LP in 2D

In the problem, a company manufactures two iPod player models, both with 3.5-inch LCD but have different memory capacities:

- 16GB – two 8GB chips
- 8GB – one 8GB chip

Weekly resources are limited to

- 800 units of 3.5-inch antiglare LCDs
- 1000 units of 8GB memory chips
- 50 hours of total labor time. It takes 3 minutes of labor for each 16GB player, and 4 minutes of labor for each 8GB player.

For marketing reasons,

- Total production cannot exceed 700
- # 16GB players cannot exceed # 8GB players by more than 350

Profit, while remaining within the marketing guidelines, can be computed as

- \$16 each 16GB player
- \$10 each 8GB player

The current weekly production plan consists of 450 16GB players and 100 8GB players, make a profit of \$16*450 + \$10*100 = \$8200.

Management is seeking a new production plan that will increase the profit.

Variables:

- x_1: weekly produced units of 16GB players
- x_2: weekly produced units of 8GB players

Objective: to maximize the weekly profit $16x_1 + 10x_2$

Constraints:

- $x_1, x_2 \geq 0$
- LCD: $x_1 + x_2 \leq 8000$
- Memory: $2x_1 + x_2 \leq 1000$
- Labor: $3x_1 + 4x_2 \leq 3000$
- Marketing total: $x_1 + x_2 \leq 7000$
- Marketing mix: $x_1 - x_2 \leq 350$

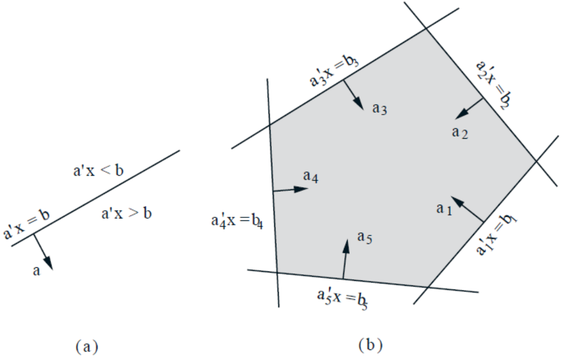
Graphical optimization

- 2D plot of the variables, constraints, and level curves of the objective
- feasible region is a polyhedron, possibly empty or unbounded
- three types of feasible points: interior, boundary, and extreme points
- level curves of the objective are parallel lines
- if there is a solution, there is an extreme point solution
- it is possible that the problem is feasible but has an unbounded $-\infty$ optimal objective
- they can be infinitely many solutions

Geometric concepts

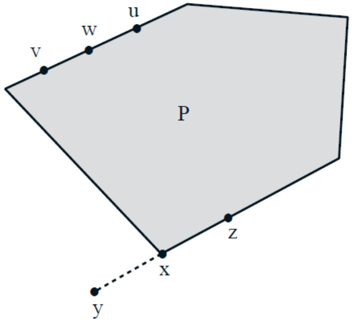
- A set S is convex if any $\boldsymbol{x}, \boldsymbol{y} \in S$, $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in S$, $\forall \alpha \in [0,1]$
- Let $S := \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K\}$.
  - $\text{span}(S) = \{\sum_{k=1}^K \lambda_k \boldsymbol{x}^k : \lambda_k \in \mathbb{R}, \forall k\}$
  - $\text{aff}(S) = \{\sum_{k=1}^K \lambda_k \boldsymbol{x}^k : \sum_{k=1}^K \lambda_k = 1, \lambda_k \in \mathbb{R}, \forall k\}$, called **affine hull**
  - $\text{cone}(S) = \{\sum_{k=1}^K \lambda_k \boldsymbol{x}^k : \lambda_k \in \mathbb{R}_+, \forall k\}$, called **convex cone**
  - $\text{convex}(S) = \{\sum_{k=1}^K \lambda_k \boldsymbol{x}^k : \sum_{k=1}^K \lambda_k = 1, \lambda_k \in \mathbb{R}_+, \forall k\}$, called **convex hull**
- The intersection of convex sets is a convex set

- consider $\mathbb{R}^n$
- $\{\boldsymbol{x} : \boldsymbol{a}^\top \boldsymbol{x} = b\}$ is called a **hyperplane**
- $\{\boldsymbol{x} : \boldsymbol{a}^\top \boldsymbol{x} \geq b\}$ is called a **halfspace**
- The intersection of finitely many halfspaces is called a **polyhedron**

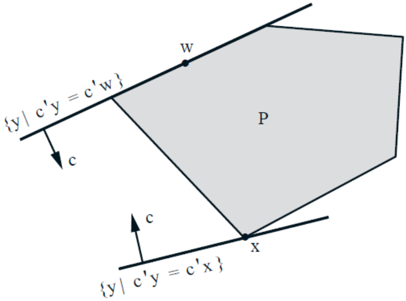


Consider the polyhedron $P := \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\} \subset \mathbb{R}^n$.

- $\boldsymbol{x} \in P$ is **an extreme point** of P if $\nexists\, \boldsymbol{y}, \boldsymbol{z} \in P, \boldsymbol{y} \neq \boldsymbol{x}, \boldsymbol{z} \neq \boldsymbol{x}, 0 < \lambda < 1$, such that $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$.
- an extreme point is not strictly within the line segment connecting two other points in P



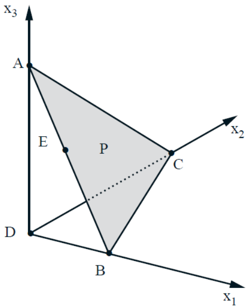
- $\boldsymbol{x} \in P$ is **a vertex** of P if $\exists \boldsymbol{c}, \ni \boldsymbol{c}^\top \boldsymbol{x} < \boldsymbol{c}^\top \boldsymbol{z}, \forall \boldsymbol{z} \in P \setminus \{\boldsymbol{x}\}$.
- a vertex is the *unique* minimizer of some linear function over P.



The standard simplex in $\mathbb{R}^3$:

$$P := \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_c \geq 0\}.$$

- Points A,B,C: each has 3 active (i.e., "=") constraints
- Point E: 2 active constraints. If add a constraint: $2x_1 + 2x_2 + 2x_3 = 2$. Then, 3 constraints are active at E, but they are *not* linearly independent.



A vertex or extreme point has n linearly independent active constraints

Standard form

- variable $\boldsymbol{x} \in \mathbb{R}^n$
- cost vector $\boldsymbol{c} \in \mathbb{R}^n$
- right-hand side vector $\boldsymbol{b} \in \mathbb{R}^m$
- coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
- standard form

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

Any non-standard form LP can be reformulated to the standard form. The standard form simplifies algorithms and unifies analysis.

Conversion to the standard form

Consider

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

Introduce *surplus* or *dummy variables* s_i.

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i \;\Leftrightarrow\; a_{i1}x_1 + \cdots + a_{in}x_n - s_i = b_i,\; s_i \geq 0$$

New form

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & [\boldsymbol{A}, -\boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0,\; \boldsymbol{s} \geq 0. \end{aligned}$$

General methods:

- "maximize" objective: minimize its negative
- $\leq$ constraint: add nonnegative slack variable
- $\geq$ constraint: subtract nonnegative slack variable
- $x_i \leq 0$: substitute x_i by $-x_i$ throughout
- free x_i: introduce $u_i, v_i \geq 0$ and substitute x_i by $u_i - v_i$ throughout
- constraint $|x_i| \leq b_i$: replace by $x_i \leq b_i$ and $-x_i \leq b_i$
- objective $|x_i|$: introduce $u_i, v_i \geq 0$ and substitute
  - x_i by $u_i - v_i$
  - $|x_i|$ by $u_i + v_i$

Example

$$\text{maximize } x_2 - x_1$$
$$\text{subject to } 3x_1 = x_2 - 5$$
$$|x_2| \le 2$$
$$x_1 \le 0.$$

Steps:

1. change to $\text{minimize}\;\; x_1 - x_2$
2. substitute x_1 by $-x_1$
3. write $|x_2| \le 2$ by $x_2 \le 2$ and $-x_2 \le 2$
4. introduce s_1 and s_2 and rewrite $x_2 + s_1 = 2$ and $-x_2 + s_2 = 2$
5. split $x_2 = u_2 - v_2$, $u_2, v_2 \ge 0$.

We obtain:

$$\begin{aligned} \text{minimize } & x_1 - x_2 \\ \text{subject to } & 3x_1 + u_2 - v_2 = 5 \\ & u_2 - v_2 + s_1 = 2 \\ & v_2 - u_2 + s_2 = 2 \\ & x_1, u_2, v_2, s_1, s_2 \geq 0. \end{aligned}$$