# Math 273a: Optimization
# 1D search methods


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based on Chong-Zak, 4th Ed.

# Goal

Develop methods for solving the one-dimensional problem

$$\underset{x \in \mathbb{R}}{\text{minimize}} \;\; f(x)$$

under the following cases:

- (0th order info) only objective value $f$ is available,
- (1st order info) $f'$ is available, but not $f''$,
- (2nd order info) both $f'$ and $f''$ are available.

Higher-order information tends to give more powerful algorithms.

These methods are also used in multi-dimensional optimization as line search for determine how far to move along a given direction.

# Iterative algorithm

Most optimization problems cannot be solved in a closed form (a single step).

For them, we develop iterative algorithms:

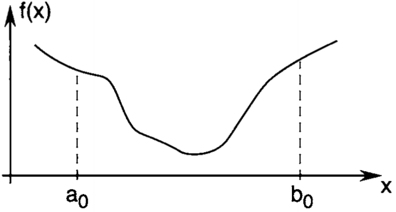
- start from an initial candidate solution: $x^{(0)}$
- generate a sequence of candidate solutions (iterates): $x^{(1)}, x^{(2)}, \ldots$,
- stop when a certain condition is met; return the candidate solution

In a large number of algorithms, $x^{(k+1)}$ is generated from $x^{(k)}$, that is, using the information of $f$ at $x^{(k)}$.

In some algorithms, $x^{(k+1)}$ is generated from $x^{(k)}, x^{(k-1)}, \ldots$. But, for time and memory consideration, most history iterates are not kept in memory.

# Golden section search

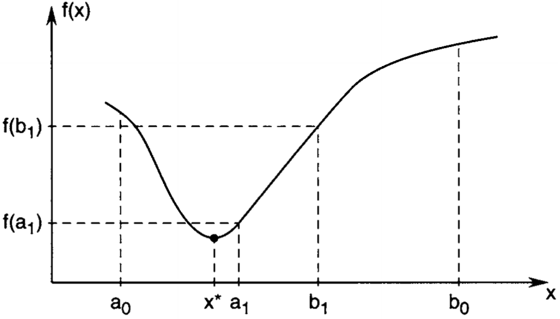
**Problem:** given a closed interval $[a_0, b_0]$, a *unimodal* function (that is, having one and only one local minimizer in $[a_0, b_0]$), and only objective value information, find a point that is no more than $\epsilon$ away from that local minimizer.




Why make the unimodal assumption? Just for ease of illustration.

# Golden section search

- **Mid-point:** evaluate the mid-point, that is, compute $f(\frac{a_0+b_0}{2})$. But, it cannot determine which half contains the local minimizer. **Does not work.**
- **Two-point:** evaluate at $a_1, b_1 \in (a_0, b_0)$, where $a_1 < b_1$.
  1. if $f(a_1) < f(b_1)$, then $x^* \in [a_0, b_1]$;
  2. if $f(a_1) > f(b_1)$, then $x^* \in [a_1, b_0]$;
  3. if $f(a_1) = f(b_1)$, then $x^* \in [a_1, b_1]$. This case will rarely occur, so we can include it in either case 1 or case 2.



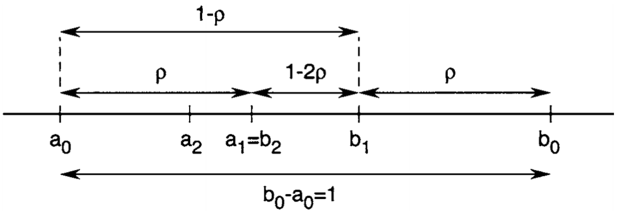
# How to choose intermediate points

- **Symmetry:** length(left piece) = length(right piece), that is,

$$(a_1 - a_0) = (b_0 - b_1)$$

- **Consistency:** Let $\rho := \frac{a_1 - a_0}{b_0 - a_0} < \frac{1}{2}$. Such a ratio is maintained in subsequent iterations.

- **Reusing evaluated points:** the 1st iteration evaluates $a_1, b_1$
  - if the 1st iteration narrows to $[a_0, b_1]$, one of the next two points $b_2$ shall equal $a_1$
    $\Longrightarrow \rho = \frac{b_1 - b_2}{b_1 - a_0} = \frac{b_1 - a_1}{b_1 - a_0}$
  - if the 1st iteration narrows to $[a_1, b_0]$, one of the next two points $a_2$ shall equal $b_1$
    $\Longrightarrow \rho = \frac{a_2 - a_1}{b_0 - a_1} = \frac{b_1 - a_1}{b_0 - a_1}$




Putting together, using $b_1 - a_0 = 1 - \rho$, $b_1 - a_1 = 1 - 2\rho$, and $\rho < \frac{1}{2}$, we get

$$\boxed{\rho(1-\rho) = 1 - 2\rho} \implies \rho = \frac{3-\sqrt{5}}{2} \approx 0.382$$

Interestingly, observe the factor

$$\frac{1-\rho}{1} = \frac{\rho}{1-\rho} = \frac{\sqrt{5}-1}{2} \approx 0.618,$$

which is the golden ratio by ancient Greeks. (Two segments with the ratio of the longer to the sum equals the ratio of the shorter to the longer.)

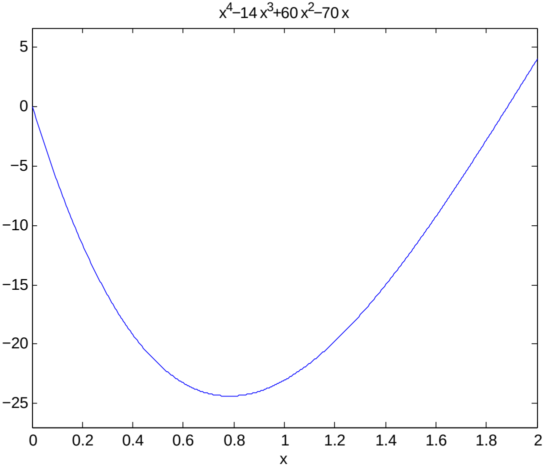
# Example of golden section search

$$\text{minimize } \ f(x) = x^4 - 14x^3 + 60x^2 - 70x.$$

(one often omits the constant in the objective unless it helps simplify it)

- Initial interval: $[0, 2]$.
- Required accuracy $\epsilon = 0.3$.
- 1st iteration: narrow to $[a_0, b_1] = [0, 1.236]$

  ......
- 4th iteration: narrow to $[a_4, b_3] = [0.6525, 0.9443]$
- STOP since $b_3 - a_4 = 0.2918 < \epsilon$.
- No coincidence: $b_3 - a_4 = 2(1 - \rho)^4$. We can predict the number of iterations.




$$x^* \approx 0.7809$$

# Algorithm complexity: How many iterations are needed?

Starting with an interval of length $L$, to reach an interval with length $\leq \epsilon$, we need $N$ iterations, where $N$ is the first integer such that

$$L(1-\rho)^N \leq \epsilon \implies N \log(1-\rho) \leq \log \frac{\epsilon}{L} \implies N \log \frac{1}{1-\rho} \geq \log \frac{L}{\epsilon}.$$

Therefore,

$$N = \left\lceil C \log \frac{L}{\epsilon} \right\rceil,$$

where $C = \left(\log \frac{1}{1-\rho}\right)^{-1}$.

We can write $N = O(\log \frac{1}{\epsilon})$ to emphasize its logarithmic dependence on $\frac{1}{\epsilon}$.

# Bisection method

**Problem:** given an interval $[a_0, b_0]$, a continuously differentiable, *unimodal* function, and derivative information, find a point that is no more than $\epsilon$ away from the local minimizer.

**Mid-point works with derivative!**

Let $x^{(0)} = \frac{1}{2}(a_0 + b_0)$

- If $f'(x^{(0)}) = 0$, then $x^{(0)}$ is the local minimizer;
- If $f'(x^{(0)}) > 0$, then narrow to $[a_0, x^{(0)})$;
- If $f'(x^{(0)}) < 0$, then narrow to $(x^{(0)}, b_0]$.

Every evaluation of $f'$ reduces the interval by half. The reduction factor is $\frac{1}{2}$. Need $N$ iterations such that $L(1/2)^N \leq \epsilon$, where $L$ is the initial interval size and $\epsilon$ is the targeted accuracy.

The *previous example* only needs 3 bisection iterations.

# Newton's method

**Problem:** given a *twice continuously differentiable* function and objective, derivative, and 2nd derivative information, find an approximate minimizer.

Newton's method does not need intervals but must start sufficiently close to $x^*$

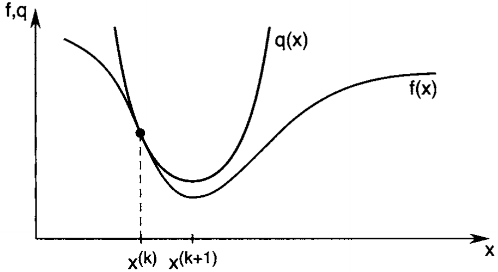
**Iteration:** minimize the *quadratic* approximation

$$x^{(k+1)} \leftarrow \arg\min q(x) := f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2} f''(x^{(k)})(x - x^{(k)})^2.$$

("$\arg\min$" returns the minimizer; "$\min$" returns the minimal value.)

This iteration in a closed form

$$\boxed{x^{(k+1)} := x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}}$$




Quadratic approximation with $f''(x) > 0$

$f$ and $q$ have the same value, tangent, and curvature at $x^{(k)}$

# Example

The previous example

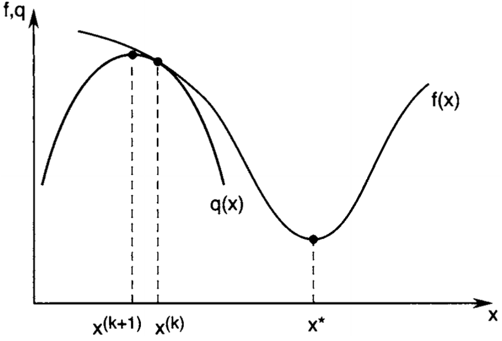
$$f(x) = x^4 - 14x^3 + 60x^2 - 70x.$$

- True solution $x^* \approx 0.7809$
- Initial point: $x^{(0)} = 0$.
- $x^{(1)} = 0.5833$
- $x^{(2)} = 0.7631$
- $x^{(3)} = 0.7807$
- $x^{(4)} = 0.7809$

Can produce highly accurate solutions in just a few steps.

Need just $N = O(\log\log\left(\frac{1}{\epsilon}\right))$ if $f''(x^*) > 0$ and $x^{(0)}$ is sufficiently close.

# What if $f''(x) < 0$?




$f''(x^{(k)}) < 0$ causes an ascending step

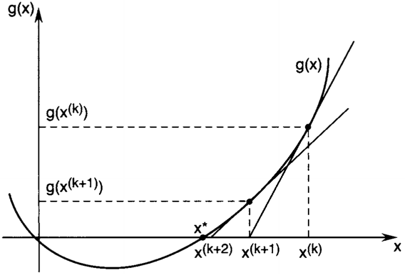
In general, Newton's iteration may diverge.

# Newton's method for finding zero

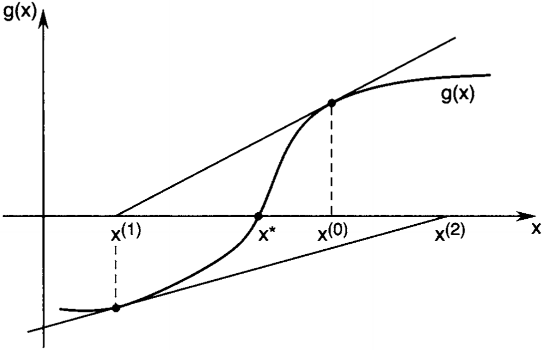
If we set $g(x) := f'(x)$ and thus $g'(x) = f''(x)$, then Newton's method

$$x^{(k+1)} := x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})}$$

is a way to solve $g(x) = 0$, that is, finding the zero of $g$. Instead of a quadratic approximation to $f$, the iteration uses a linear (tangent) approximation to $g$.



# A failed example




If $x^{(0)}$ is not sufficiently close to $x^*$, Newton's sequence diverges unboundedly.

Note: $g''(x)$ change signs between $x^*$ and $x^{(0)}$.

# Secant method

Recall Netwon's method for minimizing $f$ uses $f''(x^{(k)})$:

$$x^{(k+1)} := x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}.$$

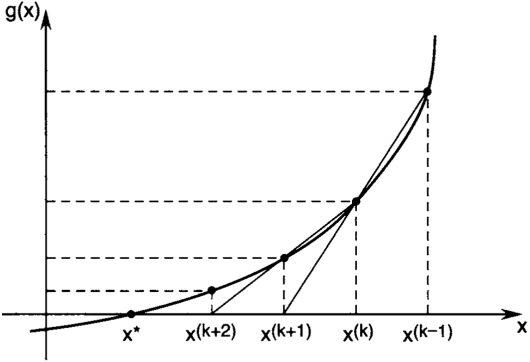
If $f''$ is not available or is expensive to compute, we can approximate

$$f''(x^{(k)}) \approx \frac{f'(x^{(k)}) - f'(x^{(k-1)})}{x^{(k)} - x^{(k-1)}}.$$

This gives the iteration of the *secant method:*

$$x^{(k+1)} := x^{(k)} - \frac{(x^{(k)} - x^{(k-1)})f'(x^{(k)})}{f'(x^{(k)}) - f'(x^{(k-1)})} = \frac{x^{(k)}f'(x^{(k-1)}) - x^{(k-1)}f'(x^{(k)})}{f'(x^{(k)}) - f'(x^{(k-1)})}.$$

Note: the method needs *two initial points*.




The secant method is slightly slower than Newton's method but is cheaper.

# Comparisons of different 1D search methods

Golden section search (and Fibonacci search):

- one objective evaluation at each iteration,
- narrows search interval by less than half each time.

Bisection search:

- one derivative evaluation at each iteration,
- narrows search interval to exactly half each time.

Secant method:

- two points to start with; then one derivative evaluation at each iteration,
- must start near $x^*$, has *superlinear* convergence

Netwon's method:

- one derivative and one second derivative evaluations at each iteration,
- must start near $x^*$, has *quadratic* convergence, fastest among the five

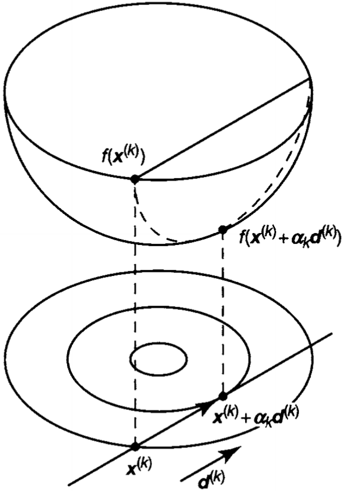
# Line search for multi-dimensional optimization

- One-dimensional search methods are used as an important part in multi-dimensional optimization, often dubbed as the line search method.
- At $\boldsymbol{x}^{(k)}$, a method generates a search direction $\boldsymbol{d}^{(k)}$.
  - The gradient method sets $\boldsymbol{d}^{(k)} = -\nabla f(\boldsymbol{x}^{(k)})$.
  - Newton's method sets $\boldsymbol{d}^{(k)} = -(\nabla^2 f(\boldsymbol{x}^{(k)}))^{-1}\nabla f(\boldsymbol{x}^{(k)})$.
  - ... many other ways to set $\boldsymbol{d}^{(k)}$ ...
- Line search determines $\alpha_k$ that *minimizes*

$$\phi(\alpha) = f(\boldsymbol{x}^{(k)} + \alpha\boldsymbol{d}^{(k)})$$

  and then sets

$$\boldsymbol{x}^{(k+1)} := \boldsymbol{x}^{(k)} + \alpha_k\boldsymbol{d}^{(k)}.$$




At minimizer $\alpha_k$, $\nabla f(\boldsymbol{x}^{(k)} + \alpha \boldsymbol{d}^{(k)}) \perp \boldsymbol{d}^{(k)}$

# Practical line search

Line search is *inexact* in practice to save time.

Some practical acceptance rules for $\alpha_k$:

- *Armijo condition*:

  $$\phi_k(\alpha_k) \leq \phi_k(0) + c\alpha_k \phi_k'(0)$$

  where $c \in (0, 1)$ is often small, e.g., $c = 0.1$.

- *Armijo backtracking* starts from large $\alpha$ and decreases $\alpha$ geometrically until the Armijo condition is met.

- *Goldstein condition*:

$$\phi_k(\alpha_k) \geq \phi_k(0) + \eta \alpha_k \phi_k'(0)$$

  where $\eta \in (c, 1)$, prevents tiny $\alpha_k$.

- *Wolfe condition*:

$$\phi_k'(\alpha_k) \geq \eta \phi_k'(0)$$

  prevents $\alpha_k$ from lending on a steep descending position.