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On the continuity of the polyconvex, quasiconvex and rank-one-convex envelopes with respect to growth condition

Wilfrid Gangbo

Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland

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Synopsis

Let Cf, Pf, Qf and Rf be respectively the convex, polyconvex, quasi-convex and rank-one-convex envelopes of a given function f. If $f_p: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ and $f_q(\xi)$ behaves as $|\xi|^q$ at infinity $q \in (1,\infty)$, we show that $\lim_{p\to q} Cf_p = Cf_q$, $\lim_{p\to q} Qf_p = Qf_q$, $\lim_{p\to q} Rf_p = Rf_q$. This is the case for $(Pf_p)_p$ provided that $q \neq 1, \ldots, \min(N, M)$, otherwise $\liminf_{p\to q} Pf_p \neq Pf_q$. In the last part of this work, we show that $f(\xi) = g(|\xi|)$ does not imply in general $Pf = Qf$.

1. Introduction

Let $f_p: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ be a Borel measurable function which behaves at infinity as $|\xi|^p$, $p \geqq 1$. Let Cf_p, Pf_p, Qf_p and Rf_p be, respectively, the convex, polyconvex, quasiconvex and rank-one-convex envelopes of f_p. (For a precise definition, see the end of the introduction.) We want to study the continuity with respect to p of these envelopes. As is well known, they are discontinuous at $p = 1$. We show that Cf_p, Qf_p and Rf_p are, however, continuous at $p > 1$. (The result for Cf_p is elementary.) In the case of Pf_p, we prove that it is discontinuous provided that $p = 2, \ldots, \min(N, M)$, and otherwise continuous. We next give two examples, the first one being elementary.

EXAMPLE 1.1. For $0 < p < 1$, let

$$f_p(\xi) = |\xi|^p, \quad 0 \leqq p \leqq 1, \quad \xi \in \mathbb{R}^{N\times N}.$$

We find that

$$\liminf_{p\to 1} Cf_p(\xi) = \liminf_{p\to 1} Pf_p(\xi) = \liminf_{p\to 1} Qf_p(\xi) = \liminf_{p\to 1} Rf_p(\xi) \equiv 0$$
$$< Cf_1(\xi) = Pf_1(\xi) = Qf_1(\xi) = Rf_1(\xi) = |\xi|.$$

EXAMPLE 1.2. Recall first that a polyconvex function with a subquadratic growth is necessarily convex. (See Remark 3.5.) For $1 \leqq p < 2$, $\xi \in \mathbb{R}^{2\times 2}$, let

$$f_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{if} \quad |\xi| \neq 0, \\ 0 & \text{if} \quad |\xi| = 0. \end{cases}$$

In view of the above remark, we find $Pf_p = Cf_p$ for every $1 \leqq p < 2$. Kohn and Strang in **[9]** proved that $Rf_2 = Qf_2 = Pf_2$ and $Cf_2(\xi) \neq Pf_2(\xi)$ if and only if

$0<|\xi|^2+2|det(\xi)|<1$ and $det(\xi)\neq 0$. Computing Cf_p, and using the result of Kohn and Strang:

$$\liminf_{p\to 2} Pf_p \neq Pf_2.$$

We now describe the contents of this paper. In Section 2 we show an elementary result: for every $q\in(1,\infty)$, $\lim_{p\to q} Cf_p = Cf_q$. In Section 3 we show that:

(i) for every $q\in(1,\infty)$, $q\neq 2,\dots,\min(N,M)$ $\lim_{p\to q} Pf_p = Pf_q$;

(ii) in some examples the result is false provided that $q\in\{2,\dots,\min(N,M)\}$.

In Section 4 we prove that for every $q\in(1,\infty)$, $\lim_{p\to q} Qf_p = Qf_q$. To achieve this, we first approximate $Qf_p(\xi)$ by $1/|Q|\int_Q f_p(\xi+\nabla\phi^p)$, where $\phi^p\in W_0^{1,p}(Q)^M$. (See [4].) The proof is based on Gehring's lemma on reverse Hölder inequality in [6] and a result of Giaquinta and Modica in [7]. We deduce, as a byproduct, that there exist quasiconvex functions $f:\mathbb{R}^{N\times M}\mapsto\mathbb{R}$ for every $N, M>1$ integers, that are not polyconvex. (See [14, 1].) We also obtain a general method of constructing such functions. We conclude this section by studying some examples such as:

$$f_p(\xi)=|\xi|^p+a\,|det(\xi)|^{p/2},\quad a>0,$$

and we prove that $Pf_p\neq Qf_p$ for p near 2.

In Section 5, we show that for every $q\in(1,\infty)$, $\lim_{p\to q} Rf_p = Rf_q$. To achieve this, we make an additional hypothesis on the family of functions $(f_p)_p$ and assume that there exists a constant $K>0$ such that $Rf_p(\xi)=f_p(\xi)$ for every $|\xi|\geqq K$. In some examples this is satisfied. In Section 6, we turn our attention to the following question: if f is a function such that $f(\xi)=g(|\xi|)$, does this always imply that $Pf=Qf$? Note that in the example of Kohn and Strang (Example 1.2 above) $Pf_2=Qf_2$. We show that, in general, if $p<2$ then $Pf_p<Qf_p$. Here, we use an interesting method (similar to one of Boccardo and Gallouet in an article in preparation) to obtain strong convergence of a certain weakly convergent sequence.

We conclude this introduction by giving some definitions used above.

Definitions 1.3 (see [4]). Let $f:\mathbb{R}^{N\times M}\mapsto\mathbb{R}$ be a Borel measurable function

(a) f is said to be *convex* if $f(\lambda\xi+(1-\lambda)\eta)\leqq\lambda f(\xi)+(1-\lambda)f(\eta)$ for every $\xi,\eta\in\mathbb{R}^{N\times M}$ and every $\lambda\in(0,1)$.

(b) f is said to be *polyconvex* if there exists a function $h:\mathbb{R}^{\tau(N,M)}\mapsto\mathbb{R}$, convex such that $f(\xi)=h(T(\xi))$ for every $\xi\in\mathbb{R}^{N\times M}$, where $\tau(N,M)=\sum_{1\leqq s\leqq\min(N,M)}\binom{M}{s}\binom{N}{s}$, $T(\xi)=(adj_1\xi,\dots,adj_{\min(N,M)}\xi)$ and $adj_s\xi$ stands for the matrix of all $s\times s$ minors of ξ. If $N=M=2$ then $T(\xi)=(\xi, det(\xi))$.

(c) f is said to be *quasiconvex* if $\frac{1}{|\Omega|}\int_\Omega f(\xi+\nabla\phi)\geqq f(\xi)$ for every $\xi\in\mathbb{R}^{N\times M}$, every $\Omega\subset\mathbb{R}^N$ (or equivalently for some $\Omega\subset\mathbb{R}^N$) and every $\phi\in W_0^{1,\infty}(\Omega)^M$.

(d) f is said to be *rank-one-convex* if $f(\lambda\xi+(1-\lambda)\eta)\leqq\lambda f(\xi)+(1-\lambda)f(\eta)$ for every $\xi,\eta\in\mathbb{R}^{N\times M}$ with $rank(\xi-\eta)\leqq 1$ and every $\lambda\in(0,1)$.

It is a well-established fact, following the work of Morrey [11, 12] and later of Ball [2] that, in general, one has:

$$f \text{ convex}\Rightarrow f \text{ polyconvex}\Rightarrow f \text{ quasiconvex}\Rightarrow f \text{ rank-one-convex}.$$

The different envelopes are defined as:

$$Cf=\sup\{g, g\leqq f, g \textit{ convex}\},$$
$$Pf=\sup\{g, g\leqq f, g \textit{ polyconvex}\},$$
$$Qf=\sup\{g, g\leqq f, g \textit{ quasiconvex}\},$$
$$Rf=\sup\{g, g\leqq f, g \textit{ rank-one-convex}\}.$$

2. Continuity of Cf_p with respect to p

We start with the main result of this section.

Theorem 2.1. *Let $[\alpha,\beta]\subset(1,\infty)$, $F, G, \gamma_0>0$, $C\geqq 1$ $N, M\geqq 1$ be two integers. Let $w:[0,\infty)\mapsto[0,\infty)$ with $\lim_{t\to 0} w(t)=w(0)=0$, $\sup\{w(t), t\in[0,\beta]\}\leqq G$ and $f_p:\mathbb{R}^{N\times M}\mapsto\mathbb{R}$ lower semicontinuous, $p\in[\alpha,\beta]$, such that:*

$$|\xi|^p\leqq f_p(\xi)\leqq C(1+|\xi|^p)\quad \textit{for every}\quad p\in[\alpha,\beta]\quad \textit{and every}\quad \xi\in\mathbb{R}^{N\times M},\tag{2.1}$$

$$|f_p(\xi)-f_q(\xi)|\leqq\frac{F}{\gamma}w(p-q)(1+|\xi|^{p+\gamma})\quad \textit{for every}\quad \gamma\in(0,\gamma_0),\tag{2.2}$$

$$\textit{every}\quad \xi\in\mathbb{R}^{N\times M},\quad \textit{and every}\quad p,q\in[\alpha,\beta]\quad \textit{with}\quad p>q.$$

Then

$$\lim_{p\to q} Cf_p(\xi)=Cf_q(\xi),\quad \textit{for every}\quad \xi\in\mathbb{R}^{N\times M}\quad \textit{and every}\quad q\in(\alpha,\beta).\tag{2.3}$$

Before we prove this theorem, let us begin with some remarks.

Remarks 2.2. (a) In general, we have $\lim_{p\to 1} Cf_p<Cf_1$. Indeed, if $f_p(\xi)=|\xi|^p$, $\xi\in\mathbb{R}^{N\times M}$, then $0\equiv\lim_{p\to 1} Cf_p<Cf_1=f_1$.

(b) Theorem 2.1 is still true if we replace the condition (2.1) by $a+b|\xi|^p\leqq f_p(\xi)\leqq C(1+|\xi|^p)$ where $a\in\mathbb{R}$ $b>0$ are two constants.

(c) To prove (2.3), we will show that $|Cf_p(\xi)-Cf_q(\xi)|$ and $|f_p(\xi)-f_q(\xi)|$ have the same modulus of continuity.

Examples 2.3. The following examples satisfy the hypotheses of the theorem:

(1)

$$f_p(\xi)=\begin{cases}1+|\xi|^p & \text{for}\quad |\xi|\neq 0,\\ 0 & \text{for}\quad |\xi|=0,\end{cases}\quad \xi\in\mathbb{R}^{N\times M};$$

(2)

$$f_p(\xi)=|\xi|^p+a\,|det(\xi)|^{p/N},\quad \xi\in\mathbb{R}^{N\times N},\quad a>0.$$

We get that $(f_p)_p$ verifies (2.1) and (2.2). Hence Theorem 2.1 leads to $\lim_{p\to q} Cf_p = Cf_q$, for every $q > 1$.

To prove Theorem 2.1, we begin with an elementary lemma.

LEMMA 2.4. *Let $N, M \geqq 1$ be two integers, $\alpha, \beta \in (1, \infty)$, $C > 0$ a constant and $f: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ lower semicontinuous such that:*

$$|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p) \quad \text{for some} \quad p \in [\alpha, \beta] \quad \text{and every} \quad \xi \in \mathbb{R}^{N\times M}.$$

Then, there exists a constant $D > 0$ depending only on α, β and C such that, for every $\xi, \xi^*, \eta \in \mathbb{R}^{N\times M}$,

$$Cf(\eta) = \langle \eta, \xi^*\rangle - f^*(\xi^*), \quad Cf(\xi) = \langle \xi, \xi^*\rangle - f^*(\xi^*) \quad \text{implies} \quad |\eta| \leqq D(1 + |\xi|). \tag{2.4}$$

Proof. Using (2.1), we get that, for every $\xi^* \in \mathbb{R}^{N\times M}$,

$$C \sup\left\{0, \frac{p-1}{(Cp)^{\acute{p}}}|\xi^*|^{\acute{p}} - 1\right\} \leqq f^*(\xi^*) \leqq \frac{p-1}{(p)^{\acute{p}}}|\xi|^{\acute{p}}, \quad \text{where} \quad \acute{p} = \frac{p}{p-1}. \tag{2.5}$$

Adding (2.1) and (2.4), we deduce that there exists a constant $E > 0$ such that

$$|\xi^*| \leqq E(1 + |\xi|). \tag{2.6}$$

Then (2.1) implies that there exists an $s \in [0, 1]$ such that

$$Cf(\eta) = \langle \eta, \xi^*\rangle - f^*(\xi^*) = s\,|\eta|^p + (1-s)(1 + |\eta|^p). \tag{2.7}$$

Hence

$$\left|\,|\eta|^{p-1}\left(s + (1-s)C - \left\langle \frac{\eta}{|\eta|}; \xi^*\right\rangle\right)\right| \leqq |(1-s)C + f^*(\xi^*)|^{(p-1)/p},$$

or

$$|\eta| \leqq |(1-s)C + f^*(\xi^*)|^{1/p}.$$

Adding (2.5) and (2.6) to these previous inequalities, we conclude the proof.

We now prove Theorem 2.1.

Proof of Theorem 2.1. Let $\xi \in \mathbb{R}^{N\times M}$; (2.1) implies that there exists

$$\lambda_1^p, \ldots, \lambda_{NM+1}^p \in [0, 1], \quad \xi_1^p, \ldots, \xi_{NM+1}^p \in \mathbb{R}^{N\times M}$$

such that

$$\sum_{i=1}^{NM+1} \lambda_i^p = 1, \quad \sum_{i=1}^{NM+1} \lambda_i^p \xi_i^p = \xi \quad \text{and} \quad \sum_{i=1}^{NM+1} \lambda_i^p f_p(\xi_i^p) = Cf_p(\xi).$$

Let $\xi^* \in \mathbb{R}^{N\times M}$ such that $Cf_p(\xi) = \langle \xi, \xi^{*,p}\rangle - f_p^*(\xi^{*,p})$. It is obvious that

$$Cf_p(\xi_i^p) = f_p(\xi_i^p) \quad \text{and} \quad Cf_p(\xi_i^p) = \langle \xi_i^p, \xi^{*,p}\rangle - f_p^*(\xi^{*,p}), \quad i = 1, \ldots, NM+1.$$

By Lemma 2.4, we find that there exists a constant $D > 0$ depending only on α, β, C such that

$$|\xi_i^p| \leqq D(1 + |\xi|) i = 1, \ldots, NM+1 \quad \text{for every} \quad p \in [\alpha, \beta].$$

Then we conclude that there exists a constant $H > 0$ depending only on α, β, C such that

$$|Cf_p(\xi) - Cf_q(\xi)| \leqq \frac{H}{\gamma} w(p-q)(1 + |\varepsilon|^{p+\gamma}) \quad \text{for every} \quad \gamma \in (0, \gamma_0).$$

Hence Theorem 2.1 is proved. □

3. Continuity and discontinuity of Pf_p with respect to p

We start with the main result of this section.

THEOREM 3.1. *Let $[\alpha, \beta] \subset (1, \infty)$, $F, G, \gamma_0 > 0$, $C \geqq 1$ and $N, M > 1$ be two integers. Let $w: [0, \infty) \mapsto [0, \infty)$ with $\lim_{t\to 0} w(t) = w(0) = 0$, $\sup\{w(t), t \in [0, \beta]\} \leqq G$ and $f_p: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ lower semicontinuous, $p \in [\alpha, \beta]$, such that:*

$$|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p) \quad \text{for every} \quad p \in [\alpha, \beta] \quad \text{and every} \quad \xi \in \mathbb{R}^{N\times M}; \tag{3.1}$$

$$|f_p(\xi) - f_q(\xi)| \leqq \frac{F}{\gamma} w(p-q)(1 + |\xi|^{p+\gamma}) \quad \text{for every} \quad \gamma \in (0, \gamma_0), \tag{3.2}$$

every $\xi \in \mathbb{R}^{N\times M}$, and every $p, q \in [\alpha, \beta]$ with $p > q$. Then

$$\text{in general} \quad \lim_{p\to q} Pf_p < Pf_q, \quad \text{for} \quad q = 2, \ldots, \min(N, M); \tag{3.3}$$

$$\lim_{p\to q} Pf_p(\xi) = Pf_q(\xi), \quad \text{for every}$$

$$\xi \in \mathbb{R}^{N\times M} \quad \text{and every} \quad q \in (\alpha, \beta)\, q \neq 2, \ldots, \min(N, M). \tag{3.4}$$

Before proving Theorem 3.1, let us begin with some remarks.

Remarks 3.2. (a) In general, we also have $\lim_{p\to 1} Pf_p < Pf_1$. Indeed, if $f_p(\xi) = |\xi|^p$, $\xi \in \mathbb{R}^{N\times M}$, then $0 \equiv \lim_{p\to 1} Pf_p < Pf_1 = f_1$.

(b) Theorem 3.1 is still true if we change the condition (3.1) to $a + b\,|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p)$, where $a \in \mathbb{R}$, $b > 0$ are two constants.

NOTATION. For $\xi \in \mathbb{R}^{N\times M}$, $q \in \mathbb{N}$, $adj_q(\xi)$ stands for the matrix of all $q \times q$ minors of ξ. (3.5)

EXAMPLES 3.3. (1) Let $f_p(\xi)=|\xi|^p+a\,|adj_q(\xi)|^{p/q}$, $\xi\in\mathbb{R}^{N\times N}$, $a>0$ and $q\in\{2,3,4,\dots\}$. We get that $(f_p)_p$ verifies (3.1) and (3.2). Hence Theorem 3.1 leads to $\lim_{p\to q} Pf_p = Pf_q$, if $q\neq 2,\dots,\min(N,M)$. We show (see Step 4 of the proof of Theorem 3.1) that $\lim_{p\to q} Pf_p < Pf_q$ for suitable values of a.

(2) Let

$$f_p(\xi)=\begin{cases}1+|\xi|^p & \text{for } |\xi|\neq 0,\\ 0 & \text{for } |\xi|=0,\end{cases}\qquad \xi\in\mathbb{R}^{2\times 2}.$$

We get that $(f_p)_p$ verifies (3.1) and (3.2). Hence Theorem 3.1 leads to $\lim_{p\to q} Pf_p = Pf_q$, for every $q>1$, $q\neq 2$.

Knowing that $Pf_p = Cf_p$ for every $0<p<2$ and $Cf_2 < Pf_2$ (see [9]), we can deduce that $\lim_{p\to 2^-} Pf_p < Pf_2$. We also get that $\lim_{p\to 1^-} Pf_p \equiv 0 < Pf_1 = f_1$.

To prove Theorem 3.1, let us now begin with the following lemma:

LEMMA 3.4. *Let $N, M \geqq 1$ be two integers, $p\in[1,\min(N,M)]$, $C>0$ a constant and $f:\mathbb{R}^{N\times M}\mapsto\mathbb{R}$ lower semicontinuous such that:*

$$|\xi|^p \leqq f_p(\xi)\leqq C(1+|\xi|^p)\quad \textit{for every}\quad \xi\in\mathbb{R}^{N\times M}.$$

Then the three following assertions are equivalent:

- *f is polyconvex;*
- $\lambda_1,\dots,\lambda_r\in[0,1]$, $\xi_1,\dots,\xi_r\in\mathbb{R}^{N\times M}$ $\sum_{i=1}^r\lambda_i=1$, $\sum_{i=1}^r \lambda_i R(\xi_i)=R\left(\sum_{i=1}^r\lambda_i\xi_i\right)$ (3.6)

  implies

  $$\sum_{i=1}^r \lambda_i f(\xi_i)\geqq f\left(\sum_{i=1}^r \lambda_i\xi_i\right), \tag{3.7}$$

  where

  $$R(\xi)=(adj_1(\xi),\dots,adj_{[p]}(\xi)),\quad \xi\in\mathbb{R}^{N\times M},$$

  $$r=1+\sum_{i=1}^{[p]}\binom{N}{i}\binom{M}{i},\ \textit{and } [p] \textit{ is the integer part of } p;$$

- $f(\xi)=h(R(\xi))$ (3.8)

  where for

  $$X\in\mathbb{R}^{r-1},\ h(X)=\inf\left\{\sum_{i=1}^r\lambda_i f(\xi_i),\ \lambda_i\in[0,1]\xi_i\in\mathbb{R}^{N\times M} i=1,\dots,r\ \sum_{i=1}^r\lambda_i=1,\ \sum_{i=1}^r\lambda_i R(\xi_i)=X\right\}.$$

Proof. The proof of Lemma 3.4 is a direct adaptation of the proof of the representation theorem of the polyconvex envelope. (See [4, p. 201, Theorem 1.1]). □

Remarks 3.4. (a) We can see that $h:\mathbb{R}^{r-1}\mapsto\mathbb{R}$ is convex.
(b) An immediate consequence of Lemma 3.4 is that a polyconvex function $f:\mathbb{R}^{N\times M}\mapsto\mathbb{R}$ with subquadratic growth is convex.

We now start with the proof of Theorem 3.1.

Proof of Theorem 3.1. We divide the proof into four steps. Let $\xi\in\mathbb{R}^{N\times M}$, and $p\in[\alpha,\beta]$.

Step 1. We prove here that $\limsup_{p\to q} Pf_p(\xi)\leqq Pf_q(\xi)$. Let $\varepsilon>0$. There exist $\lambda_1,\dots,\lambda_{\tau+1}\in[0,1]$, $\xi_1,\dots,\xi_{\tau+1}\in\mathbb{R}^{N\times M}$, such that

$$\sum_{i=1}^{\tau+1}\lambda_i=1,\quad \sum_{i=1}^{\tau+1}\lambda_i T(\xi_i)=T(\xi)\quad\text{and}\quad \sum_{i=1}^{\tau+1}\lambda_i f_q(\xi_i)<-\varepsilon+Pf_q(\xi),$$

where

$$T(\xi)=(adj_1(\xi),\dots,adj_{\min(N,M)}(\xi)),\quad \xi\in\mathbb{R}^{N\times M},$$

$$\tau=\sum_{i=1}^{\min(N,M)}\binom{N}{i}\binom{M}{i}$$

(See [4].) Using (3.2), we get

$$Pf_q(\xi)>-\varepsilon+Pf_q(\xi)-\frac{F}{\gamma}w(p-q)\sum_{i=1}^{\tau+1}\lambda_i(1+|\xi_i|^{p+\gamma}+|\xi_i|^{q+\gamma})$$

for every $p\in[\alpha,\beta]$. Hence

$$\limsup_{p\to q} Pf_p(\xi)\leqq Pf_q(\xi). \tag{3.9}$$

Step 2. We suppose in this step that $q>\min(N,M)$ and prove that $\liminf_{p\to q} Pf_p(\xi)\geqq Pf_q(\xi)$. Recalling that f_p is lower semicontinuous and verifies (3.1), we deduce that for every $p>\min(N,M)$ there exist

$$\lambda_1^p,\dots,\lambda_{\tau+1}^p\in[0,1],\quad \xi_1^p,\dots,\xi_{\tau+1}^p\in\mathbb{R}^{N\times M}\ \sum_{i=1}^{\tau+1}\lambda_i^p=1,$$

such that

$$\sum_{i=1}^{\tau+1}\lambda_i^p T(\xi_i^p)=T(\xi)\quad\text{and}\quad\sum_{i=1}^{\tau+1}\lambda_i^p f_p(\xi_i^p)=Pf_p(\xi).$$

Using the fact that there exists a constant $D>0$ that depends only on N, M such that

$$|T(\xi_i^p)|\leqq D\,|\xi_i^p|^{\min(N,M)}\quad\text{for every}\quad i=1,\dots,\tau+1,$$

adding (3.1) and the fact that $p>\min(N,M)$, we can suppose without restriction that the sequence $(|\xi_i^p|)_p$ is bounded with respect to p. By the fact that f_q is lower semicontinuous and $(f_p)_p$ verifies (3.2), we find that

$$\liminf_{p\to q} Pf_p(\xi) \geqq Pf_q(\xi) \quad \text{for every} \quad p > \min(N, M). \tag{3.10}$$

Step 3. We suppose here that $q < \min(N, M)$, $q \neq 2, \ldots, \min(N, M)$ and prove that $\liminf_{p\to q} Pf_p(\xi) \geqq Pf_q(\xi)$. Using Lemma 3.4, knowing that $[p] = [q]$ for p close to q and replacing $T(\xi)$ by $R(\xi) = (adj_1(\xi), \ldots, adj_{[p]}(\xi))$, τ by $r - 1 = \sum_{i=1}^{[p]} \binom{N}{i}\binom{M}{i}$, $\min(N, M)$ by $[p]$ in the previous step, where $[p]$ is the integer part of p, we obtain by the same arguments as those we used in Step 2 that

$$\liminf_{p\to q} Pf_p(\xi) \geqq Pf_q(\xi) \quad \text{for every} \quad p < \min(N, M), \quad q \neq 2, \ldots, \min(N, M). \tag{3.11}$$

Now (3.9), (3.10) and (3.11) imply that

$$\lim_{p\to q} Pf_p(\xi) = Pf_q(\xi) \quad \text{for every} \quad q \in [\alpha, \beta], \quad q \neq 2, \ldots, \min(N, M).$$

Step 4. We suppose in this step that $q \in \{1, \ldots, \min(N, M)\}$ and prove that $\liminf_{p\to q} Pf_p(\xi) < Pf_q(\xi)$.

(i) Let $R(\xi) = (adj_1(\xi), \ldots, adj_{q-1}(\xi))$ and $f_p(\xi) = |\xi|^p + a\,|adj_q(\xi)|^{p/q}$ $a > 0$. Using the same arguments as in Step 3, knowing that $[p] = q - 1$ for every $p \in (q-1, q)$ and combining with (3.8) of Lemma 3.4, we get that

$$\liminf_{p\to q-} Pf_p(\xi) = \gamma_q(\xi),$$

where

$$\gamma_q(\xi) \equiv \inf\left\{\sum_{i=1}^{s+1} \lambda_i f_q(\xi_i),\ \lambda_i \in [0, 1]\xi_i \in \mathbb{R}^{N\times M} i = 1, \ldots, s+1,\right.$$
$$\left.\sum_{i=1}^{s+1} \lambda_i = 1 \sum_{i=1}^{s+1} \lambda_i R(\xi_i) = R(\xi)\right\},$$
$$s = \sum_{i=1}^{q-1} \binom{N}{i}\binom{M}{i}.$$

We can see that the infimum is a minimum.

(ii) Applying Lemma 3.4 to f_q, we get that for suitable ξ and $a > 0$ we obtain that $|\xi|^p + a\,|adj_q(\xi)| > \gamma_q(\xi)$. Hence

$$\liminf_{p\to q} Pf_p(\xi) < Pf_q(\xi).$$

This completes the proof of Theorem 3.1. □

4. Continuity of Qf_p with respect to p

We start with the main theorem of this section.

Theorem 4.1. *Let $[\alpha, \beta] \subset (1, \infty)$, $F, G, \gamma_0 > 0$, $C \geqq 1$ $N, M > 1$ be integers and $Q = \{x \in \mathbb{R}^N; |x_i| \leqq 3, i = 1, \ldots, N\}$. Let $w: [0, \infty) \mapsto [0, \infty)$ with $\lim_{t\to 0} w(t) = w(0) = 0$, $\sup\{w(t), t \in [0, \beta]\} \leqq G$ and $f_p: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ lower semicontinuous, $p \in [\alpha, \beta]$, such that:*

$$|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p) \quad \textit{for every} \quad p \in [\alpha, \beta] \quad \textit{and every} \quad \xi \in \mathbb{R}^{N\times M}, \tag{4.1}$$

$$|f_p(\xi) - f_q(\xi)| \leqq \frac{F}{\gamma} w(p - q)(1 + |\xi|^{p+\gamma}) \quad \textit{for every} \quad \gamma \in (0, \gamma_0), \tag{4.2}$$

every $\xi \in \mathbb{R}^{N\times M}$, and every $p, q \in [\alpha, \beta]$ with $p > q$. Then

$$Qf_p(\xi) = \inf\left\{\frac{1}{|Q|}\int_Q f_p(\xi + \nabla\phi),\ \phi \in W_0^{1,p}(Q)^M\right\} \tag{4.3}$$

for every $\xi \in \mathbb{R}^{N\times M}$ and every $p \in [\alpha, \beta]$, and

$$\lim_{p\to q} Qf_p(\xi) = Qf_q(\xi), \quad \textit{for every} \quad \xi \in \mathbb{R}^{N\times M} \quad \textit{and every} \quad q \in (\alpha, \beta). \tag{4.4}$$

Before proving this theorem, let us begin with some remarks.

Remarks 4.2. (a) The assumption (4.1) says that $f_p(\xi)$ behaves at infinity as $|\xi|^p$ and can be replaced by $C_1(|\xi|^p - 1) \leqq f_p(\xi) \leqq C(1 + |\xi|^p)$.

(b) The assumption (4.2) stands for "continuity" of f_p with respect to p. This continuity is stronger than the usual continuity and weaker than the uniform continuity.

(c) (4.3) is the result of (4.1) and the characterisation of the quasiconvex envelope. (See [4].)

(d) The idea of the proof of (4.4) is the following: for p fixed, we use (4.3) and approximate $Qf_p(\xi)$ by $1/|Q| \int_Q f_p(\xi + \nabla\phi_n)$ where $\phi_n \in W_0^{1,p}(Q)^N$. By Gehring's lemma, we deduce that $(\phi_n)_n$ is bounded in $W_{\text{loc}}^{1,p+\varepsilon}(Q)^N$ for $\varepsilon > 0$ independent of p. This will lead to (4.4).

Examples 4.3. (a) Let

$$f_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{for} \quad |\xi| \neq 0, \\ 0 & \text{for} \quad |\xi| = 0, \end{cases} \quad \xi \in \mathbb{R}^{2\times 2}.$$

(2) Let

$$f_p(\xi) = |\xi|^p + a\,|det(\xi)|^{p/2}, \quad a > 2.$$

Using Theorem 4.1 we show that in Examples 4.3 (1) and (2) $Qf_p > Pf_p = Cf_p$ for p near 2.

Notation.

- Let $R > 0$, $a \in \mathbb{R}^N$. We define: $Q_R(a) = \{x \in \mathbb{R}^N, |x_i - a_i| \leqq R,\ i = 1, \ldots, N\}$, $B_R(a) = \{x \in \mathbb{R}^N, \sum_{1\leqq i\leqq N} (x_i - a_i)^2 \leqq R^2\}$, $Q = Q_3(0)$; (4.5)
- for every $x \in \mathbb{R}^N$, $|x|_p^p = \sum_{1\leqq i\leqq N} |x_i|^p$ if $1 \leqq p < \infty$

and $|x|_\infty = \max\{|x_i|, i = 1, \ldots, N\}$; (4.6)

- $dist(x, y) = |x - y|_\infty$ for every $x, y \in \mathbb{R}^N$; (4.7)
- $⨍_Q u = \frac{1}{|Q|}\int_Q u$, $u_R(a) = ⨍_{Q_{R(a)}} u$, $\|u\|_r^r = \int_Q |u|^r$ for every $r \geqq 1$, (4.8)
  every $u \in L^r(Q)$, every $a \in Q$, and every $R > 0$ "small";
- $\mathbb{R}^{N\times N}$ is the set of the $N \times N$ real matrices. (4.9)

Lemma 4.4. *Let $N \geqq 2$ be an integer, $\beta \in (1, \infty)$ and $\Omega \subset \mathbb{R}^N$ a bounded open set with Lipschitz boundary. There exists a constant $C > 0$ depending only on Ω and β such that:*

$$\int_\Omega |u|^p \leqq C\left(\int_\Omega |u|^\mu\right)^{p/\mu} \tag{4.10}$$

for every $p \in [1, \beta]$, $u \in W^{1,p}(\Omega)$ with $\int_\Omega u = 0$ and for $\mu = \max\{1, (Np)/(N+p)\}$.

Remark 4.5. Lemma 4.4 is exactly Poincaré's inequality and Sobolev's theorem. We want to show that Sobolev's constant C corresponding to the embedding of $W^{1,\mu}(\Omega)$ to $L^p(\Omega)$ remains bounded when $p \in K = [1, \beta] \subset \mathbb{R}$. This result is not surprising and is easily proved.

Proof of Lemma 4.4. We divide the proof into two parts.

Part 1. Suppose that $1 \leqq p \leqq N/(N-1) = \bar{p}$. Using Sobolev's embedding theorem, we find two constants $\bar{C}_1, \bar{C}_2 > 0$ depending only on Ω such that:

$$\int_\Omega |u|^{\bar{p}} \leqq \bar{C}_1\left(\int_\Omega (|\nabla u| + |u|)\right)^{\bar{p}} \quad \text{for every} \quad u \in W^{1,1}(\Omega), \tag{4.11}$$

$$\int_\Omega |u| \leqq \bar{C}_2\left(\int_\Omega |\nabla u|\right) \quad \text{for every} \quad u \in W^{1,1}(\Omega) \quad \text{verifying} \quad \int_\Omega u = 0. \tag{4.12}$$

(See [3, p. 168] and [12].) By Hölder's inequality,

$$u \in W^{1,p}(\Omega) \quad \text{implies} \quad \int_\Omega |u|^p \leqq (1 + |\Omega|)\left(\int_\Omega |u|^{\bar{p}}\right)^{p/\bar{p}}. \tag{4.13}$$

From (4.11), (4.12) and (4.13) we find a constant $C_1 > 0$ depending only on Ω such that $1 \leqq p \leqq \bar{p}$, $u \in W^{1,p}(\Omega)$, $\int_\Omega u = 0$ imply

$$\int_\Omega |u|^p \leqq C_1\left(\int_\Omega |\nabla u|^\mu\right)^{p/\mu}. \tag{4.14}$$

Part 2. We now carry out an induction on i_0 and suppose that there exist constants $C_1, \ldots, C_{i0}$ such that

$$p \leqq \frac{Np}{N + ip} < \bar{p}, \quad i = 1, \ldots, i_0, \quad u \in W^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega u = 0$$

imply that

$$\int_\Omega |u|^p \leqq C_i\left(\int_\Omega |\nabla u|^\mu\right)^{p/\mu}.$$

Let

$$u \in W^{1,p}(\Omega), \quad p \in [1, \beta],$$

such that

$$\int_\Omega u = 0, \quad \frac{Np}{N + (i_0+1)p} < \bar{p} \leqq \frac{Np}{N + i_0 p} \quad \text{and} \quad \mu_1 = \max\left\{1, \frac{Np}{N + (i_0+1)p}\right\}.$$

We find: $\mu_1 = (Np)/(N + (i_0+1)p) < \bar{p}$, $\mu = (Np)/(N+p)$. By our induction hypothesis, we get:

$$\int_\Omega |u|^\mu \leqq C_{i_0}\left(\int_\Omega |\nabla u|^{\mu_1}\right)^{\mu/\mu_1}. \tag{4.15}$$

Let $P: W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^N)$ be the extension operator. (See [3, pp. 158–162].) We obtain

$$\|u\|_p \leqq \|Pu\|_{L^p(\mathbb{R}^N)} \leqq \beta\, \|\nabla Pu\|_{L^\mu(\mathbb{R}^N)} \leqq \beta \bar{C}_3(\|u\|_\mu + \|\nabla u\|_\mu), \tag{4.16}$$

with $\bar{C}_3$ depending only on Ω and $\|u\|_\mu$ denoting $\|u\|_{L^\mu(Q)}$. Using Hölder's inequality in (4.15) and adding (4.16), we can conclude that there exists a constant $C_{i_0+1} > 0$ depending only on Ω, β, i_0 such that: $\int_\Omega |u|^p \leqq C_{i_0+1}(\int_\Omega |\nabla u|^\mu)^{p/\mu}$. Assuming that $C = \max\{C_1, \ldots, C_i\}$ with $(N\beta)/(N + i\beta) < \bar{p} = N/(N-1)$, we obtain the existence of a constant $C > 0$ depending only on Ω and β such that: $p \in [1, \beta]$, $u \in W^{1,p}(\Omega)$ and $\int_\Omega u = 0$ imply $\xi_\Omega |u|^p \leqq C(\int_\Omega |\nabla u|^\mu)^{p/\mu}$. □

Lemma 4.6. *let $a \in \mathbb{R}^N$, $R > 0$ be real, $v > 1$ be an integer. Assume that $A_0 = Q_R(a)$, $A_i = \{x \in \mathbb{R}^N: dist(x, Q_R(a)) < (ir)/v\}$, $i = 1, \ldots, v$. Then there exist $\phi_i \in C_0^1(A_i)$, $i = 1, \ldots, v$ such that*

$$0 \leqq \phi_i(x) \leqq 1, \quad x \in A_i, \quad \phi_i(x) = 1, x \in A_{i-1}, \quad |\nabla\phi_i(x)| \leqq \frac{v+1}{R}, \quad x \in A_i. \tag{4.17}$$

The proof of Lemma 4.6 is elementary.

Lemma 4.7. *Let $b, q > 1$, $r > q$, $N > 0$ an integer, $\theta < 1/(a_1(q)) = 1/(30^N(q-1))$ $((q - 1/(5q))^q$ and $g, h: Q = \{x \in \mathbb{R}^N, |x_i| \leqq 3, i = 1, \ldots, N\} \mapsto [0, \infty)$ be two functions such that $g \in L^q(Q)$ and $h \in L^r(Q)$. Suppose that for every $x_0 \in Q$, every $0 < R < \frac{1}{2}dist(x_0, \partial Q)$*

$$⨍_{Q_R(x_0)} g^q \leqq b\left\{\left(⨍_{Q_{2R}(x_0)} g\right)^q + ⨍_{Q_{2R}(x_0)} h^q\right\} + \theta ⨍_{Q_{2R}(x_0)} g^q.$$

Then:

$$\int_{D_k} g^t \leqq C(t, q)(3)^{Nt/q}(2^k)^{Nt/q}\, \|g\|_q^t \left[\int_Q g^q + \int_Q h^t\right] \quad \text{for every} \quad t \in [q, q+\varepsilon), \tag{4.18}$$

where

$$\varepsilon = \min\left\{r - q, \frac{q-1}{a-1}\right\}, \quad a \equiv a(q,b,\theta) = \frac{a_1(q)+a_2(q)}{1-\theta a_1(q)} b, \quad a_2(q) = 2^N\left(\frac{5q}{q-1}\right)^{q-1}, \tag{4.19}$$

$$C_{-1} = \{x \in \mathbb{R}^N, |x_i| \leqq 1, i = 1, \ldots, N\}, \tag{4.20}$$

$$C_k = \left\{x \in \mathbb{R}^N, \frac{1}{2^k} < dist(x, \partial Q) \leqq \frac{1}{2^{k-1}}\right\} \quad D_k = \bigcup_{i=-1}^{i=k} C_k \quad k = -1, 0, 1, \ldots \tag{4.21}$$

$$\text{and} \quad C(t,q) = \max\left\{1, \frac{q-1}{aq-(a-1)t-1}, \frac{a(t-q)}{aq-(a-1)t-1}\right\}. \tag{4.22}$$

Proof. The proof of the above lemma has been given in **[7]** by Giaquinta and Modica. It is based on Gehring's lemma. □

As an illustration, we give the following lemma in the case $N = M$. For the general case ($N, M > 1$) see **[10]**.

Lemma 4.8. *Let $p \in [\alpha, \beta] \subset (1, \infty)$, $C \geqq 1$, $N \geqq 2$, $0 \leqq \eta \leqq 1$ $f: \mathbb{R}^{N\times N} \mapsto \mathbb{R}$ be a Borel measurable function, $Q = \{x \in \mathbb{R}^N, |x_i| \leqq 3,\ i = 1, \ldots, N\}$, $u \in W^{1,p}(Q)^N$ such that:*

$$|\xi|^p \leqq f(\xi) \leqq C(1 + |\xi|^p) \quad \text{for every} \quad \xi \in \mathbb{R}^{N\times N}, \tag{4.23}$$

$$F(A,u) \leqq F(A,v) + \eta \int_A |\nabla u - \nabla v|, \tag{4.24}$$

for every $A \Subset Q$ an open set and every $v \in u + W_0^{1,p}(A)^N$ with $F(A,v) = \int_A f(\nabla v)$. Then

$$\oint_{Q_R(x_0)} |\nabla u|^p \leqq b\left\{\left(\oint_{Q_{2R}(x_0)} |\nabla u|^\mu\right)^{p/\mu} + \oint_{Q_{2R}(x_0)} h^{p/\mu}\right\} + \frac{2^{p+1}C}{v} \oint_{Q_{2R}(x_0)} |\nabla u|^p, \tag{4.25}$$

for every $x_0 \in Q$, every $0 < R < \frac{1}{2}dist(x_0, \partial Q)$ and every $v > 1$ integer, where $\mu = \max\{1, (Np)/(Np+1)\}$, $h = (3 + \|\nabla u\|_2)^{\mu/p}$, $b = \{2^{p+1}D(p)(v+1)^p + 1\} \times \{2^{p-1+N}C + 1\}$ and $D(p)$ is defined (see Lemma 4.4) by $\int_\Omega |u|^p \leqq D(p)(\int_Q |\nabla u\|^\mu)^{p/\mu}$ for every $u \in W^{1,p}(Q)$ such that $\int_Q u = 0$. Further, there exist two constants m_3, $E > 0$ depending only on α, β, C such that:

$$\int_{D_k} |\nabla u|^s \leqq E(3)^{Ns/p}(2^k)^{Ns/p} \|\nabla u\|_p^s \left[\int_Q |\nabla u|^p + \int_Q (3 + |\nabla u|)^{s/p}\right], \tag{4.26}$$

for every $k = -1, 0, 1, \ldots$ and every $s \in [p, p + m_3]$.

Proof. The proof of (4.25) has been given in **[10]** by Marcellini and Sbordone. It is easy to reproduce their proof assuming that f satisfies (4.23), f depends only on ∇u and f is Borel measurable. We now prove (4.26). Using (4.25), we get:

$$\oint_{Q_R(x_0)} |\nabla u|^p \leqq b\left\{\left(\oint_{Q_{2R}(x_0)} |\nabla u|^\mu\right)^{p/\mu} + \oint_{Q_{2R}(x_0)} h^{p/\mu}\right\} + \frac{2^{p+1}C}{v} \oint_{Q_{2R}(x_0)} |\nabla u|^p, \tag{4.27}$$

for every $x_0 \in Q$, every $0 < R < \frac{1}{2}dist(x_0, \partial Q)$ and every $v > 1$ integer. Let

$$A_1 = \sup\{a_1(q), q \in (1, \beta]\}, \quad v > 2^{\beta+2}CA_1. \tag{4.28}$$

We obtain

$$\theta < \frac{1}{2A_1}, \quad 0 < a(q,b,\theta) \leqq 4A_1 b. \tag{4.29}$$

Assuming that

$$g = |\nabla u|^\mu, \quad q = \frac{p}{\mu}, \quad t = \frac{s}{\mu},$$

$$m_3 = \frac{1}{2}\min\left\{(\alpha - 1), \frac{\alpha-1}{A_1-1}, \frac{\alpha^2}{N+\beta}\right\},$$

$$E = \sup\left\{\left|c\left(\frac{s}{\mu}, \frac{p}{\mu}\right)\right| + 1, p \in [\alpha, \beta], s \in [p, p+m_3]\right\}$$

and using Lemma 4.7, we find (4.26). □

Remark 4.9. One can see that

$$s \leqq p^2 \quad \text{for every} \quad s \in [p, p + m_3) \tag{4.30}$$

and, by Hölder's inequality, (4.26) implies:

$$\int_{D_k} |\nabla u|^s \leqq E 2^\beta |Q| (3.2^k)^{Ns/p} \|\nabla u\|_p^s [1 + 3^\beta + 2\|\nabla u\|_p^\beta][1 + \|\nabla u\|_p^\beta]. \tag{4.31}$$

We now proceed with the proof of Theorem 4.1.

Proof of Theorem 4.1. To illustrate, we give the proof in the case $N = M$. Fix $\xi \in \mathbb{R}^{N\times N}$.

Part 1. The Proof of (4.3) is elementary.

Part 2. We prove (4.4). We decompose the proof into six steps.

Step 1. Let

$$V = W_0^{1,1}(Q)^N,$$

$$\bar{F}_p(\phi) = \begin{cases} \dfrac{1}{|Q|}\displaystyle\int_Q f_p(\xi + \nabla\phi) & \text{if} \quad \phi \in W_0^{1,p}(Q)^N, \\ \infty & \text{if} \quad \phi \in V - W_0^{1,p}(Q)^N. \end{cases} \tag{4.32}$$

Here, we show that there exists a sequence $(\phi_n^p)_n \in V$ such that:

$$\int_{D_k} |\xi + \nabla\phi_n^p|^s \leqq J 2^{Nk\beta}[1 + \|\xi + \nabla\phi_n^p\|_p^{2\beta}] \quad \text{and} \quad \lim_{n\to\infty} \bar{F}_p(\phi_n^p) = Qf_p(\xi) \tag{4.33}$$

for every $s \in [p, p + m_3]$ and every $k = -1, 0, 1, \ldots$, where D_k, m_3 are defined in Lemma 4.7, Lemma 4.8 and J is a constant depending only on Q, β. Observe that

$$f_p \text{ is lower semicontinuous} \Rightarrow \bar{F}_p \text{ is lower semicontinuous.} \tag{4.34}$$

By (4.3) and (4.34) we deduce that there exists a sequence

$$(\psi_n^p)_n \in V \quad \text{with} \quad F_p(\psi_n^p) \leqq \inf\{\bar{F}_p(\phi), \phi \in V\} + \frac{1}{n} = Qf_p(\xi) + \frac{1}{n}. \tag{4.35}$$

Using the variational principle of Ekeland (see [5]), we obtain a sequence

$$(\phi_n^p)_n \in V \quad \text{such that} \quad \bar{F}_p(\phi_n^p) \leqq \bar{F}_p(\psi_n^p) \quad \text{with} \quad \int_Q |\nabla\psi_n^p - \nabla\phi_n^p| \leqq 1 \tag{4.36}$$

and

$$\bar{F}_p(\phi_n^p) \leqq \bar{F}_p(\phi) + \frac{1}{n}\int_Q |\nabla\phi - \nabla\phi_n^p| \quad \text{for every} \quad \phi \in V. \tag{4.37}$$

Further, for every A an open set, $A \Subset Q$ and every $\phi \in \phi_n^p + W_0^{1,p}(A)^N$, we obtain

$$\bar{F}_p(A, \phi_n^p) \leqq \bar{F}_p(A, \phi) + \frac{1}{n}\int_Q |\nabla\phi - \nabla\phi_n^p|, \tag{4.38}$$

where

$$\bar{F}_p(A, \phi) = \frac{1}{|Q|}\int_A f_p(\xi + \nabla\phi) \quad \text{if} \quad \phi \in W^{1,p}(A)^N. \tag{4.39}$$

In the next four steps, we suppose that $q \in (\alpha, \beta)$ is a fixed number.

Step 2. Let n be a fixed integer. We show that for every sequence $S \subset [\alpha, q]$ there exists a subset $\bar{S} \subset S$, a subsequence $(\phi_n^r)_{r\in\bar{S}}$ and a $\phi_n \in W_0^{1,q}(Q)^N$ such that:

$$\phi_n^r \xrightarrow[r\to q]{\text{weakly}} \phi_n W^{1,p}(Q)^N \quad \text{for every} \quad p \in \bar{S}. \tag{4.40}$$

By (4.2), (4.35) and (4.36), for every $p \in S$ such that p is near q, we find that:

$$\frac{1}{|Q|}\int_Q |\xi + \nabla\phi_n^p|^p \leqq f_p(\xi) + 1 \leqq f_q(\xi) + 1 + \frac{FG}{\gamma_0}(1 + |\xi|^{q+\gamma_0}) = H(\xi). \tag{4.41}$$

We choose with respect to p a subsequence in the following way. First, we fix $p_1^1 \in S$. Using Hölder's inequality in (4.41), we deduce that there there exists a sequence $p_1^1 < p_2^1 < p_3^1 <, \dots,$ in S such that

$$\phi_n^{p_i^1} \xrightarrow[i\to\infty]{\text{weakly}} l_n^{p_1^1} W^{1,p_1^1}(Q)^N \quad \text{and} \quad p_i^1 \xrightarrow[i\to\infty]{} q. \tag{4.42}$$

Then assume that $p_1^2 = p_1^1$, $p_2^2 = p_2^1$. Using Hölder's inequality again in (4.41), we deduce that there exists $p_1^2 < p_2^2 < p_3^2 <, \dots,$ in S such that

$$\phi_n^{p_i^2} \xrightarrow[i\to\infty]{\text{weakly}} l_n^{p_2^2} W^{1,p_2^2}(Q)^N, \; p_i^2 \xrightarrow[i\to\infty]{} q \quad \text{and} \quad l_n^{p_1^1} = l_n^{p_2^2}. \tag{4.43}$$

Now suppose that we have found the numbers $p_1^1 < p_2^2 < \dots < p_k^k$ and an increasing subsequence $(p_i^k)_{i\in\mathbb{N}}$ such that

$$\phi_n^{p_i^k} \xrightarrow[i\to\infty]{\text{weakly}} l_n^{p_k^k} W^{1,p_1^1}(Q)^N, \dots, W^{1,p_k^k}(Q)^N.$$

Assume that $p_1^{k+1} = p_1^k, \dots, p_{k+1}^{k+1} = p_{k+1}^k$. Using Hölder's inequality again in

(4.41), we can obtain an increasing subsequence $(p_i^{k+1})_{i\geqq k+2}$ from $(p_i^k)_{i\geqq k+2}$ such that

$$\phi_n^{p_i^{k+1}} \xrightarrow[i\to\infty]{\text{weakly}} l_n^{p_{k+1}^{k+1}} W^{1,p_{k+1}^{k+1}}(Q)^N \quad \text{and} \quad l_n^{p_k^k} = l_n^{p_{k+1}^{k+1}}.$$

Assume that $\phi_n = l_n^{p_{k+1}^{k+1}}$ and $\bar{S} = \{p_k^k, k \in \mathbb{N}\}$. Using (4.41), it is easy to deduce that $\phi_n \in W_0^{1,q}(Q)^N$ and

$$\phi_n^r \xrightarrow[r\to q]{\text{weakly}} \phi_n W^{1,p}(Q)^N \quad \text{for every} \quad p \in \bar{S}. \tag{4.44}$$

Step 3. We show that

$$\liminf_{p\to q-} Qf_p(\xi) \geqq Qf_q(\xi), \tag{4.45}$$

where $\liminf_{p\to q-} Qf_p(\xi)$ is defined by $\liminf_{p\to q, p<q} Qf_p(\xi)$. To show (4.45), we suppose that

$$\liminf_{p\to q-} Qf_p(\xi) < Qf_q(\xi) \tag{4.46}$$

and we get a contradiction. Now (4.46) implies that there exists a sequence $(p_i)_i \subset [\alpha, q]$ such that $\lim_{i\to\infty} Qf_{p_i}(\xi) < Qf_q(\xi)$. Let m_3 and D_k be defined as in Lemma 4.7 and assume that $\gamma = \min\left\{\gamma_0, \frac{m_3}{8}\right\}$. By (4.35) and (4.36), we get $Qf_p(\xi) \geqq \frac{1}{|Q|}\int_Q f_{p_i}(\xi + \nabla\phi_n^{p_i}) + \frac{1}{n}$. Assuming that $|p - q| < m_3$ (m_3 is defined in Lemma 4.8) and using (4.2) and (4.33), we deduce that for every $k = -1, 0, 1, \dots$

$$Qf_{p_i}(\xi) \geqq \frac{1}{|Q|}\int_{D_k} f_q(\xi + \nabla\phi_i^p n) - \frac{1}{n} - \frac{FG}{\gamma|Q|} w(p_i - q)J(\beta, k, \gamma, \xi)$$

and $\lim_{i\to\infty} Qf_{p_i}(\xi) \geqq \liminf_{i\to\infty} 1/|Q| \int_{D_k} f_q(\xi + \nabla\phi_n^{p_i}) - 1/n$. Using the fact that Qf_q is quasiconvex (see [4]), we obtain

$$\lim_{i\to\infty} Qf_{p_i}(\xi) \geqq \frac{1}{|Q|}\int_{D_k} Qf_q(\xi + \nabla\phi_n) - \frac{1}{n}. \tag{4.47}$$

Recalling that $\phi_n \in W_0^{1,q}(Q)^N$ and that (4.47) is true for every $k = -1, 0, 1, \dots$, we conclude that

$$\lim_{i\to\infty} Qf_{p_i}(\xi) \geqq Qf_q(\xi). \tag{4.48}$$

Therefore (4.46) leads to a contradiction. This implies that (4.46) is false and so (4.45) is proved.

Step 4. We show that

$$\liminf_{p\to q-} Qf_p(\xi) \leqq Qf_q(\xi). \tag{4.49}$$

By (4.35) and (4.36), we get $Qf_q(\xi) \geqq Qf_p(\xi) - (1/|Q| \int_Q u_n^p) - (1/n)$, where

$u_n^p = |f_q(\xi + \nabla\phi_n^q) - f_p(\xi + \nabla\phi_n^q)|$. Using a subsequence of (u_n^p) with respect to p, we obtain that

$$u_n^p \xrightarrow[p\to q-]{} 0 L^1(Q).$$

Thus for every $n \in \mathbb{N}$, $Qf_q(\xi) \geqq \liminf_{p\to q-} Qf_p(\xi) - (1/n)$. This leads to (4.49).

Step 5. We show that

$$\liminf_{p\to q+} Qf_p(\xi) \geqq Qf_1(\xi), \tag{4.50}$$

where $\liminf_{p\to q+} Qf_p(\xi)$ is defined by $\liminf_{p\to q, p>q} Qf_p(\xi)$. By (4.35) and (4.36) we find that $Qf_p(\xi) \geqq 1/|Q| \int_Q f_p(\xi + \nabla\phi_n^p) - (1/n)$ and $(\phi_n^p)_p$ is bounded in $W^{1,q}(Q)^N$. Using a subsequence of $(\phi_n^p)_p$ with respect to p, we deduce that

$$\phi_n^p \xrightarrow[p\to q+]{\text{weakly}} \psi_n W^{1,q}(Q)^N.$$

We proceed as in Steps 3 and 4 to conclude that $\liminf_{p\to q+} Qf_p(\xi) \geqq Qf_q(\xi)$, which proves (4.50).

Step 6. It is very easy to prove that

$$\limsup_{p\to q} Qf_p(\xi) \leqq Qf_q(\xi) \tag{4.51}$$

In conclusion, (4.45), (4.49), (4.50) and (4.51) imply that:

Step 6. It is very easy to prove that

$$\limsup_{p\to q} Qf_p(\xi) \leqq Qf_q(\xi) \tag{4.51}$$

In conclusion, (4.45), (4.49), (4.50) and (4.51) imply that:

$$\lim_{p\to q} Qf_p(\xi) = Qf_q(\xi).$$

and Theorem 4.1 is completely proved. □

We now use Theorem 4.1 to find some quasiconvex functions $f: \mathbb{R}^{N\times N} \to \mathbb{R}$ which are not polyconvex. We study such functions below.

EXAMPLE 4.10. Let $C, F > 0$, $\beta > 2$, $f_p(\xi) = |\xi|^p + a h_p(det(\xi))$, $\xi \in \mathbb{R}^{2\times 2}$, $p \in [\alpha, \beta] \subset (1, \infty)$ and $a \geqq 0$ such that $h_p(x)$ behaves as $|x|^{p/2}$. This means that:

$$h_p: \mathbb{R} \to \mathbb{R} \text{ is lower semicontinuous;} \tag{4.52}$$

$$h_p(x) \leqq h_q(x) \quad \text{for} \quad p < q \quad \text{and} \quad |x| \geqq 1; \tag{4.53}$$

$$|x|^{p/2} \leqq h_p(x) \leqq C(1 + |x|^{p/2}) \quad \text{for every} \quad x \in \mathbb{R} \quad \text{and every} \quad p \in [\alpha, \beta]; \tag{4.54}$$

$$h_2 \text{ is convex and } h_2(1) > h_2(0); \tag{4.55}$$

$$|h_p(x) - h_q(x)| \leqq \frac{F}{\gamma} |p - q| (1 + |x|^{(q/2)+\gamma}) \quad \text{for every} \quad p < q \in [\alpha, \beta] \tag{4.56}$$

and every $\gamma \in [0, \frac{1}{2}]$. Then for every $\alpha > \dfrac{2}{h_2(1) - h_2(0)}$ there exists a $p_0 \in (1, 2)$ such that

$$Qf_p > Pf_p \quad \text{for} \quad p_0 < p < 2.$$

Proof. (We note that $h_p(x) = |x|^{p/2}$ satisfies the hypotheses above.) First, h_2 convex and $a \geqq 0$ imply that f_2 is polyconvex; $a > \dfrac{2}{h_2(1) - h_2(0)}$ implies that f_2 is not convex. Using (4.53) and (4.56), we find that:

$$\limsup_{p\to 2} Cf_p \leqq Cf_2. \tag{4.57}$$

By Theorem 4.1 and (4.56) we find that

$$\liminf_{p\to 2} Qf_p = Qf_2 > Cf_2 \geqq \limsup_{p\to 2} Cf_p = \limsup_{p\to 2} Pf_p.$$

Knowing that $Pf_2 = f_2$ and $Pf_p = Cf_p$ for every $p \in (1, 2)$, we conclude that there exists a $p_0 \in (1, 2)$ such that

$$Qf_p > Pf_p \quad \text{for} \quad p \in (p_0, 2). \quad \square$$

EXAMPLE 4.11. A particular case of this example has been studied in **[8]** by Kohn. Let $N > 1$ be an integer, $\beta > 2$, F, γ_0, $d > 0$, $M \geqq 1$, I the identity matrix of $\mathbb{R}^{N\times N}$, $p \in [\alpha, \beta] \subset (1, \infty)$ and

$$f_p(\xi) = \min(|\xi + I|^p, |\xi - I|^p) + h_p(|det(\xi)| - 1), \quad \xi \in \mathbb{R}^{N\times N}, \quad (h_p \equiv 0 \text{ in } [8]),$$

with $h_p(x)$ behaving as $|x|^{p/N}$. This means that:

$$h_p: \mathbb{R} \to \mathbb{R} \text{ is lower semicontinuous;} \tag{4.58}$$

$$h_p(0) = 0, \ \sup\{|h_p(x)|; |x| \leqq M\} < M\} < F \quad \text{for every} \quad p \in [\alpha, \beta]; \tag{4.59}$$

$$h_p(x) \leqq h_q(x) \quad \text{for} \quad p < q \quad \text{and} \quad |x| \geqq M; \tag{4.60}$$

$$0 \leqq h_p(x) \leqq d(1 + |x|^{p/N}) \quad \text{for every} \quad x \in \mathbb{R} \quad \text{and every} \quad p \in [\alpha, \beta]; \tag{4.61}$$

$$|h_p(x) - h_q(x)| \leqq \frac{F}{\gamma} |p - q| (1 + |x|^{(q/N)+\gamma}) \quad \text{for every} \quad p < q \in [\alpha, \beta] \tag{4.62}$$

and every $\gamma \in (0, \gamma_0]$. Then there exists a $p_0 \in (1, 2)$ such that

$$Qf_p > Pf_p \quad \text{and} \quad p \in (p_0, 2)$$

Proof. (We first note that $h_p(x) = |x|^{p/N}$ satisfies the hypotheses above.) We find that $PF_2(\xi) = 0 \Rightarrow \xi = I, -I$ and $Cf_2(tI) = 0$ for every $t \in [0, 1]$. Therefore

$$Pf_2 > Cf_2. \tag{4.63}$$

Assuming that $g_p(\xi) = f_p(\xi) + N^{\beta/2}$, we find that g satisfies the hypotheses of Theorem 4.1. Thus $\liminf_{p\to 2} Qf_p = Qf_2$. (4.59), (4.61) and (4.60) imply that $\limsup_{p\to 2} Cf_p \leqq Cf_2$. We then conclude that:

$$\liminf_{p\to 2} Qf_p = Qf_2 \geqq Pf_2 > Cf_2 \geqq \limsup_{p\to 2} Cf_p.$$

Therefore there exists a $p_0 \in (1, 2)$ such that

$$Qf_p > Pf_p \quad \text{for every} \quad p \in (p_0, 2). \quad \square$$

5. Continuity of Rf_p with respect to p

We first start with the main theorem of this section.

THEOREM 5.1. *Let $[\alpha, \beta] \subset (1, \infty)$, $F, G, K, \gamma_0 > 0$, $C \geqq 1$ and $N, M > 1$ be two integers. Let $w: [0, \infty) \mapsto [0, \infty)$ with $\lim_{t \to 0} w(t) = w(0) = 0$, $\sup\{w(t), t \in [0, \beta]\} \leqq G$ and $f_p: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ lower semicontinuous, $p \in [\alpha, \beta]$, such that:*

$$|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p) \quad \text{for every} \quad p \in [\alpha, \beta] \quad \text{and every} \quad \xi \in \mathbb{R}^{N\times M}; \tag{5.1}$$

$$|f_p(\xi) - f_q(\xi)| \leqq \frac{F}{\gamma} w(p - q)(1 + |\xi|^{p+\gamma}) \quad \text{for every} \quad \gamma \in (0, \gamma_0), \tag{5.2}$$

every $\xi \in \mathbb{R}^{N\times M}$, and every $p, q \in [\alpha, \beta]$ with $p > q$;

$$f_p(\xi) = Rf_p(\xi) \quad \text{for every} \quad |\xi| \geqq K. \tag{5.3}$$

Then

$$\lim_{p \to q} Rf_p(\xi) = Rf_q(\xi), \quad \text{for every} \quad \xi \in \mathbb{R}^{N\times M} \quad \text{and every} \quad q \in (\alpha, \beta). \tag{5.4}$$

Before proving this theorem, let us begin with some remarks.

Remarks 5.2. (a) In general, we have $\liminf_{p \to 1^-} Rf_p < Rf_1$. Indeed, if $f_p(\xi) = |\xi|^p$, $\xi \in \mathbb{R}^{N\times M}$ then $0 \equiv \liminf_{p \to 1^-} Rf_p < Rf_1 = f_1$.

(b) Theorem 5.1 is still true if we replace the condition (5.1) by $a + b|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p)$, where $a \in \mathbb{R}$ and $b > 0$ are two constants.

(c) We do not need to use (5.3) to prove that $\limsup_{p \to q} Rf_p \leqq Rf_q$ for $q \in [\alpha, \beta]$. The most difficult part in this case is to prove that $\liminf_{p \to q} Rf_p \geqq Rf_q$. We were unable to prove this inequality without assuming (5.3).

(d) If we keep only hypotheses (5.1) and (5.2), we can show that

$$\lim_{p \to q} R_k f_p = R_k f_q \quad \text{for every} \quad k \in \mathbb{N}, \quad \text{and every} \quad q \in [\alpha, \beta].$$

Where $R_0 f = f$, $R_{k+1} f(\xi) = \inf\{tR_k f(\eta) - (1 - t)R_k f(\mu),\ t \in (0, 1),\ \eta, \mu \in \mathbb{R}^{N\times M},\ rank(\eta - \mu) \leqq 1,\ \xi = t\eta + (1 - t)\mu\}$. One knows that $\lim_{k \to \infty} R_k f = Rf$. (See [**4, 9**].)

EXAMPLE 5.3. Let

$$f_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{for} \quad |\xi| \neq 0, \\ 0 & \text{for} \quad |\xi| = 0, \end{cases} \quad \xi \in \mathbb{R}^{N\times M}.$$

Knowing that $Cf_p(\xi) = 1 + |\xi|^p$ if $|\xi| \geqq (1/(p - 1))^{1/p}$, we find that $(f_p)_p$ verifies (5.1), (5.2) and (5.3). Hence Theorem 5.1 leads to $\lim_{p \to q} Rf_p = Rf_q$, for every $q > 1$.

To prove Theorem 5.1, let us now begin with the following lemma:

LEMMA 5.4. *Let $N, M \geqq 1$ be two integers, $C, K > 0$ two real constants and $f: \mathbb{R}^{N\times M} \mapsto \mathbb{R}$ lower semicontinuous such that:*

$$|\xi|^p \leqq f_p(\xi) \leqq C(1 + |\xi|^p) \quad \text{for every} \quad \xi \in \mathbb{R}^{N\times M}; \tag{5.5}$$

$$Rf(\xi) = f(\xi) \quad \text{for every} \quad |\xi| \geqq K. \tag{5.6}$$

Then, for every $\xi \in \mathbb{R}^{N\times M}$, such that $|\xi| \leqq K$, there exist $t \in [0, 1]$, $\xi_1, \xi_2 \in \mathbb{R}^{N\times M}$ verifying

$$rang(\xi_1 - \xi_2) \leqq 1, \quad \xi = t\xi_1 + (1 - t)\xi_2, \quad R_1 f(\xi) = tR_1 f(\xi_1) + (1 - t)R_1 f(\xi_2),\ |\xi_1|, |\xi_2| \leqq K. \tag{5.7}$$

Proof. The proof of Lemma 5.4 is left to the reader. □

We now prove Theorem 5.1.

Proof of Theorem 5.1. Let $\xi \in \mathbb{R}^{N\times M}$ and $q \in [\alpha, \beta]$. Using the same arguments as in Step 1 of the proof of Theorem 3.1, we prove that $\limsup_{p \to q} Rf_p(\xi) \leqq Rf_q(\xi)$.

Let us now prove that $\liminf_{p \to q} Rf_p(\xi) \geqq Rf_q(\xi)$. Using (5.1), the result is obvious if $|\xi| \geqq K$. If $|\xi| \leqq K$, by (5.1), (5.2), (5.3) and Lemma 5.4, we get that, for every $k \in \mathbb{N}$, for every $p \in [\alpha, \beta]$, there exist $\lambda_i^p \in (0, 1)$, $\xi_i^p \in \mathbb{R}^{N\times M}$, $i = 1, \ldots, 2^k$ such that

$$|\xi_i^p| \leqq K, \quad i = 1, \ldots, 2^k, \quad Rf_p(\xi) = \sum_{i=1}^{2^k} \lambda_i^p f_p(\xi_i^p) \quad \text{and} \quad Rf_q(\xi) \leqq \sum_{i=1}^{2^k} \lambda_i^p f_q(\xi_i^p).$$

We deduce from the previous relations that

$$Rf_p(\xi) \geqq Rf_q(\xi) - \frac{F}{\gamma} w(p - q)(1 + M^{p+\gamma} + M^{p+\gamma}),$$

and so

$$\liminf_{p \to q} Rf_p(\xi) \geqq Rf_q(\xi).$$

Hence Theorem 5.1 is proved. □

6. Examples of f such that $f(\xi) = g(|\xi|)$ and $Pf \neq Qf$

THEOREM 6.1. *Let $f: \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ be a Borel measurable function, $a, b, c, \alpha > 0$, $d \in \mathbb{R}$ and $q \in (1, 2)$ such that*

$$a|\xi| + f(0) \leqq f(\xi) \quad \text{for every} \quad \xi \in \mathbb{R}^{2\times 2}, \quad \text{with equality for} \quad \xi' = \frac{\alpha}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{6.1}$$

$$b|\xi|^q + d \leqq f(\xi) \leqq c(1 + |\xi|^q) \quad \text{for every} \quad \xi \in \mathbb{R}^{2\times 2}; \tag{6.2}$$

there exists $t_0 \in (0, 1)$ for which $f(\xi^*) \neq a|\xi^*| + f(0)$ with $\xi^* = t_0\xi'$. (6.3)

Then

$$Qf(\xi^*) > Pf(\xi^*). \tag{6.4}$$

Before proceeding to the proof, we make the following remarks.

Remarks 6.2. (a) By (6.1) the graph of f and one of Cf intersect. This plays an important role in the proof of Theorem 6.1.

(b) Using Theorem 4.1, we can prove Theorem 6.1 only for q near 2. But here we will conclude for every $q \in (1, 2)$. This theorem implies $Qf_p > Pf_p$ for every $p \in (1, 2)$ where

$$f_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{if} \quad |\xi| \neq 0, \\ 0 & \text{if} \quad |\xi| = 0, \end{cases} \quad \xi \in \mathbb{R}^{2\times 2}.$$

Note that in **[9]** Kohn and Strang proved that $Pf_2 = Qf_2$.

LEMMA 6.2. *Let $N \geqq 1$ be an integer, $\Omega \subset \mathbb{R}^N$ a bounded open set, $\beta, \gamma > 0$ and $r > 1$. Let $(v_n)_n \subset L^r(\Omega)$ such that $\gamma \leqq \|v_n\|_1$, $\|v_n\|_r \leqq \beta$ for all $n \in \mathbb{N}$. Then there exist $k, l > 0$ such that $|\{x \in \Omega : |v_n(x)| \geqq k\}| \geqq l$ for every $n \in \mathbb{N}$.*

Proof. The proof of Lemma 6.3 is elementary. □

LEMMA 6.4. *Let $f: \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ a Borel measurable function, $a, b, c, \alpha > 0$, $d \in \mathbb{R}$, $q \geqq 1$ and $\Omega \subset \mathbb{R}^2$ a bounded open set such that*

$$a\,|\xi| + f(0) \leqq f(\xi) \quad \textit{for every} \quad \xi \in \mathbb{R}^{2\times 2} \quad \textit{with equality for} \quad \xi_0 = \frac{\alpha}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{6.5}$$

$$b\,|\xi|^q + d \leqq f(\xi) \quad \textit{for every} \quad \xi \in \mathbb{R}^{2\times 2}. \tag{6.6}$$

Then

$$Cf(t\xi_0) = a\,|t\xi_0| + f(0) \quad \textit{for every} \quad t \in [0, 1]. \tag{6.7}$$

If for a fixed $\xi \in \mathbb{R}^{2\times 2}$

$$Qf(\xi) = a\,|\xi| + f(0) = \lim_{n\to\infty} \frac{1}{|\Omega|} \int_\Omega f(\xi + \nabla\phi_n), \tag{6.8}$$

where $(\phi_n)_n \subset W_0^{1,\infty}(\Omega)^2$, then, up to a subsequence, the following hold:

$$\phi_n \rightharpoonup 0 W_0^{1,q}(\Omega)^2, \tag{6.9}$$

$$f(\xi + \nabla\phi_n) - a\,|\xi + \nabla\phi_n| - f(0) \to 0 L^1(\Omega), \tag{6.10}$$

$$|\xi + \nabla\phi_n| + |\xi| - |2\xi + \nabla\phi_n| \to 0 L^1(\Omega). \tag{6.11}$$

But note that the following does not hold:

$$\phi_n \to 0 W^{1,1}(\Omega) \quad \textit{if} \quad f(\xi) \neq a\,|\xi| + f(0) \textit{ and } f \textit{ is continuous at } \xi. \tag{6.12}$$

Proof. We first establish (6.7). We find using (6.5) that: $a\,|\xi| + f(0) \leqq Cf(\xi)$ for every $\xi \in \mathbb{R}^{2\times 2}$. Let $\xi \in \mathbb{R}^{2\times 2}$ be such that $\xi = t\xi_0$ with $t \in [0, 1]$. We have $Cf(\xi) \leqq tf(\xi_0) + (1 - t)f(0) = a\,|\xi| + f(0)$ and we obtain (6.7).

We now prove (6.9), (6.10). Let $\xi \in \mathbb{R}^{2\times 2}$ be fixed and $(\phi_n)_n \subset W_0^{1,\infty}(\Omega)^2$ such that $a\,|\xi| + f(0) = Qf(\xi) = \lim_{n\to\infty} \frac{1}{|\Omega|} \int_\Omega f(\xi + \nabla\phi_n)$. Using (6.6), we find that $\int_\Omega |\xi + \nabla\phi_n|^q \leqq \frac{1}{b}(\int_\Omega [f(\xi + \nabla\phi_n) - d])$ and that $(\phi_n)_n$ is bounded in $W^{1,q}(\Omega)^2$. Up to a subsequence, we can suppose that

$$\phi_n \xrightarrow[n\to\infty]{\text{weakly}} \phi W_0^{1,q}(\Omega)^2. \tag{6.13}$$

This implies that

$$\begin{aligned} 0 &\leqq \int_\Omega |f(\xi + \phi_n) - a\,|\xi + \nabla\phi_n| - f(0) \\ &= \int_\Omega f(\xi + \nabla\phi_n) - a\,|\xi + \nabla\phi_n| - f(0) \\ &\leqq \int_\Omega f(\xi + \nabla\phi_n) - a\,|\xi| - f(0) \to 0 \end{aligned}$$

by (6.8). We therefore conclude that (6.10) is true. Using (6.8) we find $|\Omega|\,(a\,|\xi| + f(0)) = \lim_{n\to\infty} \int_\Omega f(\xi + \nabla\phi_n) = \lim_{n\to\infty} \int_\Omega [a\,|\xi + \nabla\phi_n| + f(0)] \geqq \int_\Omega [a\,|\xi + \nabla\phi| + f(0)]$. This immediately gives $\nabla\phi = 0$. Using (6.13), we now find (6.9).

We now establish (6.11): up to a subsequence we can suppose that $\lim_{n\to\infty} \int_\Omega |2\xi + \nabla\phi|$ exists. (6.9) implies that

$$\int_\Omega |2\xi| \leqq \lim_{n\to\infty} \int_\Omega |2\xi + \nabla\phi|. \tag{6.14}$$

Note that $f(\eta) \geqq a\,|\eta| + f(0)$ for every $\eta \in \mathbb{R}^{2\times 2}$ and using (6.10) we obtain:

$$\lim_{n\to\infty} \int_\Omega |\xi + \nabla\phi_n| = \int_\Omega |\xi|. \tag{6.15}$$

using (6.14) and (6.15) we find $0 \leqq \lim_{n\to\infty} \int_\Omega |\,|\xi + \nabla\phi_n| + |\xi| - |2\xi + \nabla\phi_n|\,| = \lim_{n\to\infty} \int_\Omega |\xi + \nabla\phi_n| + |\xi| - |2\xi + \nabla\phi_n| \leqq 0$. We therefore obtain (6.11).

We finally prove (6.12): assume that f is continuous at ξ and $f(\xi) \neq a\,|\xi| + f(0)$. Now we suppose that $\phi_n \to 0$ strongly in $W^{1,1}(\Omega)$ and show that this leads to a contradiction. Using a subsequence we can find an $x \in \Omega$ such that $\lim_{n\to\infty} \nabla\phi_n(x) = 0$ and $\lim_{n\to\infty} f(\xi + \nabla\phi_n(x)) - a\,|\xi + \nabla\phi_n(x)| - f(0) = 0$. Thus $f(\xi) - a\,|\xi| - f(0) = 0$, which is a contradiction to our hypotheses. Thus, $\phi_n \to 0 W^{1,1}(\Omega)$ is false. This completes the proof of Lemma 6.3. □

Proof of Theorem 6.1. The hypotheses on f imply that $Pf = Cf$. To conclude, it suffices to show that $Qf(\xi^*) > Cf(\xi^*)$. Recall that by (6.7) $Cf(\xi^*) = a\,|\xi^*| + f(0)$.

To obtain a contradiction, we suppose that $Qf(\xi^*) = a|\xi^*| + f(0)$. Assuming that

$$u_n = 2(|\partial_2\phi_1^n| + |\partial_1\phi_2^n|) + |\partial_1\phi_1^n - \partial_2\phi_2^n|, \quad \varepsilon \in (0, q-1) \quad \text{and} \quad v_n = u_n^{1+\varepsilon},$$

we get $v_n \in L^r(\Omega)$, where $\partial_1\phi_1$ denotes $\partial\phi_1/\partial x_1$ and $r = q/(1+\varepsilon) > 1$. Two cases may occur:

Case 1. $v_n \to 0$ $L^1(\Omega)$. It follows that $\Delta\phi_n \to 0$ $W^{-1,1/\varepsilon}(\Omega)$, which implies that $\phi_n \to 0$ $W^{1,1+\varepsilon}(\Omega)$ and then $\phi_n \to 0$ $W^{1,1}(\Omega)$ (for more details see **[13]**). But by Lemma 6.4 $\phi_n \to 0$ $W^{1,1}(\Omega)$ does not hold. We therefore have a contradiction.

Case 2. $v_n \to L^1(\Omega)$ does not hold. Using a subsequence, we then find that there exists a constant $\gamma > 0$ such that, for every $n \in \mathbb{N}$, $\|v_n\|_1 \geqq \gamma$. Since $(v_n)_n$ is bounded in $L^r(\Omega)$ and $r > 1$, by Lemma 6.3, there exist two constants $\bar{k} > 0$ and $l > 0$ such that $|\{x \in \Omega: v_n(x) \geqq \bar{k}\}| \geqq l$ for all $n \in \mathbb{N}$. We immediately conclude that there exist $B > 0$, $k > 0$ such that $|\{x \in \Omega: u_n(x) \geqq k$ and $|\nabla\phi_n(x)| > B\}| \geqq l/2$. We now write:

$$A_n = \{x \in \Omega: u_n(x) \geqq k, |\nabla\phi_n(x)| \leqq B\},$$

$$K = \{\eta \in \mathbb{R}^{2\times2}: |\eta| \leqq B, 2(|\eta_{12}| + |\eta_{21}|) + |\eta_{11} - \eta_{22}| \geqq k\},$$

$$F(\eta) = \tfrac{1}{2}|\xi + \eta| + \tfrac{1}{2}|\xi| - \tfrac{1}{2}|2\xi + \eta| \ \eta \in \mathbb{R}^{2\times2}.$$

K is compact in $\mathbb{R}^{2\times2}$, F is continuous in $\mathbb{R}^{2\times2}$ and we find $0 < \beta = \min\{F(\eta): \eta \in K\}$ because $F(\eta) \leqq 0$ implies that $\eta = \begin{pmatrix} \eta_{11} & 0 \\ 0 & \eta_{22} \end{pmatrix} \notin K$. But $\int_\Omega F(\nabla\phi_n) \geqq \int_{A_n} F(\nabla\phi_n) \geqq \beta|A_n| \geqq \dfrac{l\beta}{2}$. Furthermore, using (6.11) in Lemma 6.4, we obtain $0 = \lim_{n\to\infty} \int_\Omega F(\nabla\phi_n)$, a contradiction. We therefore deduce that $v_n \to 0$ $L^1(\Omega)$. Since the two cases do not apply, we conclude that

$$Qf(\xi^*) \neq Cf(\xi^*),$$

which is equivalent to

$$Qf(\xi^*) > Pf(\xi^*).$$

This finishes the proof of Theorem 6.1. □

COROLLARY 6.5. *Let* $p \in (1, 2)$, $t \in (0, \alpha)$, $\alpha = [1/(p-1)]^{1/p}$, $\xi_t = \dfrac{t}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and*

$$f_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{if } |\xi| \neq 0, \\ 0 & \text{if } |\xi| = 0, \end{cases} \quad \xi \in \mathbb{R}^{2\times2}.$$

Then

$$Pf_p(\xi_t) < Qf_p(\xi_t).$$

Proof. It is easy to see that

$$Cf_p(\xi) = \begin{cases} 1 + |\xi|^p & \text{if } |\xi| \geqq \alpha, \\ a|\xi| & \text{if } |\xi| < \alpha, \end{cases}$$

where $a = p^{1/p}p'^{1/p'}$ and $p' = p/(p-1)$. Additionally the following relations hold:

(a) $Rf_p(\xi) = Qf_p(\xi) = Pf_p(\xi) = Cf_p(\xi) = f(\xi)$ for every $|\xi| \geqq \alpha$;

(b) $a|\xi| \leqq f(\xi)$ with equality if and only if $|\xi| = 0$ or α;

(c) $0 < |\xi_t| = t < \alpha$.

These are the hypotheses of Theorem 6.1. Thus, Corollary 6.5 is proved. □

Remark 6.6. To construct a function $f: \mathbb{R}^{2\times2} \mapsto \mathbb{R}$ satisfying the hypotheses of Theorem 6.1, it suffices to construct a continuous function $g: [0, \infty[\to \mathbb{R}$ such that:

$$g(0) = 0, \quad g(x) = 1 + x^p \quad \text{if} \quad x \geqq \alpha \quad ax < g(x) < 1 + x^p \quad \text{if} \quad x \in (0, \alpha),$$

where

$$1 < p < 2, \quad a = p^{1/p}p'^{1/p'}, \quad p' = \frac{p}{p-1}, \quad \text{and} \quad \alpha = \left[\frac{1}{p-1}\right]^{1/p}.$$

Assuming that $f(\xi) = g(|\xi|)$, we find that

$$Qf(\xi_t) > Pf(\xi_t)$$

for every $\xi_t = \dfrac{t}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $t \in (0, \alpha)$.

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