ON THE WEAK LOWER SEMICONTINUITY OF ENERGIES WITH POLYCONVEX INTEGRANDS

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Abstract. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \to [0, \infty)$ be a Borel measurable function such that $f(x, u, \xi) = a(x, u) g(x, \xi)$ and $g(x, \cdot)$ is polyconvex in the last variable $\xi$ for almost every $x \in \Omega$. It is shown that if $f$ is continuous, if $a$ is bounded away from zero and if $F(u) := \int_\Omega a(x, u) g(x, \nabla u(x)) dx$, $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, then $F$ is weakly lower semicontinuous in $W^{1,p}$, $p > N - 1$, in the sense that $F(u) \leq \liminf_{\nu \to u} F(\nu)$ for $u$, $\nu \in W^{1,N}(\Omega, \mathbb{R}^N)$ such that $u \to \nu$ in $W^{1,p}$. On the contrary if $g$ is only a Carathéodory function then in general $F$ is not weakly lower semicontinuous in $W^{1,p}$ for $N > p > N - 1$. Precisely, it is shown that if $F(u) := \int_K \det(\nabla u(x)) dx$ where $K$ is a compact set, then $F$ is weakly lower semicontinuous in $W^{1,p}$, $N > p > N - 1$ if and only if $\text{meas}(\partial K) = 0$.

1. Introduction

Let $N \geq 2$ be an integer number, let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \to [0, \infty)$ be a Borel measurable function. We set

$$ F(u) := \int_\Omega f(x, u(x), \nabla u(x)) dx, \quad u \in W^{1,p}(\Omega, \mathbb{R}^N) := W^{1,p}. $$

If one uses the direct method of the calculus of variations to obtain existence of minima for $F$, one needs to show that $F$ is weakly lower semicontinuous in $W^{1,p}$. Since Morrey's works ([Mo1], [Mo2]) and later Acerbi-Fusco ([AF], Marcellini [Ma2]) and others, it is well known that if $1 \leq p < \infty$ and if

$$ 0 \leq f(x, u, \xi) \leq a + b |\xi|^p, \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times N} $$

then $F$ is weakly lower semicontinuous in $W^{1,p}$ if only if $f$ is quasiconvex with respect to the last variable $\xi$. We recall that $f$ is said to be quasiconvex if it verifies the following Jensen's inequality:

$$ \frac{1}{|\Omega|} \int_\Omega f(x_0, u_0, \xi + \nabla u(x)) dx \geq f(x_0, u_0, \xi) $$

for almost every $x_0 \in \Omega$, for every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$ and for every $\xi \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. As it is very hard to check whether or not a given function is quasiconvex,
We give some definitions relevant for this work.

**Definition 1.1.** Let $N, M \geq 1$ be two integer numbers and let $\Omega \subset \mathbb{R}^M$ be an open set. A function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times M} \to \mathbb{R}$ is said to be a Carathéodory function if $f(\cdot, u, \psi)$ is measurable for every $(u, \psi) \in \mathbb{R}^N \times \mathbb{R}^{N \times M}$ and $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.

**Definition 1.2.** (See [Da].) Let $f : \mathbb{R}^{N \times M} \to \mathbb{R}$ be a Borel measurable function defined on the set of the $N \times M$ real matrices.

- $f$ is said to be **convex** if $f(\lambda \xi + (1 - \lambda) \eta) \leq \lambda f(\xi) + (1 - \lambda) f(\eta)$ for every $\xi, \eta \in \mathbb{R}^{N \times M}$ and every $\lambda \in (0, 1)$.

- $f$ is said to be **polyconvex** if there exists a function $h : \mathbb{R}^r(N, \mathbb{M}) \to \mathbb{R}$ convex such that $f(\xi) = h(T(\xi))$ for every $\xi \in \mathbb{R}^{N \times M}$, where $T(N, \mathbb{M}) = \sum_{1 \leq s \leq \min(N, M)} M_s \mathbb{M}_s$.

Let $T(\xi) = (\text{adj}_1 \xi, \ldots, \text{adj}_{\min(N, M)} \xi)$ and $\text{adj}_s \xi$ stands for the matrix of all $s \times s$ minors of $\xi$.

When $N = M = 2$ then $T(\xi) = (\xi, \text{det}(\xi))$.

- $f$ is said to be **quasiconvex** if $\frac{1}{|\Omega|} \int_{\Omega} f(\xi + \nabla \phi) \geq f(\xi)$ for every $\xi \in \mathbb{R}^{N \times M}$, for every $\Omega \subset \mathbb{R}^N$ open bounded set and for every $\phi \in W^{1, \infty}(\Omega)^M$ (it is equivalent to assume that the previous inequality holds for any open bounded, $\Omega \subset \mathbb{R}^N$).

For completeness we state the following well known result.

**Proposition 1.3.** Let $N, M \geq 2$ be two integer numbers, let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{N \times M} \to \mathbb{R}$ be a continuous function such that $f(x, u, \cdot)$ is quasiconvex for each $(x, u) \in \Omega \times \mathbb{R}^M$. Furthermore assume that $f$ satisfies

$$-\alpha (|u|^p + |\xi|^p) - \gamma (x) \leq f(x, u, \xi) \leq \alpha (|u|^p + |\xi|^p) + \gamma (x),$$

where $\alpha > 0$, $\gamma \in L^1(\Omega)$, $1 \leq q < p < \infty$,

$$|f(x, u, \xi) - f(x, u, \eta)| \leq \beta (1 + |u|^p + |\xi|^p + |u|^{p-1} + |\xi|^{p-1} + |\eta|^{p-1})$$

$$\times (|u - \eta| + |\xi - \eta|)$$

where $\beta > 0$ and

$$|f(x, u, \xi) - f(y, u, \xi)| \leq \nu (|x - y|)(1 + |u|^p + |\xi|^p),$$

where $\nu$ is a continuous increasing function with $\nu(0) = 0$. Let

$$F(u) := \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx,$$

$u \in W^{1, p}(\Omega, \mathbb{R}^M)$.

Then $F$ is weakly lower semicontinuous in $W^{1, p}$.

**Proof.** For the proof we refer the reader to Theorem 2.4 in [Da].

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Lemma 1.4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $\psi \in C_0^\infty (\Omega, \mathbb{R}^N)$. If $p > N - 1$, if $u_\nu, u \in W^{1,N} (\Omega, \mathbb{R}^N)$ and if $u_\nu \rightharpoonup u$ in $W^{1,p}$ then
\[ \lim_{\nu \to \infty} \int_\Omega \phi'(u^\nu) \ldots \phi'(u^\nu_N)(\psi; T(\nabla u_\nu))dx = \int_\Omega \phi'(u^1) \ldots \phi'(u^N)(\psi; T(\nabla u))dx. \]
Moreover the results stands for $p = N - 1, N = 2$. Here $\langle ; , \rangle$ is the scalar product in $\mathbb{R}^r$ and $\tau = \sum_{1 \leq s \leq N} \binom{N}{s}^2$.\\
Proof. Lemma 1.4 is obtained as a slight modification of the proof of Lemma 1 in [DM].

2. The case of continuous integrands

Let us first state the main result of this section.

Theorem 2.1. Let $N \geq 2$ be an integer number, let $\gamma > 0$, let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $\tau = \sum_{1 \leq s \leq N} \binom{N}{s}^2$. Let $a : \Omega \times \mathbb{R}^N \to [0, \infty)$ and $g : \mathbb{R}^N \times \mathbb{R}^r \to [0, \infty)$ be two continuous functions such that $g(x, \cdot)$ is convex for each $x \in \Omega$. Let:
\[ F(u) := \int_\Omega a(x, u(x)) g(x, T(\nabla u(x))) dx, \quad u \in W^{1,p} (\Omega, \mathbb{R}^N). \]
Then,
\[ F(u) \leq \lim_{\nu \to \infty} \inf F(u_\nu), \]
if $u_\nu, u \in W^{1,N} (\Omega, \mathbb{R}^N)$ and $u_\nu \rightharpoonup u$ in $W^{1,p}$, $p > N - 1$. Moreover, if $N = 2$ the result is true even if $p = N - 1 = 1$.

We recall that $T(\nabla u)$ stands for the matrix of all minors of $\nabla u$.

Remark 2.2. If $p \geq N$ it is easy to prove Theorem 2.1 even in the general case where $F(u) := \int_\Omega f(x, u(x), T(\nabla u(x))) dx, f$ continuous, $f(x, u, T) \geq 0, f(x, u, \cdot)$ convex. Indeed, $h$ being a fixed real number, we truncate the sequence $u_\nu, u$ and get a sequence $u_\nu, v$ so that $|v_\nu(x)|, |v(x)| < h$ for almost every $x \in \Omega$. As $f$ is convex in the last variable $T$, using Lemma 2.3 we can approximate $f$ on $\Omega \times [-h, h]^N \times \mathbb{R}^r$ by a non decreasing sequence of smooth functions $f_j$ such that
\[ 0 \leq f_j(x, u, T) \leq C_j(x, u)(1 + |T|), \]
where $C_j(x, u) = 0$ for every $x \in \Omega$ such that $\text{dist}(x, \partial \Omega) < \frac{1}{l_j}$, for suitable $l_j \in \mathbb{N}$.

Then we apply Proposition 1.3 to $f_j(x, u, T)$ and to the sequence $v_\nu, v$. Letting $j$ go to infinity and then $h$ go to infinity, we obtain
\[ F(u) \leq \lim_{\nu \to \infty} \inf F(u_\nu). \]
2. – In the general case \( F(u) := \int_\Omega f(x, u(x), T(\nabla u(x))) \, dx \), (2.2) would be true if we knew that the sequence \( \{\det(\nabla u_\nu)\} \) is bounded in \( L^1 \). Indeed, as indicated above, by De Giorgi’s Lemma, we approximate \( f(x, u, T) \) from below by a function \( g(x, u, T) \) which is smooth and grows linearly in the variable \( T \). We can assume without loss of generality that there exist constants \( h > 0 \) such that \( |u_\nu(x)|, |u(x)| \leq h \) for almost every \( x \in \Omega \). Then, we fix a compact set \( K \) in \( \Omega \times [-h, h]^N \) and by Weierstrass’s Approximation Lemma we obtain:

\[
\begin{align*}
|g(x, u, T) - g_n(x, u, T)| \leq \varepsilon (1 + \max_{(y, v) \in K} g(y, v, T)) \\
\forall T \in \mathbb{R}^r, \quad \forall (x, u) \in K,
\end{align*}
\]

where \( g_n(x, u, T) \) has the form

\[
g_n(x, u, T) = \sum_{k=0}^n a_k^n(u_1) \ldots a_k^n(u_N) h_k^n(x, T).
\]

Then we can show with the relaxed assumptions

\[
a_k^1(u_1), \ldots, a_k^N(u_N), h_k^n(x, T) \geq 0
\]

that if \( F_n(u) := \int_\Omega g_n(x, u(x), T(\nabla u(x))) \, dx \), then,

\[
F_n(u) \leq \lim \inf_{n \to \infty} F_n(u_\nu),
\]

which together with (2.3) and the fact that \( \{\det(\nabla u_\nu)\} \) is bounded in \( L^1 \), yields

\[
F(u) \leq \lim \inf_{n \to \infty} F(u_\nu).
\]

However \( \{\det(\nabla u_\nu)\} \) is not necessarily bounded in \( L^1 \). [DM] provides an example where \( u_\nu \rightharpoonup u \) in \( W^{1,p} \), \( N > p > N - 1 \) and \( \{\det(\nabla u_\nu)\} \) is not bounded in \( L^1 \) (cf. also [BM]). For instance if \( N = 2, \quad \Omega = (0, 1)^2, \quad 1 < p \leq 2, \quad u_\nu \equiv \nu^{p-1} (1-y)^p (\sin \nu x, \cos \nu x) \rightharpoonup (0, 0) \) in \( W^{1,p} \).

Then \( \det(\nabla u_\nu) = -\nu^{p-1} (1-y)^p (\sin \nu x, \cos \nu x) \rightharpoonup (0, 0) \) in \( W^{1,p} \).

3. – The assumption that \( u_\nu \in W^{1,N} \) is important. It can be useful to extend the definition of \( F(u) \) to functions \( u \in W^{1,p} \), \( p < N \) (cf. [Mal]). Also Theorem 2.1 is false if one omits this assumption (cf. [BM]).

4. – If \( 1 \leq p < N - 1 \), and if \( N \geq 3 \), then \( F \) is not necessarily weakly lower semicontinuous (cf. [Mal]). But if \( p = N - 1, N \geq 3 \), the question to know whether or not \( F \) is weakly lower semicontinuous is still open. However Malý proved in [Mal] that if \( u, u_\nu \in W^{1,N-1} \) are sense preserving diffeomorphisms such that \( u_\nu \rightharpoonup u \) in \( W^{1,N-1} \), then \( F(u) \leq \lim \inf_{n \to \infty} F(u_\nu) \).

5. – The basic idea to prove Theorem 2.1 is the following: in the first step, we approximate \( f \) from below by a sum of functions of the form \( c(x) b^1(u_1) \ldots b^N(u_N) g(x, T(\nabla u)), \) with \( c(x), b^1(u_1), \ldots, b^N(u_N) \geq 0 \). This can be done using Weierstrass’s Approximation Theorem (see Lemma 2.5). In the second step, changing variables we write \( c(x) b^1(u_1) \ldots b^N(u_N) g(x, T(\nabla u)) \) in the form
$h = h(x, T(\nabla v))$. Then, following the idea of Dacorogna and Marcellini in their study of the integrands of the form $h = h(T(\nabla v))$ (see [DM]), we conclude the Theorem.

**Lemma 2.3.** (De Giorgi’s Lemma). – Let $N$, $\tau \geq 1$ be two integer numbers, let $\Omega \in \mathbb{R}^N$ be a open bounded set, and let $g : \Omega \times \mathbb{R}^\tau \to [0, \infty)$ be a continuous function such that $g(x, \cdot)$ is convex for each $x \in \Omega$. There exists a non decreasing sequence of functions $(g_i)_i$ of class $C^\infty(\Omega \times \mathbb{R}^\tau)$ such that:

i) $g_i \geq -1$;

ii) $(g_i)_i$ converges uniformly to $g$ in every compact subset of $\Omega \times \mathbb{R}^\tau$;

iii) $g_i(x, \cdot)$ is convex;

iv) $g_i(x, T) = 0$ if $\text{dist}(x, \partial \Omega) \leq \frac{1}{i}$;

v) On every compact subset $K$ of $\Omega$, $D_T g_i(x, T)$ is bounded in $K \times \mathbb{R}^\tau$ by a constant which depend only on $l$, $g$ and $K$, where $D_T g_i = \left( \frac{\partial}{\partial T_1} g_i, \ldots, \frac{\partial}{\partial T_\tau} g_i \right)$.

**Proof.** – For the proof we refer the reader to [Ma2].

**Remark 2.4.** – One can deduce from Lemma 2.3 that there exists a constant $C = C(l, g)$ such that $|D_T g_i(x, T)| \leq C$ for every $(x, T) \in \Omega \times \mathbb{R}^\tau$.

**Lemma 2.5.** – (Weierstrass’s Approximation Theorem)

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Then, for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $n \geq n_0(\varepsilon)$ implies

$$\left| f(u) - \sum_{0 \leq k \leq n} \binom{n}{k} f\left(\frac{k}{n}\right) u^k (1-u)^{n-k} \right| \leq \varepsilon (1 + \max_{0 \leq t \leq 1} |f(t)|),$$

for every $u \in [0, 1]$.

**Proof.** – For the proof we refer the reader to [Kl].

**Proof of Theorem 2.1.** – We give the proof of Theorem 2.1 only in the case where $N > p > N - 1$ since the case $p \geq N$ is easily obtained (see Remark 2.2). In the first step of the proof, we truncate the functions $(u_\nu)_\nu$ and $u$ to get a new sequence which is uniformly bounded in $L^\infty$. Then we write $f$ as a sum of functions of the form $c(x) b_1(u^1) \cdots b_N(u^N) g(x, T(\nabla u))$, where $c$ and $b_1, \ldots, b_N$ are smooth. In the second step we study the particular case where $f$ has the form $c(x) b_1(u^1) \cdots b_N(u^N) g(x, T(\nabla u))$. In the last step we study the general case where $f$ satisfies the hypotheses of Theorem 2.1. Clearly (2.2) is true if

$$\lim \inf_{\nu \to \infty} \int_{\Omega'} a(x, u_\nu) g(x, T(\nabla(u_\nu))) = \infty.$$

Assume that

$$M := \lim \inf_{\nu \to \infty} \int_{\Omega'} a(x, u_\nu) g(x, T(\nabla(u_\nu))) < +\infty.$$

Fix $h > 0$, $E = [-h, h]^N$, $l_0 \in \mathbb{N}$ and $\Omega' = \left\{ x \in \Omega, \text{dist}(x, \partial \Omega) > \frac{1}{l_0 + 1} \right\}$. 

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First step.

a) Factorization of \( a(x, u) \)

Using explicitly Weierstrass’s Approximation Theorem and the fact that \( a(x, u) \geq \gamma > 0 \), it is easy to deduce that there are two sequences \((b_k^n)_{k \leq n}\) and \((c_k^n)_{k \leq n}\) such that for every \( \varepsilon > 0 \), there is \( n(\varepsilon) \in \mathbb{N} \) depending only on \( \varepsilon, \Omega' \) and \( h \) verifying

\[
(2.4) \quad 0 \leq a(x, u) - \sum_{k=0}^{n} c_k^n(x) b_k^n(u) \leq \varepsilon \quad \forall (x, u) \in \Omega' \times E := K, \quad \forall n \geq n(\varepsilon),
\]

\[
b_k^n(x, u_1, \ldots, u_N) = b_k^{1,n}(u_1) \cdots b_k^{N,n}(u_N) \quad k = 1, \ldots, n, \quad n \geq 0.
\]

\[
b_k^n \in C^\infty(\mathbb{R}), \quad b_k^{j,n} \geq 0 \quad j = 1, \ldots, N, \quad k = 1, \ldots, n, \quad n \geq 1,
\]

\[
c_k^n \in C^\infty(\Omega'), \quad c_k^n \geq 0 \quad k = 1, \ldots, n, \quad n \geq 1,
\]

\[
c_0^0(0) = 1, \quad b_0^0(u) = -\varepsilon.
\]

b) Truncation of \( u \) and \( u_0 \).

Fix \( \delta(h) \ll 1 \). Truncate \( u \) and \( u_0 \) by considering \( \phi(u) \) and \( \phi(u_0) \) respectively where \( \phi \) is given by

\[
(2.5) \quad \phi(u) = \prod_{i=1}^{N} \psi(u^i), \quad \phi'(u) = \prod_{i=1}^{N} \psi'(u^i) \quad \text{with} \quad \psi'(t) = \frac{d\psi}{dt}(t),
\]

and \( \psi \in C^\infty(\mathbb{R}, \mathbb{R}) \) is defined in the following way

\[
\psi(t) = \begin{cases} 
- h & \text{if } t < -h - \delta(h), \\
\quad t & \text{if } |t| \leq h, \\
\quad h & \text{if } t > h + \delta(h),
\end{cases}
\]

\( 0 \leq \psi'(t) \leq 1 \) for every \( t \in \mathbb{R} \) and \( \psi'(t) = 0 \) if and only if \( |t| \geq h + \delta(h) \).

We apply Lemma 2.3 to \( g(x, T) \).

We obtain a sequence \((g_l)_l \) which has properties i), i), v) of Lemma 2.3. Recall that

\[
(2.6) \quad g(x, T) = \lim_{l \to \infty} g_l(x, T) \quad \forall (x, T) \in \Omega \times \mathbb{R}^r.
\]

Since \((g_l)_l \) is uniformly bounded below on \( \Omega \times \mathbb{R}^r \), we can assume without loss of generality that \( g_l \geq 0 \).

Second step. For \( l = l_0, k \geq 1 \) we show that

\[
(2.7) \quad \lim \inf_{l \to \infty} \int_{\Omega'} \phi'(u_0) c_k^n(x) b_k^n(u_0) g_l(x, T(\nabla u_0))
\]

\[
\geq \int_{\Omega'} \phi'(u) c_k^n(x) b_k^n(u) g_l(x, T(\nabla u)).
\]

- If \( \lim \inf_{l \to \infty} \int_{\Omega'} \phi'(u_0) c_k^n(x) b_k^n(u_0) g_l(x, T(\nabla u_0)) = \infty \) then (2.7) is trivial.

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Assume that \( \inf_{\nu \to \infty} \int_{\Omega^\nu} \phi'(u_\nu) c_k^\nu(x) b_k^\nu(u_\nu) g_l(x, T(\nabla u)) < \infty \). We may assume without loss of generality that
\[
u \in C^\infty(\Omega, \mathbb{R}^N).
\]
If this wasn't the case then it would suffice to replace \( u \) by \( u_\nu \in C^\infty(\Omega, \mathbb{R}^N) \) such that \( \| u_\nu - u \|_{W^{1,N}} \leq \epsilon \), following the proof with necessary modifications. Since
\[
gl(x, \cdot) \equiv 0 \quad \text{if} \quad \text{dist}(x, \partial \Omega) \leq \frac{1}{l},
\]
\[
| D_T g_l(x, T) | \leq C \equiv C(l, h) \quad \text{for every} \quad (x, T) \in \Omega \times \mathbb{R}^r,
\]
gl \( \in C^\infty(\Omega \times \mathbb{R}^r, [0, \infty)) \) and \( gl(x, \cdot) \) is convex,
\[
c_k^\nu \in C^\infty(\tilde{\Omega}^\nu),
\]
\[
b_k^\nu \in C^\infty(\mathbb{R}^N),
\]
and
\[
\phi' \in C^\infty(\mathbb{R}, \mathbb{R})
\]
we deduce that
\[
\liminf_{\nu \to \infty} \int_{\Omega^\nu} c_k^\nu(x) b_k^\nu(u_\nu) \phi'(u_\nu) g_l(x, T(\nabla u))
\geq \liminf_{\nu \to \infty} \int_{\Omega^\nu} c_k^\nu(x) b_k^\nu(u_\nu) \phi'(u_\nu) g_l(x, T(\nabla u))
\]
\[+ \int_{\Omega^\nu} c_k^\nu(x) b_k^\nu(u_\nu) \phi'(u_\nu)(D_T g_l(x, T(\nabla u)); T(\nabla u) - T(\nabla u))
\]\[\geq \int_{\Omega^\nu} c_k^\nu(x) b_k^\nu(u) \phi'(u) g_l(x, T(\nabla u))
\]
\[+ \int_{\Omega^\nu} c_k^\nu(x) b_k^\nu(u_\nu) \phi'(u_\nu)(D_T g_l(x, T(\nabla u)); T(\nabla u) - T(\nabla u)),
\]
where we used Fatou's Lemma and the fact that
\[
c_k^\nu(x) b_k^\nu(u_\nu) \phi'(u_\nu) \to c_k^\nu(x) b_k^\nu(u) \phi'(u) \quad \text{a.e.}
\]
For \( T \in \mathbb{R}^r \), we set \( T = (\bar{T}, t), t \in \mathbb{R} \). For fixed \( x \in \Omega \), let \( D_T g_l(x, \cdot) \) denote the matrix of the partial derivatives of \( g_l(x, \cdot) \) with respect to the \( \tau - 1 \) first variables in \( \mathbb{R}^r \). Let \( H \) be the functional defined on \( \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \) by
\[
H(x, u, \xi) = c_k^\nu(x) b_k^\nu(u) \phi'(u)(D_T g_l(x, T(\nabla u(x))); \bar{T}(\xi) - \bar{T}(\nabla u)).
\]
It is easy to see that \( H \) and \( -H \) are quasiconvex in the last variable. Using the fact that \( u \in C^\infty(\Omega, \mathbb{R}^N) \), (2.8) and the fact that \( |\phi'(u_\nu)| \leq 1 \), we get that \( H \) and \( -H \) verify the assumptions of Proposition 1.3. We deduce that
\[
\liminf_{\nu \to \infty} \int_{\Omega^\nu} c(x) b_k^\nu(u_\nu) \phi'(u_\nu)(D_T g_l(x, T(\nabla u)); \bar{T}(\nabla u) - \bar{T}(\nabla u)) = 0.
\]
On the other hand, setting
\[ u^i = B^i_k (\psi (u^i)) \]
where \( B^i_k (t) = \int_{-h - \delta (h)}^t b^i_k (s) \psi^{-1} (s) ds \), \( |t| \leq h + \delta (h) \),
then we obtain
\[ u^i \rightarrow u^i \ \text{in} \ W^{1,p}, \]
\[ b^i_k (u^i) \psi' (u^i) \rightarrow b^i_k (u) \psi' (u) \ \text{a.e.} \]
As
\[ \frac{\partial}{\partial t} g_i (x, T \nabla u) \in C_{0}^\infty (\Omega), \]
by Lemma 1.4 we obtain:
\[
\lim_{\nu \to \infty} \inf \int_{\Omega'} c^n_k (x) b^n_k (u_\nu) \phi' (u_\nu) \frac{\partial}{\partial t} g_i (x, T (\nabla u)) \left( \det (\nabla u_\nu) - \det (\nabla u) \right)
\]
\[ \leq \lim \inf_{\nu \to \infty} \left( \int_{\Omega'} c^n_k (x) \frac{\partial}{\partial t} g_i (x, T \nabla u) \left( \det (\nabla u_\nu) - \det (\nabla u) \right) \right)
\]
\[ - \int_{\Omega'} \left( b^n_k (u_\nu) \phi' (u_\nu) - b^n_k (u) \phi' (u) \right) \frac{\partial}{\partial t} g_i (x, T \nabla u) \det (\nabla u) = 0 \]
which together with (2.9), yields (2.7).

Third step. We conclude that
\[
\int_{\Omega} a (x, u (x)) g (x, T (\nabla u (x))) \, dx \leq \lim \inf_{\nu \to \infty} \int_{\Omega} a (x, u_\nu (x)) g (x, T (\nabla u_\nu (x))) \, dx.
\]
Since \( M := \lim \inf_{\nu \to \infty} \int_{\Omega} a (x, u_\nu (x)) g (x, T (\nabla u_\nu (x))) < \infty \) and \( a (x, u) \geq \gamma > 0 \), by steps 1 and 2 we obtain
\[
\lim \inf_{\nu \to \infty} \int_{\Omega} a (x, u_\nu (x)) g (x, T (\nabla u_\nu (x))) \, dx
\]
\[ \geq \lim \inf_{\nu \to \infty} \int_{\Omega'} \left( \phi' (u_\nu) \sum_{k=0}^{n (\epsilon)} c^n_k (x) b^n_k (u_\nu (x)) g_i_0 (x, T (\nabla u_\nu)) \right) \, dx
\]
\[ \geq \sum_{k=0}^{n (\epsilon)} \int_{\Omega'} \phi' (u) c^n_k (x) b^n_k (u) g_i_0 (x, T (\nabla u)) \, dx - \epsilon S,
\]
where \( S = \frac{M + 1}{\gamma} + 3 \text{ meas } (\Omega) + \int_{\Omega} g_i_0 (x, T (\nabla u)) \, dx \).

In the previous inequalities, we used the second step to prove that
\[
\lim \inf_{\nu \to \infty} \int_{\Omega'} \phi' (u_\nu) c^n_k (x) b^n_k (u_\nu) g_i (x, T (\nabla u_\nu)) \geq \int_{\Omega'} \phi' (u) c^n_k (x) b^n_k (u) g_i (x, T (\nabla u))
\]
for \( k \neq 0 \). For \( k = 0 \), we used the fact that \( a (x, u) \geq \gamma > 0 \), and \( M < \infty \). Letting \( \epsilon \) go to zero, \( i_0 \) go to infinity and then \( h \) go to infinity in the previous inequality we obtain (2.2). \( \blacksquare \)
3. The case of Carathéodory integrands

We state the main result of this section.

**Theorem 3.1.** Let \( N \geq 2 \) be an integer number, \( N - 1 < p < N \), let \( \Omega \subset \mathbb{R}^N \) be an open bounded set, and let \( K \subset \Omega \) be a compact set. The two following assertions are equivalent:

\[
(3.10) \quad \text{meas}(\partial K) \neq 0,
\]

\[
(3.11) \quad \liminf_{\nu \to \infty} \int_K |\det(\nabla u_\nu(x))| \, dx < \int_K |\det(\nabla u(x))| \, dx
\]

for a suitable \( u_\nu, u \in W^{1,N}(\Omega, \mathbb{R}^N) \) such that \( u_\nu \rightharpoonup u \) in \( W^{1,p} \).

Before proving Theorem 3.1 we begin with some remarks.

**Remark 3.2.** Let us recall that if \( F(u) = \int_K |\det(\nabla u(x))| \, dx \) and if \( K \) is a compact set then, for \( p \geq N \), \( F \) is weakly lower semicontinuous on \( W^{1,p} \) even if \( \text{meas}(\partial K) \neq 0 \) (see [AFI]). For \( p < N - 1 \) then \( F \) is not weakly lower semicontinuous on \( W^{1,p} \) even if \( \text{meas}(\partial K) = 0 \) (see [Mal]).

The following lemma will be used to prove that (3.10) implies (3.11).

**Lemma 3.3.** Let \( N, \tau \geq 2 \) be two integer numbers, let \( \Omega \subset \mathbb{R}^N \) be an open bounded set and let \( K \subset \Omega \) be a compact set such that \( \text{meas}(\partial K) > 0 \). Let \( p < N \) be a real number. Then there is a sequence \( u_k \in W^{1,N}(\Omega, \mathbb{R}^N) \) such that

(i) \( u_k \rightharpoonup u = \text{id} \) in \( W^{1,p}(\Omega, \mathbb{R}^N) \) with \( \text{id}(x) = x \),

(ii) \( |\det(\nabla u_k(x))| \leq 1 \) on \( K \),

(iii) \( \text{meas}\{x \in \partial K : \det(\nabla u_k(x)) \neq 0\} < \frac{1}{2^k} \).

**Proof.** We divide the proof into five steps. We assume without loss of generality that \( \Omega = (0, 1)^N \).

First step. We construct the sequence \( u_k \). Let \( k \in \mathbb{N} \) be fixed. Using Vitali's Covering Theorem we find two sequences \((x^k_i)\), \((\beta^k_i)\) such that

\[
\begin{aligned}
\partial K &\subset \tilde{N}_k \bigcup \left( \bigcup_{i=1}^{k} B(x^k_i, \beta^k_i) \right), \\
B(x^k_i, \beta^k_i) \cap B(x^k_j, \beta^k_j) &= \emptyset \quad \text{for} \quad i \neq j, \quad i, j = 1, \ldots, \infty, \\
\text{meas}(\tilde{N}_k) &\leq \frac{\text{meas}(\partial K)}{2^{k+1}}, \\
\text{meas} \left( \bigcup_{i=1}^{k} B(x^k_i, \beta^k_i) \setminus \partial K \right) &\leq \frac{\text{meas}(\partial K)}{2^{k+1}}, \\
B(x^k_i, \beta^k_i) &\subset \Omega \quad \text{for} \quad i = 1, \ldots, \infty,
\end{aligned}
\]
where \( B(x, \beta) \) stands for the open ball in \( \mathbb{R}^{N} \) with center \( x \) and radius \( \beta \) and \( \tilde{N}_k \) is an open set. Since \( K \) is a compact set we have

\[
\partial K \subset \tilde{N}_k \cup \left( \bigcup_{i=1}^{T(k)} B(x_i^k, \beta_i^k) \right),
\]

where \( T(k) \) is a constant depending on \( k \). Now we want to change the centers \( x_i^k \) by other centers which belong to the complementary of \( K \). Using (3.12), (3.13), (3.14) and the fact that \( x_i^k \in \partial K \), we deduce that there are an open set \( N_k \) and two sequences \( a_i^k \in B(x_i^k, \beta_i^k) \setminus K, \quad 0 < \epsilon_i^k < \beta_i^k \), such that

\[
\left\{ \begin{array}{l}
\partial K \subset N_k \cup \left( \bigcup_{i=1}^{T(k)} B(a_i^k, \epsilon_i^k) \right), \\
B(a_i^k, \epsilon_i^k) \subset B(x_i^k, \beta_i^k) \quad i = 1, \ldots, T(k),
\end{array} \right.
\]

\[
\text{meas}(N_k) \leq \frac{\text{meas}(\partial K)}{2^k},
\]

\[
\text{meas}(\bigcup_{i=1}^{T(k)} B(a_i^k, \epsilon_i^k) \setminus \partial K) \leq \frac{\text{meas}(\partial K)}{2^k}.
\]

Since \( \Omega \setminus K \) is an open set and \( a_i^k \in B(x_i^k, \beta_i^k) \setminus K \), there is \( \delta_i^k > 0 \) such that

\[
\delta_i^k < \left( \frac{1}{T(k)(2^k \cdot \epsilon_i^k)^p} \right)^{\frac{1}{p'}} \quad i = 1, \ldots, T(k)
\]

and

\[
B(a_i^k, \delta_i^k) \subset \Omega \setminus K \quad i = 1, \ldots, T(k).
\]

We define

\[
u_k(x) = \begin{cases}
a_i^k + \frac{\delta_i^k}{\epsilon_i^k} (x - a_i^k) & x \in B(a_i^k, \delta_i^k), \\
a_i^k + \frac{\delta_i^k}{|x - a_i^k|} (x - a_i^k) & x \in B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k), \\
x & x \in \Omega \setminus \left( \bigcup_{i=1}^{T(k)} B(a_i^k, \epsilon_i^k) \right).
\end{cases}
\]

It is easy to see that \( u_k \) is a diffeomorphism from \( B(a_i^k, \delta_i^k) \) into \( B(a_i^k, \epsilon_i^k) \) and \( u_k \) maps \( B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k) \) into \( \partial B(a_i^k, \epsilon_i^k) \).

**Second step.** In this step we show that \( u_k \in W^{1, \infty}(\Omega, \mathbb{R}^{N}) \). As

\[
u_k \in C^1(B(a_i^k, \delta_i^k), \mathbb{R}^{N}),
\]

\[
u_k \in C^1(B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k), \mathbb{R}^{N})
\]

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and $u_k$ is continuous on $B(a_i^k, \varepsilon_i^k)$, we have

$$u_k \in W^{1, \infty}(B(a_i^k, \varepsilon_i^k), \mathbb{R}^N)$$  \hspace{1cm} (3.20)

and since

$$u_k(x) = x \quad \text{on} \quad \partial B(a_i^k, \varepsilon_i^k)$$  \hspace{1cm} (3.21)

we conclude that

$$u_k \in C^0(\Omega, \mathbb{R}^N).$$  \hspace{1cm} (3.22)

Using the definition of $u_k$ on $\Omega \setminus \left( \bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right)$ it is obvious that

$$u_k \in W^{1, \infty}\left( \Omega \setminus \bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right),$$  \hspace{1cm} (3.23)

which together with (3.20) and (3.22) yields

$$u_k \in W^{1, \infty}(\Omega, \mathbb{R}^N).$$  \hspace{1cm} (3.24)

\textbf{Third step.} We show that, up to a subsequence, $u_k \rightharpoonup u = \text{id}$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. Using the definition of $u_k$, we obtain:

$$|u_k(x) - x| \leq \frac{1}{2k} \quad \text{for every} \quad x \in \Omega$$  \hspace{1cm} (3.25)

and

$$\nabla u_k(x) = \begin{cases} \frac{\varepsilon_i^k}{\delta_i^k} I_N & x \in B(a_i^k, \delta_i^k), \\ I_N & x \in \Omega \setminus \left( \bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right), \\ \end{cases} \hspace{1cm} (x - a_i^k) \otimes (x - a_i^k)$$

where $I_N$ is the identity matrix in $\mathbb{R}^{N \times N}$. If $a, b \in \mathbb{R}^N, a \otimes b$ denotes the $N \times N$ matrix with component $a_i b_j$ and $|a| = \sqrt{a_1^2 + \cdots + a_N^2}$. Clearly, there exists a constant $C = C(N)$ such that

$$|\nabla u_k(x)| \leq \begin{cases} C \frac{\varepsilon_i^k}{\delta_i^k} & x \in B(a_i^k, \delta_i^k), \\ C & x \in \Omega \setminus \left( \bigcup_{i=1}^{T(k)} B(a_i^k, \varepsilon_i^k) \right), \\ \end{cases} \hspace{1cm} (3.26)$$
Thus by (3.17) and (3.18) we have:

\[
\int_{\Omega} |\nabla u_k(x)|^p \, dx \leq C_p \left( 1 + \sum_{i=1}^{T(k)} \left( \int_{B(a_i^k, \epsilon_i^k)} \left( \frac{\epsilon_i^k}{|x - a_i^k|} \right)^p \, dx \right) + \int_{B(a_i^k, \delta_i^k)} \left( \frac{\epsilon_i^k}{\delta_i^k} \right)^p \, dx \right)
\]

\[
\leq w_N C_p \left( 1 + \sum_{i=1}^{T(k)} N \left( \frac{\epsilon_i^k}{N-p} + \frac{1}{2k} \right) \right),
\]

where \( w_N = \text{meas} B(0,1) \). Recalling that \( B(a_i^k, \epsilon_i^k) \) does not intersect \( B(a_j^k, \epsilon_j^k) \) for \( i \neq j \) and \( B(a_i^k, \epsilon_i^k) \subset \Omega = (0,1)^N \) we conclude that

\[
(3.26) \quad \int_{\Omega} |\nabla u_k(x)|^p \, dx \leq w_N C_p \left( 1 + \frac{N}{w_N(N-p) + \frac{1}{2k}} \right).
\]

Therefore \( (u_k)_k \) is bounded in \( W^{1,p} \) and by (3.25) we deduce that, up to a subsequence,

\[
u_k \rightharpoonup u = \text{id} \quad \text{in} \quad W^{1,p}(\Omega, \mathbb{R}^N).
\]

Fourth step. We show that \( |\det(\nabla u_k(x))| \leq 1 \) a.e. on \( K \). Indeed (\( \Sigma \)) implies that

\[
(3.27) \quad \det(\nabla u_k(x)) = 1 \quad \text{a.e.} \quad x \in \Omega \setminus \bigcup_{i=1}^{T(k)} B(a_i^k, \epsilon_i^k).
\]

We know that \( u_k \in C^1(\bar{B}(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k), \mathbb{R}^N) \) and

\[
|u_k(x) - a_i^k| = \epsilon_i^k \quad \forall x \in \bar{B}(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k).
\]

As \( u_k \) is the identity on \( \partial B(a_i^k, \epsilon_i^k) \) we obtain

\[
u_k(\bar{B}(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k)) = \partial B(a_i^k, \epsilon_i^k).
\]

Therefore \( u_k \) is not locally invertible at any point \( x \in B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k) \). We conclude that

\[
(3.28) \quad \det(\nabla u_k(x)) = 0 \quad \text{a.e.} \quad x \in B(a_i^k, \epsilon_i^k) \setminus B(a_i^k, \delta_i^k),
\]

which, together with (3.19) and (3.27) implies that

\[
(3.29) \quad 0 \leq \det(\nabla u_k(x)) \leq 1 \quad \text{a.e.} \quad x \in K.
\]

Fifth step. We claim that \( \text{meas} \{ x \in \partial K : \det(\nabla u_k(x)) \neq 0 \} \leq \frac{\text{meas}(\partial K)}{2^k} \).

By (3.15), (3.19), (3.27) and (3.28) we have

\[
(3.30) \quad \{ x \in \partial K : \det(\nabla u_k(x)) \neq 0 \} \subset N_k
\]

and the result follows now from (3.16). 

Proof of Theorem 3.1. – We prove that (3.10) implies (3.11). Assume that \( \text{meas}(\partial K) \neq 0 \). By Lemma 3.3 there exists a sequence \( u_k \in W^{1,N}(\Omega, \mathbb{R}^N) \) such that:

(i) \( u_k \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega, \mathbb{R}^N), \quad u(x) := x \),

(ii) \( |\det(\nabla u_k(x))| \leq 1 \quad \text{a.e.} \quad \text{on} \quad K \),

\]

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(3.32)  \[ \text{mes}(\{ x \in \partial K : \det(\nabla u_k(x)) \neq 0 \}) < \frac{1}{2^k}. \]

Then (3.31) and (3.32) imply that
\[
\int_K |\det(\nabla u_k(x))| \, dx = \int_{\partial K} |\det(\nabla u_k(x))| \, dx + \int_{K \setminus \partial K} |\det(\nabla u_k(x))| \, dx \leq \frac{\text{mes}(\partial K)}{2^k} + \text{mes}(K \setminus \partial K)
\]

and so
\[
\liminf_{k \to \infty} \int_K |\det(\nabla u_k(x))| \, dx \leq \text{mes}(K \setminus \partial K) < \text{mes}(K) = \int_K |\det(\nabla u(x))| \, dx
\]

and we conclude (3.11).

In order to prove that (3.11) implies (3.10), we assume that \( \text{mes}(\partial K) = 0 \). It is easy to construct a sequence \( a_n \in C^0(\Omega, \mathbb{R}^N) \) such that (see [Ga])
\[
(3.33) \quad a_n(x) \to 1_K(x) \text{ a.e. } x \in \Omega,
\]
\[
(3.34) \quad 0 \leq a_n(x) \leq a_{n+1}(x) \leq 1_K(x) \text{ a.e. } x \in \Omega.
\]

Let \( u_k, u \in W^{1,N}(\Omega, \mathbb{R}^N) \) be such that \( u_k \to u W^{1,p}(\Omega, \mathbb{R}^N) \). Setting in Theorem 2.1
\[
a(x, u) \equiv 1, \quad g(x, \bar{T}, \bar{t}) = a_n(x) |\bar{t}|,
\]
we obtain
\[
\int_{\Omega} a_n(x) |\det(\nabla u(x))| \, dx \leq \liminf_{k \to \infty} \int_{\Omega} a_n(x) |\det(\nabla u_k(x))| \, dx \leq \liminf_{k \to \infty} \int_K |\det(\nabla u_k(x))| \, dx,
\]
for each fixed \( n \). Using (3.33), (3.34) and Fatou’s Lemma we conclude that
\[
\int_K |\det(\nabla u(x))| \, dx \leq \liminf_{k \to \infty} \int_K |\det(\nabla u_k(x))| \, dx.
\]

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