Extension theorems for vector valued maps

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Abstract

We revisit studies on extension of Lipschitz maps and obtain new results about extension of displacements of bounded strain tensors. These questions are of interest in elasticity theory, optimal designs, as well as in functional analysis.

Résumé

Nous discutons l'extension d'applications Lipschitziennes et donnons, entre autres, une nouvelle démonstration d'un théorème de Schönbeck. Puis nous étudions le problème d'extension de déplacements dont le tenseur des déformations est borné. Ces questions sont intéressantes en élasticité (cf. le problème de Michell) aussi bien qu'en analyse fonctionnelle.

Key words: Extension of Lipschitz maps, Kirszbraun theorem, Michell problem.

Mots clés: Extension d'applications Lipschitziennes, théorème de Kirszbraun, problème de Michell.

1 Introduction

1.1 Statement of the problem

In this article we deal with essentially two types of extension of vector valued maps.

Extension of Lipschitz maps. We consider two Banach spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$. We ask whenever a map $u : D \subset E \to F$ satisfying

$$\|u(x) - u(y)\|_F \leq \|x - y\|_E\,,\ x, y \in D \tag{1}$$

can be extended to the whole of E so as to preserve the inequality.
This is by now a classical problem and we revisit this question in Section 2.

Extension of maps and Michell trusses. The second problem, that we consider, concerns maps $u : D \subset \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$|\langle u(x) - u(y); x - y\rangle| \leq \|x - y\|^2, \quad x, y \in D \tag{2}$$

where $\langle .; . \rangle$ denotes the scalar product in $\mathbb{R}^d$ and $\|.\|$ the Euclidean norm. We ask the same question as before, namely, when can we extend u from D to $\mathbb{R}^d$ so as to preserve inequality (2). This question will be dealt with in Section 3.

1.2 A motivation for studying extension maps

We now motivate these two questions by considerations of optimal design.

One of the basic problem in optimal design, which has received a lot of attention (see [2], [3], [35], [36]), is the study of the variational problem

$$\inf_{\sigma}\{I[\sigma] := \int_{\overline{\Omega}} \hat{\rho}(\sigma) \; : \; \sigma \in \Sigma_F(\Omega)\}. \tag{3}$$

Here, $\hat{\rho} : \mathbb{R}^{d\times d} \to [0, +\infty]$ is a prescribed function, homogeneous of degree 1, so that $\int_{\overline{\Omega}} \hat{\rho}(\sigma)$ is well defined even if σ is a measure whose support is in $\overline{\Omega} \subset \mathbb{R}^d$. Also, $\mathbf{F} = (F_1, \cdots, F_d)$ is a system of forces in Ω that is in equilibrium. This means that $F_1, \cdots, F_d$ are signed measures of null average and moments

$$\int_{\mathbb{R}^d} (x_j dF_i(\mathbf{x}) - x_i dF_j(\mathbf{x})) = 0, \quad i, j = 1, \cdots, d. \tag{4}$$

Furthermore Ω contains the support of the F_is. The unknown in (3) is a symmetric stress tensor σ such that $\sigma_{ij} = \sigma_{ji}$ are Radon measures supported on $\overline{\Omega}$. It represents a frame to be designed and satisfies the equation

$$-\mathrm{div}(\sigma) = \mathbf{F} \;\; in \;\; \overline{\Omega}, \tag{5}$$

which prevents overall motion of the structure. We have denoted by $\Sigma_F(\Omega)$ the set of σ satisfying these conditions.

In the cases we are interested in, one can introduce a more tractable problem, dual to (3) of the form:

$$\sup_u\{\int_{\Omega} \langle \mathbf{F}; u\rangle \; : \; u \in Lip^{\Psi}(\Omega)\}. \tag{6}$$

Here, $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ which can be explicitly written in term of $\hat{\rho}$ and $Lip^{\Psi}(\Omega)$ is the set of $u : \bar{\Omega} \to \mathbb{R}^d$ such that $\Psi(x - y, u(x) - u(y)) \leq 0$ for all $x, y \in \Omega$. In this study we keep our focus on the following two cases which have attracted a lot of attention.

- *Case 1.* We assume that

$$\hat{\rho}(\eta) = \sup_{\xi \in \mathbb{R}^{d\times d}} \{|\langle \eta; \xi\rangle| \; : \; \rho(\xi) \leq 1\}, \; \rho(\xi) = \sup_{b \in \mathbb{R}^d} \{|\langle \xi b; b\rangle| \; : \; |b| = 1\} \tag{7}$$

which is the Michell case [25], referred to as the *fictive materials* or *light structures* case (see also [7]). Let $\langle\cdot;\cdot\rangle$ and $\|\cdot\|$ be respectively the Euclidean scalar product and the associated Euclidean norm on $\mathbb{R}^d$. Then

$$\Psi(a,b) := |\langle a;b\rangle| - \|a\|^2$$

is such that the values in (3) and (6) agree.

- *Case 2.* Let $||\cdot||_E$ and $||\cdot||_F$ be two norms on $\mathbb{R}^d$ and define

$$\hat{\rho}(\xi) = \sup_{a\in\mathbb{R}^d}\{||\xi a||_F \ : \ ||a||_E \leq 1\}.$$

Then

$$\Psi(a,b) = ||b||_F - ||a||_E$$

is such that the values in (3) and (6) agree. This case has been intensively studied (see [5], [6] and [7] for additional references). When the dimension $d \geq 2$ and the set $\left\{b \in \mathbb{R}^d : ||b||_F = 1\right\}$ is strictly convex then Theorem 11 gives a necessary and sufficient condition for $Lip^{\Psi}(\Omega)$ and $Lip^{\Psi}(\mathbb{R}^d)$ to coincide.

We next assume that Ω is a connected set with Lipschitz boundary or Ω is the whole space. For the class of Ψ we have considered above, the duality relation holds

$$\sup_u\{\int_\Omega \langle \mathbf{F};u\rangle \ : u \in Lip^{\Psi}(\Omega)\} = \inf_\sigma\{\int_{\overline{\Omega}} \hat{\rho}(\sigma) \ : \ -div(\sigma) = \mathbf{F} \ in \ \overline{\Omega}\}. \quad (8)$$

The inclusion $Lip^{\Psi}(\mathbb{R}^d) \subset Lip^{\Psi}(\Omega)$ and the fact that $\mathbf{F}$ is supported by Ω yield that

$$\sup_u\{\int_\Omega \langle \mathbf{F};u\rangle \ : u \in Lip^{\Psi}(\Omega)\} \geq \sup_u\{\int_{\mathbb{R}^d} \langle \mathbf{F};u\rangle \ : u \in Lip^{\Psi}(\mathbb{R}^d)\}. \quad (9)$$

In case every map $u \in Lip^{\Psi}(\Omega)$ admits an extension $\widetilde{u} \in Lip^{\Psi}(\mathbb{R}^d)$, we write $Lip^{\Psi}(\Omega) = Lip^{\Psi}(\mathbb{R}^d)$ and observe that the two expressions in (9) coincide. This, together with the fact that (8) holds also for $\Omega = \mathbb{R}^d$ would yield that

$$\inf_\sigma\{\int_{\overline{\Omega}} \hat{\rho}(\sigma) \ : \ -div(\sigma) = \mathbf{F} \ in \ \overline{\Omega}\} = \inf_\sigma\{\int_{\mathbb{R}^d} \hat{\rho}(\sigma) \ : \ -div(\sigma) = \mathbf{F} \ in \ \mathbb{R}^d\}. \quad (10)$$

For example when $\Psi(a,b) = ||b||_F - ||a||_E$, where $||\cdot||_E$ and $||\cdot||_F$ are norms induced by a scalar product on $\mathbb{R}^d$ then $Lip^{\Psi}(\Omega) = Lip^{\Psi}(\mathbb{R}^d)$.

In case $\Psi(a,b) = |\langle a,b\rangle| - \|a\|^2$, we prove that for a class of $D \subset \mathbb{R}^d$, there are maps $u \in Lip^{\Psi}(D)$ which do not admit any extension $\widetilde{u} \in Lip^{\Psi}(\mathbb{R}^d)$. This class includes the convex sets $D \subset \mathbb{R}^d$ of non empty interior.

1.3 Notation

- If $a,b \in \mathbb{R}^d$ we denote by $\langle a;b\rangle$ the standard scalar product between a and b.

- We denote by $\mathbb{R}^{d\times d}$ the set of $d\times d$ matrices. If $\xi=(\xi_{ij})_{i,j=1}^d$ and $\eta=(\eta_{ij})_{i,j=1}^d$ then

$$\xi^T=(\xi_{ji})_{i,j=1}^d,\quad \langle\xi;\eta\rangle=\sum_{i,j=1}^d \xi_{ij}\eta_{ij},\quad \|\xi\|^2=\langle\xi;\xi\rangle$$

denote respectively, the transposed of ξ, the trace of $\xi\eta^T$ and the square norm of ξ. We denote by I_d the $d\times d$ identity matrix.

- $\mathcal{H}^l$ denotes the l–dimensional Hausdorff measure.

- If X is a metric space, $\mathcal{M}(X)$ denotes the set of Borel signed-measures on X. The set of (nonnegative) Borel measures on X is denoted by $\mathcal{M}^+(X)$.

- When $\Omega\subset\mathbb{R}^d$ and $\{F_i\}_{i=1}^d\subset\mathcal{M}(\Omega)$, we set $\mathbf{F}=(F_1,F_2,\cdots,F_d)$. The moments of the force $\mathbf{F}$ is the skew-symmetric matrix

$$\int_{\mathbb{R}^d}\mathbf{F}\wedge\mathbf{x}=\left(\int_{\mathbb{R}^d}x_j dF_i(x)-\int_{\mathbb{R}^d}x_i dF_j(x)\right)_{i,j=1}^d.$$

- If $u:\Omega\subset\mathbb{R}^d\to\mathbb{R}^d$ is Borel-measurable, set

$$\int_\Omega\langle\mathbf{F};u\rangle=\sum_{i=1}^d\int_\Omega u_i(x)dF_i(x),$$

and

$$||u||_\Omega^*=\sup_{x,y\in\Omega}\{(|\langle u(x)-u(y);x-y\rangle|)/\|x-y\|^2\ :\ x\neq y\}.$$

If in addition Ω is open and u is differentiable almost everywhere, we define $e(u)$ to be the symmetric part of the gradient of u :

$$e(u)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)_{i,j=1}^d.$$

- When $\xi\in\mathbb{R}^{d\times d}$, we define

$$\rho(\xi)=\sup_{b\in\mathbb{R}^d}\{|\langle\xi b;b\rangle|\,/\,\|b\|^2\ :\ b\neq 0\}. \tag{11}$$

It is easily checked that

$$\rho(\xi)=\max_{1\le i\le d}|\lambda_i(\frac{\xi+\xi^T}{2})|, \tag{12}$$

where $\lambda_1(\xi),\cdots,\lambda_d(\xi)$ are the eigenvalues of ξ. If we denote by ρ^o the polar conjugate of ρ, then a simple computation leads to

$$\rho^o(\hat{\xi})=\begin{cases}\sum_{i=1}^d|\lambda_i(\hat{\xi})| & \text{if }\ \hat{\xi}\in S^{d\times d}\\ \\ +\infty & \text{if }\ \hat{\xi}\notin S^{d\times d}\end{cases} \tag{13}$$

where $\mathcal{S}^{d\times d}$ is the subset of $\mathbb{R}^{d\times d}$ that consists of symmetric matrices.

Clearly, ρ is a convex, lower semicontinuous function, and is a semi-norm on the set $\mathbb{R}^{d\times d}$. Since ρ^o is the supremum of the linear functions $l_\xi : \eta \to \langle \xi; \eta \rangle$ over the set of ξ satisfying $\rho(\xi) \leq 1$, one concludes that ρ^o is also convex, homogeneous of degree one and lower semicontinuous.

- It is easily seen that if $||u||^*_\Omega \leq 1$, then the map $u + id$ (as well as $-u + id$) is monotone over Ω. Moreover if $\Omega \subset \mathbb{R}^d$ is open and connected then u is differentiable everywhere, except on a $(d-1)$-dimensional Hausdorff set (see [1]), and $\rho(e(u)) \leq 1$. The well-known Korn inequality also ensures that u is continuous and so, is locally bounded (see [29]).

- If $\Omega \subset \mathbb{R}^d$ is a convex set containing 0 in its interior, we define the *Minkowski function* (or the *gauge*) associated to Ω to be

$$\rho_\Omega(x) = \inf_{t>0}\{t \ : \ x/t \in \Omega\}.$$

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2 Extension of Lipschitz functions on Banach spaces revisited

Throughout this section $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are normed spaces. We denote by S^E the unit sphere in E, namely the set of $x \in E$ such that $\|x\|_E = 1$. The convex hull of S^E is the closed ball $\overline{B}^E$ of interior B^E.

Definition 1 *(i) We say that $u : E \to F$ is a* contraction on D *or u is* 1–Lipschitz on D *if*

$$\|u(x) - u(y)\|_F \leq \|x - y\|_E \quad \textit{for all } x, y \in D.$$

In this case, we write that $u \in Lip_1(D, F)$.

(ii) When $u \in Lip_1(E, F)$, we simply say that u is a contraction.

Definition 2 *We say that $[E; F]$ has the* extension property for contractions on D *if every $u \in Lip_1(D, F)$ has an extension $\widetilde{u} \in Lip_1(E, F)$. If $[E; F]$ has the extension property for contractions for every $D \subset E$, we simply say that $[E; F]$ has the* extension property for contractions.

In the present section we discuss some necessary and sufficient conditions on the spaces E and F, which in most of our analysis will be Banach spaces, ensuring that $[E; F]$ has the extension property for contractions.

The earliest result in this direction is the celebrated *Mac Shane lemma* [24] asserting that if $\dim F = 1$, then $[E; F]$ has the extension property for contractions for any E. It turns out that this is also true for any F, if $\dim E = 1$.

At the same time Kirszbraun [21] proved that if E and F are both finite dimensional spaces whose norms are induced by a scalar product, then $[E; F]$ has the extension property for contractions. This result, known as *Kirszbraun theorem*, has been proved, and at the same time extended to Hilbert spaces, in several different ways, notably by Valentine [38], [39], Grünbaum [18], Minty [27] and others; one could also consult textbooks such as Federer [16] or Schwartz [34].

When turning to necessary conditions, it was established by Schönbeck [31] that if $\dim E, \dim F \geq 2$ and if the unit sphere S^F of F is strictly convex (see below for a precise definition), then $[E; F]$ has the extension property for contractions if and only if both E and F are Hilbert spaces. It can also be shown that $[E; F]$ has the extension property for contractions if and only if for every set $D \subset D'$ of respective cardinality $3, 4$, every map $u \in Lip_1(D, F)$ admits an extension $\widetilde{u} \in Lip_1(D', F)$. When $E = F$, one can prove some stronger results, see Edelstein and Thompson [15], Schönbeck [32] and DeFigueiredo and Karlovitz [13], [14].

It is one of our goals to give a still different, and somehow more elementary and more self contained, proof of the result of Schönbeck (see Theorem 11). The approach used to obtain this result involves the smallest norm above $\|.\|_E$ which is induced by an inner product. This norm is precisely the Minkowski function $\rho_{\Sigma^E_{\max}}$ of the ellipse $\Sigma^E_{\max}$ of maximal volume, inscribed in S^E. Similarly, one also considers the largest norm below $\|.\|_E$ which is induced by an inner product. This norm turns out to be the Minkowski function $\rho_{\Sigma^E_{\min}}$ of the ellipse of minimal volume, circumscribed about S^E. One seeks for conditions under which $\rho_{\Sigma^E_{\max}} = \|.\|_E = \rho_{\Sigma^E_{\min}}$ and $\rho_{\Sigma^F_{\max}} = \|.\|_F = \rho_{\Sigma^F_{\min}}$.

2.1 Norms induced by an inner product

We start by collecting some well known facts about inner product spaces. One can consult, as a general reference, Amir [4]. Only Lemma 6 and Lemma 8 will be used in the proofs of the next sections, we have however incorporated some other results for the sake of giving a broader panorama.

Definition 3 *An ellipse centered at* 0 *in* $\mathbb{R}^d$ *is a set*

$$\Sigma^\alpha := \{x \in \mathbb{R}^d : \sum_{i=1}^{d} \alpha_i^2 x_i^2 = 1\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d) \in (0, +\infty)^d$. *We refer to the convex hull of* Σ^α *as the region enclosed by* Σ^α *and we denote it by* B^α.

The next lemma is due to Löwner in an apparently unpublished work.

Lemma 4 (Löwner) *Assume that $d \geq 2$ and that $E = \mathbb{R}^d$. Then there exist a unique ellipse $\Sigma_{\max}$ of maximal volume inscribed in S^E and a unique ellipse of minimal volume $\Sigma_{\min}$ circumscribed about S^E. Furthermore both $\Sigma_{\max} \cap S^E$ and $\Sigma_{\min} \cap S^E$ contain at least $2d$ distinct points.*

Proof. Existence of ellipses of maximal volume. If Σ^α is inscribed in B^E, then

$$\sum_{i=1}^{d} \alpha_i^2 x_i^2 \geq \|x\|_E^2 , \tag{14}$$

for all $x \in \mathbb{R}^d$. Assume that for some $\epsilon > 0$, we have

$$\epsilon \leq vol(B^\alpha) = \frac{\omega_d}{\prod \alpha_i}, \tag{15}$$

where ω_d is the volume of the unit Euclidean ball. The set of α such that $\alpha_i > 0$, and (14)–(15) hold is a compact subset $K_\epsilon \subset \mathbb{R}^d$. Every maximizing sequence of the set of ellipses inscribed in B^E, of maximal volume, has their accumulation points in K_ϵ for some small $\epsilon > 0$. This shows that there exists an ellipse $\Sigma_{\max}$ inscribed in S^E and of maximal volume. Similarly, one obtains an ellipse $\Sigma_{\min}$ circumscribed about S^E and of minimal volume.

Uniqueness of ellipses of maximal volume. Assume that Σ^a, Σ^c are two ellipses inscribed in S^E and of maximal volume. By an affine transformation, we may assume that $c = (1, \cdots, 1)$ so that the volume of these two regions are

$$\omega_d = vol(B^c) = vol(B^a) = vol(B^c) \prod_{i=1}^{d} 1/a_i.$$

We therefore deduce that $\prod_{i=1}^d a_i = 1$.

Let $\|.\|_E^o$ be the polar conjugate of $\|.\|_E$ defined by

$$\|z\|_E^o = \sup_x \{\langle x; z\rangle \ : \ \|x\|_E \leq 1\}.$$

Denote by ρ_{Σ^a} (respectively ρ_{Σ^c}) the Minkowski function associated to B^a (respectively B^c) and $\rho_{\Sigma^a}^o$ (respectively $\rho_{\Sigma^c}^o$) be its polar. Since Σ^a, Σ^c are inscribed in S^E we have that $\|.\|_E \leq \rho_{\Sigma^a}, \rho_{\Sigma^c}$ and so, $\rho_{\Sigma^a}^o, \rho_{\Sigma^c}^o \leq \|.\|_E^o$. Hence,

$$\begin{aligned}\rho^o(z)^2 &:= \sum_{i=1}^{d} \frac{1 + 1/a_i^2}{2} z_i^2 = \frac{1}{2}\left(\sum_{i=1}^{d} \frac{z_i^2}{a_i^2} + \sum_{i=1}^{d} z_i^2\right) \\ &= \frac{1}{2}(\rho_{\Sigma^a}^o(z)^2 + \rho_{\Sigma^c}^o(z)^2) \leq \left(\|z\|_E^o\right)^2 ,\end{aligned}$$

holds for all $z \in \mathbb{R}^d$. The previous inequality yields that $\rho^2 \geq \|.\|_E^2$, which means that

$$\sum_{i=1}^{d} \frac{2}{1 + 1/a_i^2} x_i^2 \geq \|x\|_E^2 . \tag{16}$$

Letting $b_i^2 = \frac{2}{1+1/a_i^2}$, $b = (b_1, ..., b_d)$, we find from (16) that Σ^b is inscribed in S^E.

We now show that Σ^a and Σ^c coincide and we proceed by contradiction assuming that they are distinct. Then, $a_i \neq 1$ for at least one $i = 1, \cdots d$. The volume of the region enclosed by Σ^b is

$$\begin{aligned} vol(B^b) &= \omega_d(\prod_{i=1}^{d} \frac{1+1/a_i^2}{2})^{\frac{1}{2}} \\ &= \omega_d(\prod_{i=1}^{d}[\frac{(1-1/a_i)^2}{2} + 1/a_i])^{\frac{1}{2}} > \omega_d(\prod_{i=1}^{d} 1/a_i)^{\frac{1}{2}} = \omega_d. \end{aligned}$$

This contradicts the maximality of the volume of Σ^c. Thus, $\Sigma^c = \Sigma^a$ and so, we have a unique ellipse of maximal volume in S^E. Replacing ρ_Σ and $\|.\|_E$ by their polar conjugates we conclude that $\Sigma_{\min}$ is unique.

<u>Intersection of the maximal ellipse with S^E.</u> As before, we assume that $\Sigma_{\max} = \Sigma^\alpha$ where $\alpha = (1, \cdots, 1)$. Since $\Sigma_{\max}$ and S^E are compact sets, they have a non empty intersection otherwise the maximality of $\Sigma_{\max}$ would be contradicted. By symmetry there are therefore at least 2 points in $S^E \cap \Sigma_{\max}$. Let us show that if we have $2s$ points in $S^E \cap \Sigma_{\max}$, $1 \leq s < d$, then in fact we have at least $2(s+1)$ points in the intersection, showing therefore the claim. Up to a rotation, we may assume that the points $\pm p^1, ..., \pm p^s \in S^E \cap \Sigma_{\max}$ lie in the subspace generated by the first s elements $\{\mathbf{e}_1, ..., \mathbf{e}_s\}$ of the standard basis, which means that for every $j = 1, ..., s$, we have

$$p_i^j = 0, \text{ for every } i \geq s+1.$$

For $\epsilon \in (0,1)$, define $\alpha_\epsilon = (\frac{1}{1-\epsilon}, ..., \frac{1}{1-\epsilon}, (1-\epsilon)^s, 1, \cdots, 1)$. Since $\Sigma_{\max}$ is unique and

$$vol(\Sigma^{\alpha_\epsilon}) = \omega_d = vol(\Sigma_{\max}),$$

we conclude that Σ^{α_ϵ} is not inscribed in S^E. Consequently, there exists $p^\epsilon = (p_1^\epsilon, ..., p_d^\epsilon) \notin \bar{B}^E$ which is in B^{α_ϵ}, the region enclosed by Σ^{α_ϵ}, and hence we have

$$\|p^\epsilon\|_E > 1 \tag{17}$$

and

$$1 \geq \rho^2_{\Sigma^{\alpha_\epsilon}}(p^\epsilon) = \rho^2_{\Sigma_{\max}}(p^\epsilon) + (\frac{1}{(1-\epsilon)^2} - 1)\sum_{i=1}^{s}(p_i^\epsilon)^2 - (1-(1-\epsilon)^{2s})\left(p_{s+1}^\epsilon\right)^2. \tag{18}$$

Because $p^\epsilon \notin \Sigma_{\max} \subset \bar{B}^E$, (18) implies that

$$(\frac{1}{(1-\epsilon)^2} - 1)\sum_{i=1}^{s}(p_i^\epsilon)^2 \leq (1-(1-\epsilon)^{2s})\left(p_{s+1}^\epsilon\right)^2.$$

Dividing both sides of the previous inequality by ϵ we get

$$\frac{2-\epsilon}{(1-\epsilon)^2}\sum_{i=1}^{s}(p_i^\epsilon)^2 \leq \frac{1-(1-\epsilon)^{2s}}{\epsilon}\left(p_{s+1}^\epsilon\right)^2. \tag{19}$$

Let $\{p^{\epsilon_n}\}_{n=1}^{\infty}$ be a subsequence of $\{p^\epsilon\}_{0<\epsilon<1}^{\infty}$ converging, as $\epsilon_n \to 0$, to some $p \in E$. We use (17)–(19) to obtain that

$$\rho_{\Sigma_{\max}}(p) \leq 1 \leq \|p\|_E\,, \quad \text{and} \quad \sum_{i=1}^{s} p_i^2 \leq s\, p_{s+1}^2. \tag{20}$$

The first two inequalities in (20) and the fact that $\rho_{\Sigma_{\max}} \geq \|.\|_E$ yield that $p \in S^E \cap \Sigma_{\max}$. The last inequality in (20) gives that $p \notin \operatorname{span}\{\mathbf{e}_1, ..., \mathbf{e}_s\}$ (in particular, $p \neq \pm p^1, ..., \pm p^s$) and thus by symmetry, $\pm p \in S^E \cap \Sigma_{\max}$. This proves that $S^E \cap \Sigma_{\max}$ has at least $2\,(s+1)$ distinct points, if $s < d$. Iterating the process we have indeed shown that $S^E \cap \Sigma_{\max}$ has at least $2d$ distinct points. Existence of at least $2d$ distinct points in $S^E \cap \Sigma_{\min}$ is obtained in a similar manner. ■

In [19] Jordan and von Neumann gave a condition which characterizes the norm induced by an inner product.

Lemma 5 (Jordan-von Neumann) *Assume that* $\dim E \geq 2$. *Then, the norm* $\|.\|_E$ *is induced by an inner product if and only if the* parallelogram rule *holds for all* $x, y \in E$, *namely*

$$2(\|x\|_E^2 + \|y\|_E^2) = \|x+y\|_E^2 + \|x-y\|_E^2\,. \tag{21}$$

Proof. The fact that every norm induced by an inner product satisfies (21) can be checked by direct computation. Conversely, if (21) holds, one defines

$$\langle x; y\rangle = \frac{\|x+y\|_E^2 - \|x-y\|_E^2}{4}$$

and check that, for every $x, y \in E$, we have

$$\langle x; y\rangle = \langle y; x\rangle\,, \quad \langle x; x\rangle = \|x\|_E^2\,, \quad \langle x; 0\rangle = 0, \quad \langle -x; y\rangle = -\langle x; y\rangle$$

Direct computations give that if $x, y, z \in E$ then

$$\langle x+y; z\rangle + \langle x-y; z\rangle = 2\,\langle x; z\rangle\,. \tag{22}$$

In particular, if we set $x = y$, $\bar{x} = x+y$ and $\bar{y} = x-y$ in (22), we obtain that

$$\langle 2x; z\rangle = 2\,\langle x; z\rangle\,, \quad \langle \bar{x}+\bar{y}; z\rangle = \langle \bar{x}; z\rangle + \langle \bar{y}; z\rangle\,.$$

By induction, if m is an integer, we get

$$\langle mx; z\rangle = m\,\langle x; z\rangle \ \text{ and } \left\langle \frac{x}{m}; z\right\rangle = \frac{1}{m}\,\langle x; z\rangle\,.$$

We conclude that
$$\left\langle \frac{m}{n}x;z\right\rangle = \frac{m}{n}\left\langle x;z\right\rangle$$
for all m, n integers. By continuity of $\|.\|_E$ we conclude that $\langle tx;z\rangle = t\langle x;z\rangle$ for all $t\in\mathbb{R}$. Thus, $\langle\cdot;\cdot\rangle$ is an inner product that induces $\|.\|_E$. ■

The following lemma, which is a corollary of Lemma 4, will be directly used in the proof of Theorem 11.

Lemma 6 *Assume that* $\dim E \geq 2$. *If* $\|.\|_E$ *is not induced by an inner product, then there exist* $x, y, X, Y \in S^E$ *so that*
$$\|x+y\|_E^2 + \|x-y\|_E^2 < 4 < \|X+Y\|_E^2 + \|X-Y\|_E^2.$$

Proof. As usual it is enough to establish the result for $E=\mathbb{R}^2$. Let us show the first inequality, the second one being obtained dually by replacing $\Sigma_{\max}$ by $\Sigma_{\min}$. Since $\Sigma_{\max}$ is inscribed in S^E, we have
$$\|z\|_E \leq \rho_{\Sigma_{\max}}(z) \text{ for every } z\in E.$$
It is also clear that we cannot have (see below)
$$\|x+y\|_E = \rho_{\Sigma_{\max}}(x+y) \tag{23}$$
for every $x, y\in\Sigma_{\max}\cap S^E$. Therefore choose $x,y\in\Sigma_{\max}\cap S^E$ such that
$$\|x+y\|_E < \rho_{\Sigma_{\max}}(x+y).$$
Since we always have $\|x-y\|_E \leq \rho_{\Sigma_{\max}}(x-y)$ and $\rho_{\Sigma_{\max}}$ satisfies the parallelogram rule, we have indeed established the claimed inequality.

We now show, by contradiction, that (23) does not hold. Up to an affine transformation, we may assume that $\Sigma_{\max}$ is the Euclidean disk:
$$\Sigma_{\max} = \{(x_1,x_2)|\ x_1^2+x_2^2=1\}.$$
By Lemma 4, $\Sigma_{\max}\cap S^E$ contains at least four distinct points $p_1^1, p_2^1, p_3^1, p_4^1$ (ordered in the clockwise direction, in particular $p_3^1=-p_1^1$ and $p_4^1=-p_2^1$) and we denote by $\mathcal{F}_1=\{p_1^1,p_2^1,p_3^1,p_4^1\}$. Note that $\rho_{\Sigma_{\max}}(p_{i+1}^1-p_i^1)\leq\pi$ (with the convention that $p_5^1=p_1^1$) for every point in $\mathcal{F}_1$.

We next use (23) for every $x,y\in\mathcal{F}_1$ to obtain a family $\mathcal{F}_2\subset\Sigma_{\max}\cap S^E$ of eight distinct points that contains $\mathcal{F}_1$. More precisely we set
$$p_1^2=p_1^1,\ p_3^2=p_2^1,\ p_5^2=p_3^1,\ p_7^2=p_4^1$$
$$p_2^2=\frac{p_1^1+p_2^1}{\rho_{\Sigma_{\max}}(p_1^1+p_2^1)},\ p_4^2=\frac{p_2^1+p_3^1}{\rho_{\Sigma_{\max}}(p_2^1+p_3^1)},$$
$$p_6^2=\frac{p_3^1+p_4^1}{\rho_{\Sigma_{\max}}(p_3^1+p_4^1)},\ p_8^2=\frac{p_4^1+p_1^1}{\rho_{\Sigma_{\max}}(p_4^1+p_1^1)}.$$

We clearly have that $\rho_{\Sigma_{\max}}(p^2_{i+1}-p^2_i) \leq \pi/2$ (with the convention that $p^2_9 = p^2_1$). We iterate this process to inductively obtain families

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \Sigma_{\max} \cap S^E$$

such that $\mathcal{F}_n = \{p^n_i\}_{i=1}^{2^{n+1}}$ has 2^{n+1} distinct points and $\rho_{\Sigma_{\max}}(p^n_{i+1}-p^n_i) \leq \pi/2^{n-1}$ (with the convention that $p^n_{2^{n+1}+1} = p^n_1$). This gives that $\cup_{n=1}^{\infty}\mathcal{F}_n$ is dense in $\Sigma_{\max}$ and in S^E. Consequently, $\Sigma_{\max} = S^E$ and thus $\|.\|_E$ is induced by an inner product, which is the desired contradiction. ■

We immediately obtain as a corollary the following result established by Day [9], which is a refinement of the lemma of Jordan-von Neumann.

Corollary 7 *Assume that* $\dim E \geq 2$. *Then the norm* $\|.\|_E$ *is induced by an inner product if and only if*

$$\|x+y\|_E^2 + \|x-y\|_E^2 = 4 \tag{24}$$

for all $x, y \in S^E$.

Proof. The fact that (24) is a necessary condition for $\|.\|_E$ to be induced by an inner product is a by-product of the parallelogram rule (21) proved in Lemma 5. Conversely we proceed by contradiction and assume that the norm $\|.\|_E$ is not induced by an inner product. By Lemma 6 we have that (24) does not hold and thus the claim. ■

We conclude with *Nordlander inequality* [30].

Lemma 8 (Nordlander) *Assume that* $\dim E \geq 2$ *and that* $0 < t < 1$. *Then*

$$\inf\{\|x+y\|_E : (x,y) \in S_t\} \leq 2\sqrt{1-t^2} \leq \sup\{\|x+y\|_E : (x,y) \in S_t\} \tag{25}$$

where

$$S_t = \left\{(x,y) \in S^E \times S^E : \|x-y\|_E = 2t\right\}.$$

Proof. Observe that it suffices to prove that (25) holds on a subspace of E of dimension 2. For that we may assume without loss of generality that $\dim E = 2$.

We first parametrize S^E in the counterclockwise direction by $s \to u(s) = (u_1(s), u_2(s))$. Since $\|.\|_E$ is Lipschitz, we obtain that u is Lipschitz too. For each s the circle of center $u(s)$ and radius $2t$ intersects S^E at two distinct points. Let $v(s) = (v_1(s), v_2(s))$ be the "nearest point in the counterclockwise direction". One can check that v is Lipschitz. By Green formula

$$\text{area}(B^E) = \int_{S^E} u_1 du_2 = \int_{S^E} v_1 dv_2. \tag{26}$$

Let E_t be the region enclosed by the curve $s \to (u(s)+v(s))/2$. The curve $C_t : s \to (u(s)-v(s))/2$ is a closed curve contained in tS^E. Hence, it coincides with tS^E and so, the region enclosed by C_t is tB^E. We use again Green formula and (26) to obtain that

$$\text{area}(tB^E) = \int_{S^E} \frac{(u_1 - v_1)}{2} d\frac{(u_2 - v_2)}{2} = \frac{1}{2}\text{area}(B^E) - \frac{1}{4}\int_{S^E}(v_1 du_2 + u_1 dv_2) \tag{27}$$

and

$$\text{area}(E_t) = \int_{S^E} \frac{(u_1 + v_1)}{2} d\frac{(u_2 + v_2)}{2} = \frac{1}{2}\text{area}(B^E) + \frac{1}{4}\int_{S^E}(v_1 du_2 + u_1 dv_2). \tag{28}$$

We add up both sides of the equalities in (27) and (28) to conclude that

$$\text{area}(E_t) = (1 - t^2)\text{area}(B^E).$$

This last identity implies that E_t neither strictly contains nor is strictly contained in the ball of radius $\sqrt{1-t^2}$ as asserted either in the left-hand side or in the right-hand side of (25). ■

2.2 Extension from a general subset of E to E

We start with a definition that will be used in the main theorem.

Definition 9 *The unit sphere S^F is said to be strictly convex if it has no flat part, meaning that*

$$\|(1-t)x + ty\|_F < (1-t)\|x\|_F + t\|y\|_F = 1$$

for all $t \in (0,1)$ and all $x, y \in S^F$ such that $x \neq y$.

Let us recall that for $1 \leq p \leq \infty$, the *Hölder norms* $|x|_p$ over $\mathbb{R}^d$ are defined as

$$|x|_p = \begin{cases} \left[\sum_{i=1}^d |x_i|^p\right]^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq i \leq d}\{|x_i|\} & \text{if } p = \infty. \end{cases}$$

When $d \geq 2$, the unit sphere for $|\cdot|_p$ is strictly convex if and only if $1 < p < \infty$.

We now can state our main theorems. First we start with the scalar case.

Theorem 10 *(i) Let $(E, \|.\|_E)$ be a normed space, then, $[E;\mathbb{R}]$ has the extension property for contractions.*

(ii) Let $(F, \|.\|_F)$ be a Banach space, then, $[\mathbb{R}; F]$ has the extension property for contractions.

We now give a theorem which characterizes the Banach spaces for which $[E, F]$ has the extension property for contractions.

Theorem 11 *Assume that* $(E, \|.\|_E)$ *and* $(F, \|.\|_F)$ *are Banach spaces such that* $dimE, dimF \geq 2$ *and that the unit sphere in* F *is strictly convex. Then, the three following properties are equivalent:*

(i) $\|.\|_E$ *and* $\|.\|_F$ *are induced by an inner product;*

(ii) $[E;F]$ *has the extension property for contractions;*

(iii) for every $\bar{x} \in E$ *and every* $S := \{x_1, x_2, x_3\}$*, every* $u \in Lip_1(S, F)$ *has an extension* $\widetilde{u} \in Lip_1(S \cup \{\bar{x}\}, F)$*.*

Remark 12 *(i) We should point out that if* S *consists of only two points* $x, y \in E$*,* $x \neq y$*, then the extension to any third point is always possible. Indeed assume that*

$$\|u(x) - u(y)\|_F \leq \|x - y\|_E \ .$$

Let then $z \in E$ *and define*

$$t = \min\left\{1, \frac{\|z-y\|_E}{\|x-y\|_E}\right\} \text{ and } u(z) = tu(x) + (1-t)u(y) \,.$$

It is immediate to check that

$$\|u(x) - u(z)\|_F \leq \|x - z\|_E \text{ and } \|u(z) - u(y)\|_F \leq \|z - y\|_E$$

as wished.

(ii) Interestingly enough, if one drops the assumption that S^F *is strictly convex, the extension property for contractions may hold for* $[E;F]$ *even if none of the norm is induced by an inner product. Indeed, if* $F = \mathbb{R}^2$ *(or* $\mathbb{R}^d$*,* $d \geq 2$*) and* $\|.\|_F = |\cdot|_\infty$*, Mac Shane lemma (Theorem 10) applied to each component of a vector valued map ensures that* $[E;F]$ *has the extension property for contractions. It is then immediate that if* $F = \mathbb{R}^2$ *and that* $\|.\|_F = |\cdot|_1$ *then* $[E;F]$ *has the extension property for contractions for any normed space* E*. This follows from the simple observation that if*

$$R = 1/2\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

then $|Ry|_1 = |y|_\infty$ *for any* $y \in \mathbb{R}^2$*. This, together with the above argument for the* $|\cdot|_\infty$ *norm gives that* $[E;\mathbb{R}^2]$ *has the extension property for contractions for any normed space* E*.*

(iii) Proceeding by contradiction in the proof that iii) $\Longrightarrow$ *i), we will find* $S := \{x_1, x_2, x_3\}$*,* $\overline{x} \in (x_1, x_2)$ *and* $u \in Lip_1(S, F)$ *so that there is no extension* $\widetilde{u} \in Lip_1(S \cup \{\bar{x}\}, F)$*. A continuity argument can show that there is also no extension* $\widetilde{u} \in Lip_1(S \cup \{\bar{x}_\delta\}, F)$ *where for* $\delta > 0$ *small enough*

$$\bar{x}_\delta = \bar{x} + \delta(x_3 - \bar{x}) \,.$$

Observe that therefore $\bar{x}_\delta \in \operatorname{int}\operatorname{conv}\{x_1, x_2, x_3\}$*.*

In the proof of Theorem 11, we will need the following lemma.

Lemma 13 *Assume that* $\dim E, \dim F \geq 2$ *and that at least one of these norms is not induced by an inner product, then there exist* $y_1, y_2 \in F$ *and* $x_1, x_2 \in E$ *so that*

$$\|x_1\|_E = \|x_2\|_E = \|y_1\|_F = \|y_2\|_F = 1 \text{ and } \|y_1 \pm y_2\|_F < \|x_1 \pm x_2\|_E .$$

Proof. It is enough to prove the lemma when $\dim E = \dim F = 2$. We assume that $\|.\|_F$ is not induced by a scalar product; a similar argument holds if $\|.\|_E$ is not induced by a scalar product. By Lemma 6, we can therefore find $y_1, y_2 \in \mathbb{R}^2$ so that

$$\|y_1\|_F = \|y_2\|_F = 1 \text{ and } \|y_1 - y_2\|_F^2 + \|y_1 + y_2\|_F^2 < 4.$$

Let

$$s = \frac{1}{2} \|y_1 - y_2\|_F$$

and use the triangle inequality to see that $0 < s < 1$. We therefore have

$$\|y_1 + y_2\|_F < 2\sqrt{1 - s^2}.$$

We next choose $t \in (s, 1)$ so that

$$\|y_1 + y_2\|_F < 2\sqrt{1 - t^2} < 2\sqrt{1 - s^2}.$$

We then apply Nordlander inequality (25) to get that there exist $x_1, x_2 \in \mathbb{R}^2$ so that

$$\|x_1\|_E = \|x_2\|_E = 1 \text{ and } \|x_1 - x_2\|_E = 2t,\ \|x_1 + x_2\|_E \geq 2\sqrt{1 - t^2}.$$

Combining all these results we have indeed found $y_1, y_2 \in F$ and $x_1, x_2 \in E$ satisfying

$$\|y_1\|_F = \|y_2\|_F = \|x_1\|_E = \|x_2\|_E = 1,$$

$$\|y_1 - y_2\|_F = 2s < 2t = \|x_1 - x_2\|_E \text{ and } \|y_1 + y_2\|_F < 2\sqrt{1 - t^2} \leq \|x_1 + x_2\|_E ,$$

as claimed in the lemma. ∎

It is interesting to see how to construct elements satisfying the conclusions of Lemma 13 in the case of Hölder norms.

Example 14 *Assume that* $E = F = \mathbb{R}^2$, *that* $\|.\|_F = |\cdot|_q$ *and* $\|.\|_E = |\cdot|_p$, *where* $1 < p, q < \infty$. *Denote also by* p' *and* q' *the conjugate exponents of* p *and* q. *We set*

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1), \quad \mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{f}_2 = \mathbf{e}_1 - \mathbf{e}_2.$$

Case 1. If $q > p$, *we set* $x_1 = y_1 = \mathbf{e}_2$, $x_2 = y_2 = \mathbf{e}_1$ *and observe that*

$$|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p}.$$

Case 2. If $p > q$, we set $x_1 = 2^{-1/p}\mathbf{f}_1$, $x_2 = 2^{-1/p}\mathbf{f}_2$, $y_1 = 2^{-1/q}\mathbf{f}_1$, $y_2 = 2^{-1/q}\mathbf{f}_2$ and observe that

$$|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q'} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p'}.$$

Case 3. We assume here that $p = q$.

(i) If $q > p'$, we set $x_1 = 2^{-1/p}\mathbf{f}_1$ $x_2 = 2^{-1/p}\mathbf{f}_2$, $y_1 = \mathbf{e}_1$, $y_2 = \mathbf{e}_2$ and observe that

$$|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p'}.$$

(ii) If $q < p'$ we let $x_1 = \mathbf{e}_1,\ x_2 = \mathbf{e}_2$, $y_1 = 2^{-1/q}\mathbf{f}_1$, $y_2 = 2^{-1/q}\mathbf{f}_2$ to obtain that

$$|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q'} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p}.$$

We can now proceed with the proofs of the theorems stated above.

Proof. (Theorem 10). **(i)** In fact, the reader could notice that arguments used in the proof of this part of the theorem are still valid in metric spaces. The fact that $[E, \mathbb{R}]$ has the extension property for contractions is, as already discussed, Mac Shane lemma. We recall that if $D \subset E$ and $u \in Lip_1(D, \mathbb{R})$ then both of the functions below are extensions of u that belong to $Lip_1(E, \mathbb{R})$:

$$u^+(x) = \inf_{y \in D} \{u(y) + \|x - y\|_E\}, \quad u^-(x) = \sup_{y \in D} \{u(y) - \|x - y\|_E\}.$$

Furthermore, if $\widetilde{u} \in Lip_1(E, \mathbb{R})$ is another extension of u then $u^- \leq \widetilde{u} \leq u^+$.

(ii) We now check that $[\mathbb{R}, F]$ has the extension property for contractions. So we assume that we have $D \subset \mathbb{R}$ and $u : D \to F$ satisfying

$$\|u(x) - u(y)\|_F \leq |x - y| \quad \text{for all } x, y \in D.$$

We wish to show that we can find $\widetilde{u} : \mathbb{R} \to F$, an extension of u, satisfying

$$\|\widetilde{u}(x) - \widetilde{u}(y)\|_F \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

We proceed into two steps.

Step 1. If D is not closed, we extend $\widetilde{u}$ to $\overline{D}$ by continuity. More precisely let $x \in \overline{D}$ and $x_n \in D$ converging to x. Observe that $\{u(x_n)\}$ is a Cauchy sequence, since

$$\|u(x_n) - u(x_m)\|_F \leq |x_n - x_m|.$$

It therefore converges to an element of F, independent of the choice of the sequence, denoted by $\widetilde{u}(x)$. With this definition we clearly deduce that

$$\|\widetilde{u}(x) - \widetilde{u}(y)\|_F \leq |x - y| \quad \text{for all } x, y \in \overline{D}.$$

Step 2. From now on we assume that D is closed. Let

$$\alpha = \inf\{x : x \in D\} \text{ and } \beta = \sup\{x : x \in D\}.$$

then

$$\operatorname{int}\operatorname{conv} D = (\alpha, \beta) .$$

For $x \in \mathbb{R}$, we define

$$x^{+} = \inf \{y : y \in D \text{ and } y \geq x\} \text{ and } x^{-} = \sup \{y : y \in D \text{ and } y \leq x\} .$$

Since D is closed, if $x \in \operatorname{int}\operatorname{conv} D$, we deduce that $x^{\pm} \in D$. Moreover if $x \in D$, we have that $x^{\pm} = x$; while if $x \in \operatorname{int}\operatorname{conv} D$ but $x \notin D$, we find $x^{-} < x < x^{+}$. If $\alpha < x < \beta$, then $-\infty < x^{-} \leq x \leq x^{+} < +\infty$ and therefore there exists a unique $t = t(x) \in [0, 1]$ such that

$$x = tx^{+} + (1 - t) x^{-} .$$

We are now in a position to define $\widetilde{u} : \mathbb{R} \to F$ through

$$\widetilde{u}(x) = \begin{cases} u(\alpha) & \text{if } x \leq \alpha \\ tu(x^{+}) + (1 - t) u(x^{-}) & \text{if } \alpha < x < \beta \\ u(\beta) & \text{if } x \geq \beta. \end{cases}$$

In the above definition it is understood that if $\alpha = -\infty$ (respectively $\beta = +\infty$), then the first (respectively the third) possibility does not happen. Furthermore, since when $x \in D$, we have that $x^{\pm} = x$, we deduce that $\widetilde{u}$ is indeed an extension of u. The fact that $\widetilde{u} \in Lip_1(\mathbb{R}, F)$ is easily checked. ■

We continue with the proof of Theorem 11.

Proof. (Theorem 11). **(i)** $\Longrightarrow$ **(ii)**. When E and F are finite dimensional spaces, the fact that (i) implies (ii) is Kirszbraun theorem. For the sake of completeness, we provide a proof based on arguments due to Minty [27]. In the light of Remark 24 and Proposition 26, it is sufficient to prove that $[E; F]$ has the extension property for contractions for finitely many points. Without any loss of generality, we may assume that both norms are the same, denoted by $\|.\|$. Clearly, since the norm is induced by an inner product, we deduce that E, F, $(a, b) \to \Psi(a, b) = ||b||^2 - ||a||^2$ satisfy the assumptions of Proposition 26. The remaining task is to check that condition (45) of Proposition 25 holds, namely

$$\sum_{i=1}^{k} \lambda_i ||y_i - \sum_{j=1}^{k} \lambda_j y_j||^2 - \sum_{i=1}^{k} \lambda_i ||x_i - x||^2 \leq 0, \text{ for every } \lambda \in \Lambda_k$$

where

$$\Lambda_k = \{(\lambda_1, \cdots, \lambda_k) \in [0, 1]^k \ : \ \sum_{i=1}^{k} \lambda_i = 1\}.$$

So, assume that $x, x_1, \cdots, x_k \in E$, $y_1, \cdots, y_k \in F$ and

$$||y_i - y_j|| \leq ||x_i - x_j||, \tag{29}$$

for all $i, j = 1, \cdots, k$. One easily checks, since the norm is induced by an inner product, the following identity

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j ||y_i - y_j||^2 = 2 \sum_{i=1}^{k} \lambda_i ||y_i - \sum_{j=1}^{k} \lambda_j y_j||^2, \tag{30}$$

for all $\lambda \in \Lambda_k$. Similarly, the inequality

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j ||x_i - x_j||^2 \leq 2 \sum_{i=1}^{k} \lambda_i ||x_i - x||^2, \tag{31}$$

holds for all $x \in E$ and all $\lambda \in \Lambda_k$. In fact, the right-hand side of (31) is minimized by the average value $\bar{x} = \sum_{i=1}^{k} \lambda_i x_i$. We combine (29), (30) and (31) to conclude that (45) holds.

(ii) $\Longrightarrow$ **(iii)**. This implication is obvious.

(iii) $\Longrightarrow$ **(i)**. We proceed by contradiction assuming that either $\|.\|_E$ or $\|.\|_F$ is not induced by an inner product. We will construct

$$u : S := \{x_1, x_2, x_3\} \subset E \to \{u(x_1) = y_1, u(x_2) = y_2, u(x_3) = y_3\} \subset F$$

so that $u \in Lip_1(S, F)$, but there is no extension $\widetilde{u} \in Lip_1(S \cup \{\bar{x} = 0\}, F)$.
We will proceed into two steps.

Step 1. From Lemma 13 there exist $y_1, \widetilde{y}_3 \in F$ and $x_1, x_3 \in E$ so that

$$\|y_1\|_F = \|\widetilde{y}_3\|_F = \|x_1\|_E = \|x_3\|_E = 1 \text{ and } \|y_1 \pm \widetilde{y}_3\|_F < \|x_1 \pm x_3\|_E .$$

We can therefore find $\epsilon > 0$ sufficiently small so that if

$$y_3 = (1 + \epsilon)\widetilde{y}_3$$

we still have

$$\|y_1 \pm y_3\|_F \leq \|x_1 \pm x_3\|_E .$$

Letting $y_2 = -y_1$ and $x_2 = -x_1$ we find that

$$\|y_1\|_F = \|y_2\|_F = 1, \; \|y_3\|_F = 1 + \epsilon, \quad \|x_1\|_E = \|x_2\|_E = \|x_3\|_E = 1,$$

$$\|y_1 - y_2\|_F = \|2y_1\|_F = 2 = \|2x_1\|_E = \|x_1 - x_2\|_E ,$$

$$\|y_1 - y_3\|_F \leq \|x_1 - x_3\|_E ,$$

$$\|y_2 - y_3\|_F = \|y_1 + y_3\|_F \leq \|x_1 + x_3\|_E = \|x_2 - x_3\|_E .$$

Hence $u \in Lip_1(S, F)$, meaning that

$$\|y_i - y_j\|_F \leq \|x_i - x_j\|_E , \; \forall i, j = 1, 2, 3. \tag{32}$$

Step 2. The claim that there is no extension $\widetilde{u} \in Lip_1(S \cup \{\bar{x} = 0\}, F)$, will follow if we can show that no $y \in F$ can verify

$$\|y - y_j\|_F \leq \|x_j\|_E = 1, \ \forall j = 1, 2, 3,$$

which is equivalent to showing that

$$\mathcal{A} = \{y \in F : \|y - y_j\|_F \leq 1, \ \forall j = 1, 2, 3\} = \emptyset.$$

To prove this we only need to show that

$$\mathcal{B} = \{y \in F : \|y - y_1\|_F, \|y - y_2\|_F = \|y + y_1\|_F \leq 1\} = \{0\},$$

and use that $\|y_3\|_F = 1 + \epsilon$ to obtain the claim. If $y \in \mathcal{B}$, we obtain

$$1 = \|y_1\|_F = \left\| \frac{1}{2}(y_1 - y) + \frac{1}{2}(y_1 + y) \right\|_F \leq \frac{1}{2} \|y_1 - y\|_F + \frac{1}{2} \|y_1 + y\|_F \leq 1$$

and consequently

$$\|y_1\|_F = \frac{1}{2} \|y_1 - y\|_F + \frac{1}{2} \|y_1 + y\|_F = 1.$$

Since $y \in \mathcal{B}$, we get that

$$\|y_1\|_F = \|y_1 - y\|_F = \|y_1 + y\|_F = 1.$$

Since the unit sphere S^F is strictly convex we obtain

$$y_1 - y = y_1 + y \Rightarrow y = 0$$

as wished. ∎

2.3 Extension from a convex subset of E to E

In many applications, such as in Browder-Petryshyn [8], Moreau [28], Lions-Stampacchia [23], Zabreiko-Kachurovsky-Krasnoselsky [40] –to cite few of them– it is important to know if for every closed convex set $\Omega \subset E$, every 1–Lipschitz map $u : \Omega \to F$ admits a 1–Lipschitz extension over E. These questions have been investigated by DeFigueiredo and Karlovitz in [12], [13] and [14] in the case $E = F$ and $\|.\|_E = \|.\|_F$. The general case which still remains open, is apparently closely related to whether or not projections on convex sets are contractions. In this section, we address the extension property for contractions for convex sets in simple cases where E is a Hilbert space.

Throughout this subsection, we assume that E is a reflexive Banach space, and that $\Omega \subset E$ is a closed convex set. We will specify it, when we need to impose that $\partial\Omega$, the boundary of Ω, is strictly convex. This means that $(1 - t)x + ty \in \mathrm{int}\Omega$ whenever $t \in (0, 1)$ and $x, y \in \partial\Omega$, $x \neq y$. Here, $\mathrm{int}\Omega$ denotes the interior of Ω.

Lemma 15 *(i) For each $x \in E$, there exists $z_\infty \in \Omega$ minimizing $z \to \|x - z\|_E$ over Ω. Moreover if $x \notin \mathrm{int}\Omega$, then $z_\infty \in \partial\Omega$.*

(ii) If in addition either S^E is strictly convex or $\partial\Omega$ is strictly convex, then z_∞ is uniquely determined. In that case, the map $x \to p_\Omega(x) := z_\infty$ is well-defined and is referred to it as the projection map onto Ω.

Proof. (i) Let $x \in E$ and let $\{z_n\}_{n=1}^\infty \subset \Omega$ be such

$$\lim_{n\to+\infty} \|x - z_n\|_E = \inf_{z\in\Omega} \|x - z\|_E \,. \tag{33}$$

The set $\{z_n\}_{n=1}^\infty$ being bounded, it is weakly precompact and so, has a subsequence that we still label $\{z_n\}_{n=1}^\infty$, converging weakly to some $z_\infty \in \Omega$. Since $\|.\|_E$ is convex, we conclude that $\|.\|_E$ is weakly lower semicontinuous and hence,

$$\|x - z_\infty\|_E \leq \lim_{n\to+\infty} \|x - z_n\|_E \,.$$

This, together with (33) yields that z_∞ is a minimizer of $\|x - z\|_E$ over Ω.

Let us show that if $x \notin \mathrm{int}\Omega$, then $z_\infty \in \partial\Omega$. By contradiction if $z_\infty \in \mathrm{int}\Omega$, we would have for $t \in (0,1)$ small enough that

$$z_t = (1-t)\, z_\infty + tx \in \Omega$$

and thus

$$\|x - z_t\|_E = (1-t)\, \|x - z_\infty\|_E < \|x - z_\infty\|_E$$

contradicting the definition of z_∞.

(ii) Let $x \notin \Omega$ and $z_\infty, \bar{z}_\infty \in \Omega$ be two minimizers of $\|x - z\|_E$ over Ω. Since, $z_\infty, \bar{z}_\infty \in \partial\Omega$, we find that $z_o := (z_\infty + \bar{z}_\infty)/2 \in \Omega$ is another minimizer of $\|x - z\|_E$. Assume for the sake of contradiction that $z_\infty \neq \bar{z}_\infty$. If $\partial\Omega$ is strictly convex then $z_o \notin \partial\Omega$, which yields a contradiction. On the other hand if S^E is strictly convex, we have from the fact that $r = \|x - z_\infty\|_E = \|x - \bar{z}_\infty\|_E > 0$, that $\|x - z_o\|_E < r$, which yields also a contradiction. This proves that the minimizer of $\|x - z\|_E$ over Ω is unique. ■

Lemma 16 *If $\|.\|_E$ is induced by an inner product $\langle \cdot ; \cdot \rangle$ then $p_\Omega : E \to E$ is a contraction.*

Proof. Since for every $t \in [0,1]$ and $z \in \Omega$, we have

$$\|x - p_\Omega(x)\|_E^2 \leq g\,(t) := \|x - [(1-t)\, p_\Omega(x) + tz]\|_E^2 \,,$$

we find, from the fact that $g'\,(0) \geq 0$, that $p_\Omega(x)$ should satisfy

$$\langle x - p_\Omega(x); z - p_\Omega(x)\rangle \leq 0, \text{ for every } z \in \Omega. \tag{34}$$

If $x_1, x_2 \in E$, we use (34), once with $z = p_\Omega(x_2)$ and once with $z = p_\Omega(x_1)$, to obtain that

$$\langle x_1 - p_\Omega(x_1); p_\Omega(x_2) - p_\Omega(x_1)\rangle \leq 0 \text{ and } \langle x_2 - p_\Omega(x_2); p_\Omega(x_1) - p_\Omega(x_2)\rangle \leq 0.$$

Adding up these two inequalities yields that

$$\|p_\Omega(x_1) - p_\Omega(x_2)\|_E^2 \leq \langle p_\Omega(x_1) - p_\Omega(x_2); x_1 - x_2 \rangle .$$

This, together with Schwarz inequality, leads to the claim, namely

$$\|p_\Omega(x_1) - p_\Omega(x_2)\|_E \leq \|x_1 - x_2\|_E .$$

■

Corollary 17 *Assume that E is a Hilbert space and F is a normed space. Then every contraction $u : \Omega \subset E \to F$ has an extension $\widetilde{u} : E \to F$ that is still a contraction.*

Proof. Every Hilbert space is reflexive. Furthermore, the parallelogram rule (21) gives that S^E is strictly convex. Hence by Lemma 16, p_Ω is a contraction. The map $\widetilde{u} := u \circ p_\Omega$ is a contraction as a composition of two contractions. ■

Remark 18 *We assume that E is merely a normed space and consider the radial map $x \to p_E(x) = x/\max\{1, \|x\|_E\}$.*

(i) In [12], under the assumption that $dimE \geq 3$, DeFigueiredo–Karlovitz, proved the following surprising result: $p_E \in Lip_1(E, E)$ if and only if $\|.\|_E$ is induced by an inner product.

(ii) As it is well-known, we verify next that p_E satisfies

$$\|x - p_E(x)\|_E \leq \|x - z\|_E , \text{ for every } z \in \bar{B}^E. \tag{35}$$

Since the result is trivial if $x \in \bar{B}^E$, we assume that $x \in E \setminus \bar{B}^E$. We then let $\rho = \|.\|_E$ and observe that it trivially is the Minkowski function of B^E. Let ρ^o be its polar; it is then an easy exercise to see that

$$p \in \partial\rho(x) \Longrightarrow \rho^o(p) \leq 1$$

where $\partial\rho(x)$ denotes the subgradient of ρ at x. So let $p \in \partial\rho(x)$ and $z \in \bar{B}^E$; we then have

$$\|x - z\|_E \geq \|x\|_E - \langle p; z\rangle \geq \|x\|_E - \rho^o(p)\|z\|_E \geq \|x\|_E - 1 = \|x - p_E(x)\|_E$$

as claimed in (35).

3 Lack of extension of maps of bounded strains

We start with the following definition.

Definition 19 *Assume that $\Omega \subset \mathbb{R}^d$ and let $\|.\|$ be the Euclidean norm.*

(i) We define $\mathcal{U}_1(\Omega)$ to be the set of $u : \Omega \to \mathbb{R}^d$ such that $||u||_\Omega^* \leq 1$ where

$$||u||_\Omega^* = \sup_{x,y\in\Omega} \{(|\langle u(x) - u(y); x - y\rangle|)/\|x - y\|^2 \ : \ x \neq y\}.$$

(ii) We say that Ω has the extension property for displacements of bounded strains, *if every $u \in \mathcal{U}_1(\Omega)$ admits an extension $\widetilde{u} \in \mathcal{U}_1(\mathbb{R}^d)$.*

Throughout this section, unless otherwise stated, we assume that d is an integer greater than or equal to 2. We discuss the following problem. Given $X \subset Y \subset \mathbb{R}^d$ and $u : X \to \mathbb{R}^d$ with $||u||^*_X = 1$, we investigate the possibility of extending u to Y in such a way that $||u||^*_Y = 1$. In Theorem 20 we show that if $d = 2$ and X, Y consist of two, respectively three points the extension preserving the norm of u is always possible. However, we can always choose a set X of three points, a set Y of four points and a map u for which there is no extension preserving the norm of u. Thus a similar phenomenon as that of Theorem 11 happens also here.

Exhibiting counterexamples of extensions becomes much more trickier when the interior of X in $\mathbb{R}^d$ is non-empty. For instance, we prove that any convex set of non empty interior does not have the extension property for displacements of bounded strains. It suffices to show this result in $\mathbb{R}^2$ and this is achieved in Theorem 22. Our proof does not exhibit an explicit counter example. It exploits the study of Michell trusses done in [17].

Throughout this section, we set $\mathbf{e}_1 = (1, 0)$, $\vec{0} = (0, 0)$, and $\mathbf{e}_2 = (0, 1)$.

Theorem 20 *(i) Assume that $X = \{a, b\} \subset \mathbb{R}^2$, $c \in \mathbb{R}^2$, and $u : X \to \mathbb{R}^2$ satisfies $||u||^*_X = 1$. Then, u admits an extension $\bar{u} : Y = \{a, b, c\} \to \mathbb{R}^2$ satisfying $||\bar{u}||^*_Y = 1$.*

(ii) Let $X = \{a, b, c\}$, where $a = -\mathbf{e}_1$, $b = \vec{0}$, and $c = \mathbf{e}_1$. Assume that $N \notin [-2, 4]$ and define $u : X \to \mathbb{R}^2$ by

$$u(a) = a + N\mathbf{e}_2, \quad u(b) = b - N\mathbf{e}_2, \quad u(c) = c - N\mathbf{e}_2.$$

Then $||u||^*_X = 1$ and $||\bar{u}||^*_Y \geq |N - 1|/3 > 1$ for every $\bar{u} : Y := X \cup \{\mathbf{e}_2\} \to \mathbb{R}^2$ which is an extension of u.

Proof. (i) It is not a loss of generality to assume that a, b and c are distinct. Translating and rotating the plane, we may assume that $b = 0$ and that $a = (a_1, 0)$. Since if necessary we could substitute u by $u - u(b)$, we may also assume without loss of generality that $u(b) = b = 0$. If $c = \lambda a$ for some $\lambda \neq 0$ we check that setting $\bar{u}(c) = \lambda u(a)$, we have that $||\bar{u}||^*_Y \leq 1$. Assume next that *span*$\{a, c\}$ is of dimension 2. We write $c = (c_1, c_2)$, so that $c_2 \neq 0$. We define

$$\bar{u}(c) = \beta(c_1, c_2) + \alpha(c_2, -c_1),$$

where $\beta \in [-1, 1]$ and

$$\alpha = \frac{\beta \|c\|^2 + \langle u(a); a - c\rangle - a_1\beta c_1}{a_1 c_2}.$$

Note that

$$|\langle \bar{u}(c); c\rangle| = |\beta| \|c\|^2 \leq \|c\|^2, \quad |\langle \bar{u}(a) - \bar{u}(c); a - c\rangle| = 0.$$

This proves (i).

(ii) Direct computations show that

$$|\langle u(x) - u(z); x - z\rangle| = \|x - z\|^2 ,$$

for all $x, z \in X$ and so, $||u||_X^* = 1$. For $y = (y_1, y_2) \in \mathbb{R}^2$ we set

$$\begin{aligned} f(y) &= \max\left\{ \frac{|\langle u(a) - y; a - \mathbf{e}_2\rangle|}{\|a - \mathbf{e}_2\|^2}, \frac{|\langle u(b) - y; b - \mathbf{e}_2\rangle|}{\|b - \mathbf{e}_2\|^2}, \frac{|\langle u(c) - y; c - \mathbf{e}_2\rangle|}{\|c - \mathbf{e}_2\|^2} \right\} \\ &= \max\left\{ \frac{|y_1 + y_2 + 1 - N|}{2}, |y_2 + N|, \frac{|-y_1 + y_2 + 1 + N|}{2} \right\}. \end{aligned}$$

Observe that

$$||\bar{u}||_Y^* = \max\{||u||_X^*,\ f(y)\} = \max\{1,\ f(y)\}. \tag{36}$$

The triangle inequality trivially leads

$$\begin{aligned} 3f(y) &\geq \frac{|y_1 + y_2 + 1 - N|}{2} + |y_2 + N| + \frac{|-y_1 + y_2 + 1 + N|}{2} \\ &\geq |\frac{y_1 + y_2 + 1 - N}{2} - (y_2 + N) + \frac{-y_1 + y_2 + 1 + N}{2}| \\ &= |N - 1|. \end{aligned}$$

This, together with (36) yields the proof of (ii). ■

Lemma 21 *Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded, convex set of non-empty interior and that $a \in (0, 1)$. Then there exist a rotation matrix R, a vector $z_o \in \mathbb{R}^2$ and a number $\epsilon > 0$ such that for $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Tx = \epsilon Rx + z_o$, we have*

$$\pm\mathbf{e}_1 \in T\Omega \subset \{(x_1, x_2)\ :\ x_2 < a\}.$$

Proof. Up to a translation, we can assume that $(0, 0) \in \Omega$. We then define the Minkowski function

$$\rho(x) = \inf_{t>0}\{t\ :\ x/t \in \Omega\}.$$

Because Ω is convex, not only its boundary is Lipschitz but, the set of points where $\partial\Omega$ is not twice differentiable in the sense of Alexandroff, is of measure zero with respect to the 1–dimensional Hausdorff measure. Let $x_o \in \partial\Omega$ be such a point, meaning that $\nabla\rho(x_o)$ is well defined and that there exists a symmetric nonnegative matrix A such that

$$\rho(x_o + h) = \rho(x_o) + \langle\nabla\rho(x_o); h\rangle + 1/2\,\langle Ah; h\rangle > +o(|h|^2). \tag{37}$$

Rotating and rescaling coordinates if necessary, we may also assume that

$$x_o = \mathbf{e}_2 \text{ and } \nabla\rho(x_o) = \lambda\mathbf{e}_2$$

for some $\lambda > 0$.

Claim 1. We claim that $x_\epsilon^\pm := (1-\epsilon^2)\mathbf{e}_2 \pm \epsilon^{3/2}\mathbf{e}_1 \in \Omega$ for $0 < \epsilon << 1$.
Indeed, using (37), the fact that $\rho(x_o) = 1$ and that $\nabla\rho(x_o) = \lambda\mathbf{e}_2$, we have that

$$\rho(x_\epsilon^\pm) = 1 - \lambda\epsilon^2 + o(\epsilon^2) < 1$$

for $0 < \epsilon << 1$, which proves the claim.

We next use that ρ is convex and again the fact that $\rho(x_o) = 1$ and that $\nabla\rho(x_o) = \lambda\mathbf{e}_2$ to obtain that if $x = (x_1, x_2)$ and $x_2 \geq 1$ then $\rho(x) \geq 1 + \lambda(x_2 - 1) \geq 1$. This proves that

$$\Omega \subset \{(x_1, x_2) \ : \ x_2 < 1\}.$$

and so, for $r > 0$ we have that

$$\Omega_r := \{x = (x_1, x_2) \ : \ \rho(x) < r\} \subset \{(x_1, x_2) \ : \ x_2 < r\}. \tag{38}$$

Claim 2. We claim that if $r > 1/a^3$ is large enough then $u^\pm = (r-a)\mathbf{e}_2 \pm \mathbf{e}_1 \in \Omega_r$.
Indeed, setting

$$\epsilon^2 = \frac{a}{r}, \quad t = 1 - \frac{1}{(ra^3)^{1/4}}, \qquad z = (1-\epsilon^2)\mathbf{e}_2$$

we have that $t \in (0,1)$, $z, x_\epsilon^\pm \in \Omega$ and so, the convexity of Ω yields that

$$u^\pm/r = x_\epsilon^\pm + t(z - x_\epsilon^\pm) \in \Omega.$$

Thus, $u^\pm \in \Omega_r$, which proves the claim.

Fix $r > 1/a^3$ large enough as before. One can readily check that we have proven that there is a transformation T, as in the statement, such that

$$T\Omega = \Omega_r - (r-a)\mathbf{e}_2.$$

By (38), $T\Omega \subset \{(x_1, x_2) \ : \ x_2 < a\}$ and by Claim 2, $\pm\mathbf{e}_1 \in T\Omega$. ∎

There is a large class $\mathcal{C}$ of open sets $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary on which there is a map $u : \bar{\Omega} \to \mathbb{R}^2$ such that $||u||_\Omega^* \leq 1$ and $||\widetilde{u}||_{\mathbb{R}^d}^* > 1$ for any extension $\widetilde{u}$ of u. The next theorem asserts that $\mathcal{C}$ contains the non empty convex bounded sets.

Theorem 22 *Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded, convex set of non-empty interior. Then Ω does not have the extension property for displacements of bounded strains.*

Proof. Observe that if T is as in Lemma 21, then Ω has the extension property for displacements of bounded strains if and only if $T(\Omega)$ has the extension property for displacements of bounded strains. Thus, by Lemma 21, we may assume without loss of generality that

$$\pm\mathbf{e}_1 \in \Omega \subset \{(x_1, x_2) \ : \ x_2 < 1/3\}. \tag{39}$$

We first introduce a system $\mathbf{F} = (\delta_{\mathbf{e}_1} + \delta_{-\mathbf{e}_1} - 2\delta_O)\mathbf{e}_2$ of equilibrated forces:

$$\int_{\mathbb{R}^2} dF_1(x) = \int_{\mathbb{R}^2} dF_2(x) = \int_{\mathbb{R}^2} x_2 dF_1(x) - \int_{\mathbb{R}^2} x_1 dF_2(x) = 0.$$

Assume, for the sake of contradiction, that Ω does have the extension property for displacements of bounded strains, i.e. $\mathcal{U}_1(\Omega) = \mathcal{U}_1(\mathbb{R}^2)$. From the fact that $\pm\mathbf{e}_1 = (\pm 1, 0) \in \Omega$ implies $(0,0) \in \Omega$, we can conclude that $\mathbf{F}$ is supported in Ω. Hence, by absurd hypothesis,

$$\sup_{u \in \mathcal{U}_1(\Omega)} \int_{\Omega} \langle \mathbf{F}; u \rangle = \sup_{u \in \mathcal{U}_1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \langle \mathbf{F}; u \rangle . \tag{40}$$

As in [17], we conclude that for every open convex set $O \subset \mathbb{R}^2$, containing the support of $\mathbf{F}$, we have that

$$\sup_{u \in \mathcal{U}_1(O)} \int_{O} \langle \mathbf{F}; u \rangle = \inf_{\sigma \in \Sigma_F(O)} \int_{\overline{O}} \rho^o[\sigma], \tag{41}$$

where $\Sigma_F(O)$ is the set of matrices $\sigma = (\sigma_{ij})$ such that $\sigma_{ij} = \sigma_{ji}$ is a measure supported by $\overline{O}$, $\rho^o : \mathbb{R}^{2\times 2} \to [0, +\infty]$ is the convex function defined in (13) and

$$\int_{\mathbb{R}^2} \langle \sigma; \nabla u \rangle = \int_{\mathbb{R}^2} \langle \mathbf{F}; u \rangle$$

for all $u \in C^1(\mathbb{R}^2; \mathbb{R}^2)$.

Since (41) holds for $O = \Omega$ and $O = \mathbb{R}^2$, we use (40) to conclude that

$$\inf_{\sigma \in \Sigma_F(\Omega)} \int_{\overline{\Omega}} \rho^o[\sigma] = \sup_{u \in \mathcal{U}_1(\Omega)} \int_{\Omega} \langle \mathbf{F}; u \rangle = \sup_{u \in \mathcal{U}_1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \langle \mathbf{F}; u \rangle = \inf_{\sigma \in \Sigma_F(\mathbb{R}^2)} \int_{\mathbb{R}^2} \rho^o[\sigma]. \tag{42}$$

Let then σ_Ω be a minimizer, which exists cf. [17], of $\int_{\overline{\Omega}} \rho^o[\sigma]$ over $\Sigma_F(\Omega)$. Since $\Sigma_F(\Omega) \subset \Sigma_F(\mathbb{R}^2)$, we deduce, from (42), that σ_Ω is also a minimizer of $\int_{\mathbb{R}^2} \rho^o[\sigma]$ over $\Sigma_F(\mathbb{R}^2)$. But, by [17] Theorem 5.2, σ_Ω is uniquely determined and satisfies

$$\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \leq 1, x_2 \geq 1/2\} \subset \mathrm{spt}\sigma_\Omega \subset \Omega,$$

which is at variance with (39). ■

4 Appendix

4.1 Ingredients for extension property from finite to infinite sets

In this subsection, we assume that $(E, \|.\|_E)$ is a Banach space such that every closed set $D \subset E$ contains a countable set $D_c \subset D$ whose closure is D. For instance, every Banach space that is union of compact sets satisfies this property. We assume that $(F, \|.\|_F)$ is a Banach space.

Definition 23 *Let E and F be Banach spaces and assume that $\Psi : E \times F \to \mathbb{R}$ is continuous.*

*(i) We say that $[E; F]$ has the Ψ–*extension property for finite sets *if for any finite set $D \subset E$, any $\bar{x} \in E \setminus D$ and any map $u : D \to F$ satisfying*

$$\Psi(x - y, u(x) - u(y)) \leq 0, \tag{43}$$

for all $x, y \in D$, there exists an extension $\bar{u} : D \cup \{\bar{x}\} \to F$ of u, such that

$$\Psi(x - y, \bar{u}(x) - \bar{u}(y)) \leq 0, \tag{44}$$

for all $x, y \in D \cup \{\bar{x}\}$.

*(ii) We simply say that $[E; F]$ has the Ψ–*extension property *if for every $D \subset E$ and any map $u : D \to F$ satisfying (43), there exists an extension $\bar{u} : E \to F$ of u satisfying (44) for all $x, y \in E$.*

Remark 24 *Observe that $[E; F]$ has the extension property for contractions for finite sets if and only if $[E; F]$ has the Ψ–extension property for finite sets, for*

$$\Psi(a, b) = \|b\|_F - \|a\|_E \ .$$

Many extension theorems of Lipschitz maps can be derived from a principle we state in Proposition 25. It states a sufficient condition for extending Lipschitz maps from sets of cardinality k into sets of cardinality $k + 1$. For completeness of the manuscript, we incorporate a proof due to Minty [27]. The following notation is needed later. When k is an integer, we define the convex set

$$\Lambda_k = \{(\lambda_1, \cdots, \lambda_k) \in [0, 1]^k \ : \ \sum_{i=1}^{k} \lambda_i = 1\}.$$

We denote the elements of Λ_k by $\lambda = (\lambda_1, \cdots, \lambda_k)$. We next need the function $F : \Lambda_k \times \Lambda_k \to \mathbb{R}$ defined by

$$F(\lambda, \mu) = \sum_{i=1}^{k} \lambda_i \Psi \left(x_i - x, y_i - \sum_{j=1}^{k} \mu_j y_j \right),$$

where $x, x_1, \cdots, x_k \in E$, $y_1, \cdots, y_k \in F$ are kept fixed.

Proposition 25 *Assume that $\Psi : E \times F \to \mathbb{R}$ is continuous and that $b \to \Psi(a, b)$ is convex for every $a \in E$. Assume that we are given $k + 1$ points $x, x_1, \cdots, x_k \in E$ and k points $y_1, \cdots, y_k \in F$ such that*

$$F(\lambda, \lambda) \leq 0, \tag{45}$$

for all $\lambda \in \Lambda_k$. Then there exists $y \in conv\{y_1, \cdots, y_k\}$ such that

$$\Psi(x_i - x, y_i - y) \leq 0,$$

for all $i = 1, \cdots, k$.

Proof. Clearly, $\lambda \to F(\lambda,\mu)$ is concave for all μ. Also, by assumption, $\Psi(a,\cdot)$ is convex and so, $\mu \to F(\lambda,\mu)$ is convex for every λ. Since Λ_k is a convex compact set, the minimax theorem holds (see [41] pp. 458) and there exists $(\bar\lambda,\bar\mu) \in \Lambda_k \times \Lambda_k$ such that

$$\min_{\mu\in\Lambda_k}\max_{\lambda\in\Lambda_k} F(\lambda,\mu) = F(\bar\lambda,\bar\mu) = \max_{\lambda\in\Lambda_k}\min_{\mu\in\Lambda_k} F(\lambda,\mu). \tag{46}$$

One can readily conclude from (46) that $(\bar\lambda,\bar\mu)$ is a saddle point in the sense that

$$F(\lambda,\bar\mu) \le F(\bar\lambda,\bar\mu) \le F(\bar\lambda,\mu), \tag{47}$$

for all $\lambda,\mu \in \Lambda_k$. Setting $\mu=\bar\lambda$ in (47) and using (45) we obtain that

$$F(\lambda,\bar\mu) \le F(\bar\lambda,\bar\mu) \le F(\bar\lambda,\bar\lambda) \le 0,$$

for all $\lambda \in \Lambda_k$. We set $y=\sum_{j=1}^k \bar\mu_j y_j$ and choose λ^i such that $\lambda^i_j = 0$ for $j \neq i$ and $\lambda^i_i = 1$. Note that $F(\lambda^i,\bar\mu) \le 0$ is equivalent to $\Psi(x_i - x, y_i - y) \le 0$. ■

We now use Zorn lemma to extend functions from finite to infinite sets.

Proposition 26 *Let E be a Banach space such that every closed set $D \subset E$ contains a countable subset whose closure is D. Let F be a reflexive Banach space. Let $\Psi : E\times F \to \mathbb{R}$ be continuous and such that*

$$b \to \Psi(a,b) \text{ is convex for every } a \in E$$

and

$$\lim_{\|b\|_F \to +\infty} \Psi(a,b) = +\infty \text{ uniformly for } a \text{ in a bounded set of } E. \tag{48}$$

Assume that $[E,F]$ has the Ψ-extension property for finite sets, that $D \subset E$ and that $u : D \to F$ satisfies

$$\Psi(x-y, u(x)-u(y)) \le 0, \tag{49}$$

for all $x,y \in D$. Then, u has an extension $\bar u : E \to F$ such that

$$\Psi(x-y, \bar u(x) - \bar u(y)) \le 0,$$

for all $x,y \in E$.

Remark 27 *Theorem 20 (ii) shows that if $E=F=\mathbb{R}^d$ and*

$$\Psi_1(a,b) = |\langle b;a\rangle| - \|a\|^2$$

then, $[E;F]$ does not have the Ψ_1–extension property for finite sets. In contrast, if we substitute Ψ_1 by

$$\Psi_2(a,b) = \langle b;a\rangle - \|a\|^2$$

then as shown in Proposition 28 below, $[E,F]$ has the Ψ_2–extension property for finite sets but fails to have the Ψ_2–extension property for general sets. This shows that it is essential to have condition (48) in Proposition 26.

Proof. (Proposition 26). Let $D_0 \subset E$ and any map $u_0 : D_0 \to F$ satisfying (49). We define

$$P = \{(D, u_D) : D_0 \subset D \subset E,\ u_D \text{ satisfies (49)}\}.$$

We next define a partial order on P, namely

$$(D_1, u_{D_1}) \leq (D_2, u_{D_2}) \ \Leftrightarrow\ D_1 \subset D_2 \text{ and } u_{D_2}|_{D_1} = u_{D_1}\,.$$

The lemma will follow from the three following claims.

Claim 1. We claim that $(P, \leq)$ admits a maximal element $(u_{\max}, D_{\max})$.

In view of Zorn lemma, it suffices to show that any totally ordered set $Q \subset P$ possesses a maximal element. Let then Q be a totally ordered set. Define

$$D_m = \cup_{(D,u_D)\in Q} D \quad \text{and} \quad u_m(x) = u_D(x)$$

for $x \in D$. Since Q is totally ordered, u_m is well defined and hence (D_m, u_m) is a maximal element.

Claim 2. We claim that $D_{\max}$ is closed.

Assume on the contrary that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset D_{\max}$, an element $\bar{x} \in E \setminus D_{\max}$ such that $x_n \to \bar{x}$. We set $y_n = u_{\max}(x_n)$. Using the coercivity assumption (48) on Ψ, we find that the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded. Since F is a reflexive Banach space, we may extract a weakly convergent subsequence still labeled $\{y_n\}$, that converges weakly to some $\bar{y} \in F$. We next set $D = D_{\max} \cup \{\bar{x}\}$ and define

$$u(x) = \begin{cases} u_{\max}(x) & \text{if } x \in D_{\max} \\ \\ \bar{y} & \text{if } x = \bar{x}\,. \end{cases}$$

In order to prove that $(u_{\max}, D_{\max}) \leq (u, D)$ which will contradict the fact that $(u_{\max}, D_{\max})$ is maximal, we fix $\epsilon > 0$ arbitrary. Since Ψ is continuous, we deduce that for any $x \in D_{\max}$

$$\Psi(\bar{x} - x, u_{\max}(x_n) - u_{\max}(x)) \leq \epsilon + \Psi(x_n - x, u_{\max}(x_n) - u_{\max}(x)) \leq \epsilon;$$

the second inequality being true since $(u_{\max}, D_{\max}) \in P$ and $x, x_n \in D_{\max}$. Note that $\Psi(\bar{x} - x, \cdot)$ is convex and so, it is weakly lower semicontinuous. Hence for n sufficiently large we have that

$$\Psi(\overline{x} - x, \overline{y} - u_{\max}(x)) \leq \epsilon + \Psi(\overline{x} - x, u_{\max}(x_n) - u_{\max}(x)).$$

Combining the last two inequalities with the fact that ϵ is arbitrary, we obtain that $(u, D) \in P$, $(u_{\max}, D_{\max}) \leq (u, D)$ and $(u_{\max}, D_{\max}) \neq (u, D)$. This contradicts the fact that $(u_{\max}, D_{\max})$ is maximal. Consequently, $D_{\max}$ is closed.

Claim 3. We claim that $D_{\max} = E$.

The proof of this claim is very similar to the proof of Claim 2. Assume on the contrary that $E \setminus D_{\max}$ is non empty and so, it contains an element z. Let $D_c = \{x_n\}_{n=1}^{\infty} \subset E \setminus \{z\}$ be a set whose closure is $D_{\max}$. Set

$$D_c^N = \{x_n\}_{n=1}^{N}, \qquad u_N = u_{\max}|_{D_c^N}\,.$$

Since $[E;F]$ has the Ψ–extension property for finite sets, we conclude that u_N admits an extension $\overline{u}_N : D_c^N \cup \{z\} \to F$ such that

$$\Psi(z - x, \bar{u}_N(z) - u_N(x)) \leq 0,$$

for all $x \in D_c^N$. By the coercivity assumption on Ψ, the sequence $\{\bar{u}_N(z) = e_N\}_{N=1}^{\infty}$ is bounded. As F is reflexive, we may extract from $\{e_N\}_{N=1}^{\infty}$ a weakly convergent subsequence, still labeled $\{e_N\}_{N=1}^{\infty}$. Let us denote by e the weak limit of that subsequence. Letting $D = D_{\max} \cup \{z\}$ and

$$v(x) = \begin{cases} u_{\max}(x) & \text{if } x \in D_{\max} \\ \\ e & \text{if } x = z, \end{cases}$$

as in the proof of the previous claim, we readily conclude that $(v, D_{\max} \cup \{z\}) \in P$. This, contradicts the fact that $(u_{\max}, D_{\max})$ is maximal. Hence, $D_{\max} = E$. ∎

4.2 Non extension property from finite to infinite sets

Proposition 28 *Assume that $E = F = \mathbb{R}^d$ and*

$$\Psi : (a, b) \to \langle b; a \rangle - \|a\|^2\,.$$

Then

(i) $[E;F]$ has the Ψ–extension property for finite sets;

(ii) there exist a set $D \subset \mathbb{R}^d$ and a function $u : D \to \mathbb{R}^d$ such that

$$\Psi(x - y, u(x) - u(y)) \leq 0 \textit{ for every } (x, y) \in D \times D,$$

and every extension of u to $\mathbb{R}^d$ fails to preserve that property.

Proof. (i) In order to prove (i), we verify that (45), Minty condition, holds. Assume that $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathbb{R}^d$ satisfy

$$\Psi(x_i - x_j, y_i - y_j) \leq 0 \iff \langle y_i - y_j; x_i - x_j \rangle \leq ||x_i - x_j||^2, \tag{50}$$

for all $i, j = 1, ..., k$. We are to show that for every $x \in \mathbb{R}^d$ and every $\lambda \in \Lambda_k$,

$$2\sum_{i=1}^{k} \lambda_i \Psi(x_i - x, y_i - \sum_{j=1}^{k} \lambda_j y_j) \leq \sum_{i,j=1}^{k} \lambda_i \lambda_j \Psi(x_i - x_j, y_i - y_j). \tag{51}$$

This together with (50) implies (45).

We first note that for any $x \in \mathbb{R}^d$ we have

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \langle y_i - y_j ; x_i - x_j \rangle = 2 \sum_{i=1}^{k} \lambda_i \left\langle y_i - \sum_{j=1}^{k} \lambda_j y_j ; x_i - x \right\rangle . \qquad (52)$$

In other words, the expression at the right-hand side of (52) is independent of x. We combine (31), namely

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j ||x_i - x_j||^2 \leq 2 \sum_{i=1}^{k} \lambda_i ||x_i - x||^2$$

with (50) and (52) to obtain (51).

(ii) Note that if $D \subset \mathbb{R}^d$ and $u, v : D \to \mathbb{R}^d$ are two functions such that $u = -v + id$ then

$$\Psi(x - y, u(x) - u(y)) \leq 0 \;\Leftrightarrow\; \langle v(x) - v(y); x - y \rangle \geq 0$$

To prove (ii), we can assume without loss of generality that $d = 1$, and choose $v : D = (0, +\infty) \to \mathbb{R}$ to be defined by

$$v(x) = \log x.$$

The map v is monotone but cannot be extended in a monotone way to $\bar{D} = [0, +\infty)$. This proves that $[E, F]$ does not have the Ψ–extension property. ■

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