Homework Assignment 3 – Math 273a, Fall 2015

- This homework assignment is open to all textbooks (listed above or not), class notes, and Internet documents except for any solution manual or the solutions from the previous quarters.
- You are encouraged to ask questions and discuss the questions and solutions on http://piazza.com/ucla/fall2015/math273a/home. However, copying others solutions or programs is considered a **serious violation**.
- Please type your answers in Latex and **submit a single PDF file**. Note that you can insert an external PDF file to your latex file by using

```
\usepackage{pdfpages} % add this line before \begin{document}

\includepdf[pages={1,2,3}]{myfile.pdf}
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- Although not required, you are encouraged to use notation similar to the course slides.

Problem 1. (10 points)

Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots m$, proper convex functions. Let $g : \mathbb{R}^m \to \mathbb{R}$ be a proper, convex, monotonically nonincreasing function, that is, $g(x) \leq g(y)$ for any $x, y \in \mathbb{R}^m$ such that $x \leq y$. Show that $h(x) := g(f(x))$ is convex.

Answer: [Type your answer here. Make sure you clearly define all mathematical objects in the answer.]

Problem 2. (10 points)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a proper convex function. Show that for any $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 < x_2 < x_3$, it holds

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}.$$

2. Show that if $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then for every set of scalars $x_1, x_2, \dots, x_n > 0$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

   It holds with equality if and only if $x_1 = x_2 = \cdots = x_n$. (Use the fact that $-\log$ is strictly convex.)

Answer: [Type your answer here. Make sure you clearly define all mathematical objects in the answer.]

Problem 3. (10 points)

(Ekeland's variational principle) Consider a proper closed convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. Let $\bar{x} \in \mathbb{R}^n$ be an ϵ-optimal solution that satisfies

$$f(\bar{x}) \leq \inf_x f(x) + \epsilon.$$

Then, for any $\delta > 0$, there exists a vector $x' \in \mathbb{R}^n$ such that

$$\|x' - \bar{x}\| \leq \frac{\epsilon}{\delta}, \quad f(x') \leq f(\bar{x})$$

and x' is the unique solution to the perturbed problem

$$\underset{x}{\text{minimize}} \;\; F(x) := f(x) + \delta\|x - x'\|$$

Answer: [Type your answer here. Make sure you clearly define all mathematical objects in the answer.]

Problem 4. (10 points)

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function, and f^* be its conjugate. That is, $f^*(y) := \sup_x \langle x, y \rangle - f(x)$. Show that the following results hold

1. $f^*(0) = -\inf_x f(x)$
2. these three properties are equivalent: (a) f^* is lower unbounded, (b) $f \equiv \infty$, (c) $f^* \equiv -\infty$.
3. **(The Fenchel-Young inequality)** $f(x) + f^*(y) \geq \langle x, y \rangle$ for any $x, y \in \mathbb{R}^n$.

Answer: [Type your answer here. Make sure you clearly define all mathematical objects in the answer.]

Problem 5. (20 points)

For any proper closed convex function f, the proximal map $\mathbf{prox}_{\gamma f}(y)$ is the unique solution to the following problem

$$\underset{x}{\text{minimize}} \;\; f(x) + \frac{1}{2\gamma}\|x-y\|^2.$$

For functions such as ℓ_1-norm, ℓ_2-norm, and linear functions, $\mathbf{prox}_{\gamma f}$ is easy to compute. However, when both $\mathbf{prox}_f$ and $\mathbf{prox}_g$ are simple, $\mathbf{prox}_{f+g}$ can be difficult to compute. There are some exceptions.

For each of the following examples and $x \in \mathbb{R}^n$, develop an algorithm for computing $\mathbf{prox}_{f+g}$ that is not significantly more expensive than computing $\mathbf{prox}_f$ and $\mathbf{prox}_g$.

1. $f(x) = \alpha\|x\|_1$ and $g(x) = \beta\|x\|_2$, where $\alpha, \beta > 0$.
2. $f(x) = \alpha\|x\|_\infty$ and $g(x) = \beta\|x\|_2$, where $\alpha, \beta > 0$.
3. $f(x) = \alpha\|x\|_1$ and $g(x) = \beta\|Dx\|_1$, where $\alpha, \beta > 0$ and D is the forward finite difference operator (thus $\|Dx\|_1 = \sum_{i=1}^{n-1}|x_{i+1} - x_i|$ is total variation).
   (a) solve the problem by assuming that an algorithm for $\mathbf{prox}_g$ has been provided to you.
   (b) develop an iterative algorithm for $\mathbf{prox}_g$.
4. letting $x \in \mathbb{R}$, show that if $f(x) = \alpha|x|$ and $g'(0) = 0$, then $\mathbf{prox}_{f+g} = \mathbf{prox}_g \circ \mathbf{prox}_f$.

Answer: [Type your answer here. Make sure you clearly define all mathematical objects in the answer.]