Discrete Decomposition of Discrete Forces

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Abstract

We prove that any discrete equilibrated system of forces can be decomposed into finitely many bars. We provide an estimate on the cost and the norm of the endpoints of the bars of that decomposition. The questions we address here are of interest in elasticity theory, optimal designs, as well as in functional analysis.

1 Introduction

Let $\mathbf{F} = (F_1, \cdots, F_d)$ be a given a system of forces in $I\!R^d$ that is in equilibrium in the sense that $F_1, \cdots, F_d$ are signed measures of null average and the moments

$$\int_{I\!R^d} (x_j dF_i(x) - x_i dF_j(x)) = 0, \quad i, j = 1, \cdots, d. \tag{1.1}$$

Let us start with the following definitions. Assume that $\Omega \subset I\!R^d$ contains the support of $\mathbf{F}$ in its interior.

Definition 1.1 *Assume that $a_1, \cdots, a_d, f$ are Radon measures whose supports are compact and contained in $\bar\Omega$. Set $\mathbf{a} = (a_1, \cdots, a_d)$. We say that*

$$-\text{div}a = f \quad on \quad \bar\Omega$$

if

$$\int_{\bar\Omega} < \mathbf{a}; \nabla\varphi >= \int_{\bar\Omega} f\varphi$$

for every $\varphi \in C^1(\bar\Omega)$. Since a_i and f are of compact supports, there is no need to impose that $\nabla\varphi$ is of bounded support. Note that a Neumann type boundary condition has been incorporated in the definition.

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Definition 1.2 *Assume that* $\{F_i\}_{i=1}^d \subset \mathcal{M}(\Omega)$*. (i) We define* $\Sigma(\bar{\Omega})$ *to be the set of of matrices* $\sigma = \{\sigma_{ij}\}_{i,j=1}^d$ *such that* $\sigma_{ij} = \sigma_{ji} \in \mathcal{M}(\bar{\Omega})$*.*

(ii) If $\mathbf{F}$ *is an equilibrated system of forces in* Ω *and* $\sigma \in \Sigma(\bar{\Omega})$ *are of compact supports, we say that* $-\mathrm{div}\sigma = \mathbf{F}$ *on* $\bar{\Omega}$ *if for each* $i = 1, \cdots, d$ *we have that* $-\mathrm{div}\sigma_{i\cdot} = F_i$ *on* $\bar{\Omega}$*. Here,* $\sigma_{i\cdot}$ *stands for the* i*th row of* σ*. In that case, we say that* $\sigma \in \Sigma_{\mathbf{F}}(\bar{\Omega})$*.*

We want to design a frame in Ω that are static under the action of force $\mathbf{F}$. The frame is represented by a stress tensor $\sigma = \{\sigma_{ij}\}_{i,j=1}^d$ such that $\sigma_{ij} = \sigma_{ji}$ is a Radon measure supported on $\bar{\Omega}$. The equilibrium equation is the balance equation

$$-\mathrm{div}(\sigma) = \mathbf{F} \quad \text{in} \quad \bar{\Omega}, \tag{1.2}$$

(in the sense of definition 1.2) which prevents overall motion of the structure. Let $\Sigma_{\mathbf{F}}(\bar{\Omega})$ be the set of all such stresses σ that are symmetric and satisfy (1.2). It is first natural to wonder if $\Sigma_{\mathbf{F}}(\bar{\Omega})$ is nonempty, the point being to show that the restriction on σ to be symmetric still allows (1.2) to be solvable. In theorem 2.4 we shall prove that $\Sigma_{\mathbf{F}}(I\!R^d) \neq \emptyset$. Furthermore, there exists an element of $\Sigma_{\mathbf{F}}(I\!R^d)$ whose support in contained in a ball whose radius depends only on $\mathbf{F}$. Hence, if Ω is large enough, $\Sigma_{\mathbf{F}}(\bar{\Omega}) \neq \emptyset$.

The condition that $\mathbf{F}$ is in equilibrium in $I\!R^d$ means that first, the resultant of the forces is null:

$$\sum_{i=1}^k \mathbf{F}_i = \vec{0}, \tag{1.3}$$

and the first moments of $\mathbf{F}$ with respect to the origin is null (the net torque is null):

$$\sum_{i=1}^k \mathbf{F}_i \wedge M_i = \vec{0}. \tag{1.4}$$

Here if a and b are two vectors in $I\!R^d$ then $a \wedge b$ is the skew symmetric matrix $(a_i b_j - a_j b_i)_{i,j=1}^d$. Note that if A is an arbitrary point in $I\!R^d$ then the moment of $\mathbf{F}$ with respect to A is

$$\sum_{i=1}^k \mathbf{F}_i \wedge (M_i - A) = \sum_{i=1}^k \mathbf{F}_i \wedge M_i + \Big(\sum_{i=1}^k \mathbf{F}_i\Big) \wedge A,$$

and so, it is independent of A whenever the resultant of the forces $\sum_{i=1}^k \mathbf{F}_i = \vec{0}$.

An equilibrated system of forces consists of at least two forces and their points of applications. In case the system has two forces, they must be opposite to each other and we refer to it as *elementary equilibrated systems*. In case the *elementary equilibrated system* is of the form

$$\mathbf{F} = \pm(\delta_A - \delta_B)\frac{A - B}{|A - B|},$$

we call it a normalized beam.

The purpose of these notes is to find sets of points $\mathcal{A} = \{A_i\}_{i=1}^l \subset I\!R^d$ and sets of real numbers $\Lambda = \{\lambda_{ij}\}_{i,j=1}^l \subset I\!R$ such that $\mathbf{F}$ can be decomposed in

$$1/2 \sum_{i,j=1}^l \lambda_{ij} Beam(A_i, A_j) = \mathbf{F}. \tag{1.5}$$

Here,

$$Beam(A_i, A_j) = (\delta_{A_i} - \delta_{A_j}) \frac{A_i - A_j}{|A_i - A_j|_2}$$

represents the "unit beam" of endpoints A_i and A_j. That question which may be basic in a course of mechanics, is answered in theorem 2.4. There, our construction gives a constant C_d, independent of $\mathbf{F}$, such that $\mathcal{A}$ is contained in an $C_d \mathrm{diam}(I\!\!M)$–neighborhood of $I\!\!M := \{M_i\}_{i=1}^k$ and

$$\mathrm{Cost}_{\mathbf{F}}\Big(\mathcal{A}, \Lambda\Big) \leq C_d \mathrm{diam}(I\!\!M) \sum_{i=1}^k |\mathbf{F}_i|_2 \, (|M_i|_2 + 1) = C_d \mathrm{diam}(I\!\!M) \int_{I\!\!R^d} (1 + |x|_2) d|\mathbf{F}|. \tag{1.6}$$

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2 Decomposition of systems of forces in equilibruim

Assume that we are given an open set $X \subset I\!\!R^d$ which is smooth enough, and that $\mathbf{F}$ is an equilibrated system of forces in $I\!\!R^d$, whose support $I\!\!M$ is compact and contained in X.

In subsection 2.1, we prove that if $\mathbf{F}$ is an equilibrium system of forces in $I\!\!R^d$ and the cardinality of $I\!\!M$ is finite, then $\Gamma_{\mathbf{F}}(I\!\!R^d)$ contains an element γ_o which consists of a finite combination of "*bars*". Furthermore, we give an estimate on how far from $I\!\!M$, the endpoints of the bars in γ_o could be. The precise result in contained in theorem 2.4.

2.1 Decomposition into finitely many bars

In this subsection, we denote by $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d\}$ the standard orthonormal basis of $I\!\!R^d$ and make the identification

$$I\!\!R = span\{\mathbf{e}_1\}, I\!\!R^2 = span\{\mathbf{e}_1, \mathbf{e}_2\}, \cdots, I\!\!R^{d-1} = span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{d-1}\}.$$

We assume that

$$\mathbf{F} = \sum_{i=1}^k \mathbf{F}_i \delta_{\mathbf{M}_i} \tag{2.1}$$

is a system of forces in equilibrium in $I\!\!R^d$ and that the $M_i's$ are distinct. We denote by $I\!\!M$ the set which consists of the points of applications of $\mathbf{F}$: $I\!\!M = \{M_1, \cdots, M_k\}$. If

$$\mathbf{F} = \sum_{1<i<j\leq l} \lambda_{ij} \mathbf{t}_{A_i A_j} (\delta_{A_i} - \delta_{A_j}), \qquad \mathbf{t}_{A_i A_j} = \frac{A_j - A_i}{|A_j - A_i|}$$

we call $\sum_{1<i<j\leq l} \lambda_{ij} \mathbf{t}_{A_i A_j} \otimes \mathbf{t}_{A_i A_j} \mathcal{H}^1_{|[A_i, A_j]}$ a frame associated to $\mathbf{F}$ and define the volume of that frame to be

$$\sum_{1\leq i,j\leq l} |\lambda_{ij}| \ |A_i - A_j|.$$

Proposition 2.1 *Assume that* $\mathbf{F}$ *is given by (2.1) and that either*
(i) $M_1, \cdots, M_k, \mathbf{F}_1, \cdots, \mathbf{F}_k \in I\!R$
or
(ii) $M_1, \cdots, M_k, \in I\!R^{d-1}$ *and* $\mathbf{F}_i$ *is parallel to* $\mathbf{e}_d$.
Then $\mathbf{F}$ *is a linear combination of finitely many beams of controlled moments and points of applications: there exist* $k+1$ *points of application* $A_1, \cdots, A_{k+1} \in I\!R^d$, *and a symmetric matrix of real numbers* $\{\lambda_{ij}\}_{i,j=1}^{k+1}$ *such that*

$$\mathbf{F} = \sum_{1 < i < j \leq k+1} \lambda_{ij} \mathbf{t}_{A_i A_j} (\delta_{A_i} - \delta_{A_j}),$$

$$dist(A_i, I\!M) \leq diam(I\!M), \qquad \sum_{1 \leq i < j \leq k+1} |\lambda_{ij}| \leq 3 \sum_{i=1}^{k} |\mathbf{F}_i| \tag{2.2}$$

$$\sum_{1 \leq i < j \leq k+1} |\lambda_{ij}| \; |A_i - A_j| \leq diam(I\!M) \sum_{i=1}^{k} |\mathbf{F}_i| \; (2|M_i| + 1). \tag{2.3}$$

Proof: Up to a rotation and translation we may assume without loss of generality that one of the M_i say, M_k, is at the origin. In order to preserve the assumptions of this theorem, we further assume that these rotation and translation are defined from $I\!R^{d-1}$ onto $I\!R^{d-1}$.

1. We assume first that (i) holds, and write $M_i = a_i \mathbf{e}_1$ and $\mathbf{F}_i = f_i \mathbf{e}_1$ for some real numbers a_i, f_i. Note that

$$\mathbf{F} = \sum_{i=1}^{k} \mathbf{F}_i \delta_{\mathbf{M}_i} - \vec{0}\, \delta_{\vec{0}} = \sum_{i=1}^{k} f_i \mathbf{e}_1 (\delta_{\mathbf{M}_i} - \delta_{\vec{0}}).$$

Set

$$A_k = \vec{O}, \quad A_i = M_i, \qquad \lambda_{ij} = \lambda_{ji} = 0, \;\; \lambda_{ik} = f_i,$$

for all $i, j = 1, \cdots, k-1$. We have that $dist(A_i, I\!M) = 0$, that

$$\sum_{1 \leq i < j \leq k} |\lambda_{ij}| = \sum_{i=1}^{k} |\mathbf{F}_i|, \quad \text{and} \quad \sum_{1 \leq i < j \leq k} |\lambda_{ij}| \; |A_i - A_j| = \sum_{i=1}^{k} |\mathbf{F}_i| \; |M_i|.$$

This proves the proposition in case (i).

2. Assume next that (ii) holds. We select a point $\bar{O} = (0, \cdots, 0, a)$ on the vertical line passing through O, where $a \neq 0$ is a constant to be chosen later (see figure 1). We write

$$M_i = (M_i', 0), \quad F_i = f_i \mathbf{e}_d.$$

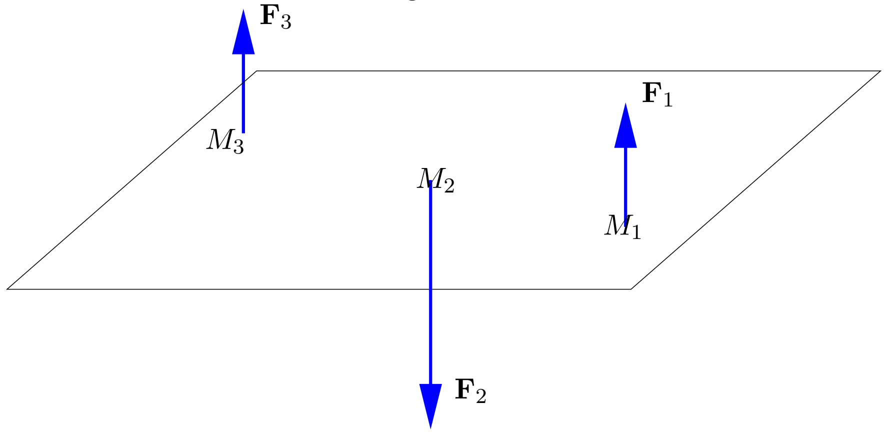
We decompose each force $\mathbf{F}_i$ into a horizontal and oblique component by setting

$$\mathbf{F}_i = \mathbf{F}_i^o + \mathbf{F}_i^h, \qquad \mathbf{F}_i^o = \Big(-\frac{f_i}{a} M_i', f_i\Big), \quad \mathbf{F}_i^h = \frac{f_i}{a} M_i.$$

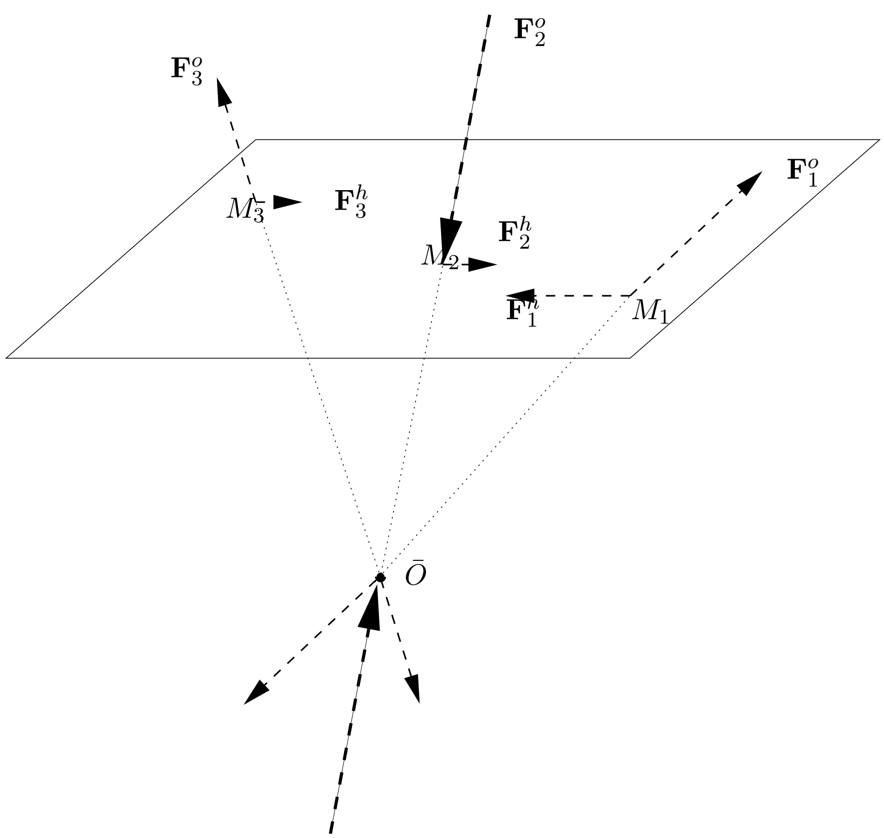
This leads us to consider a force which we will soon realize to be in equilibrium in $I\!R^{d-1}$. We define

$$\mathbf{F}^h = \sum_{i=1}^{k} \mathbf{F}_i^h \delta_{\mathbf{M}_i}, \qquad \mathbf{F}^o = \sum_{i=1}^{k} \mathbf{F}_i^o \delta_{\mathbf{M}_i}.$$

Figure 1:



Original system of forces $\mathbf{F}$



Decomposition of vertical forces:

Note that because M_i lies in the horizontal hyperplane and $\mathbf{F}_i$ is a vertical vector then the d th column $(M_i \wedge \mathbf{F}_i)_{\cdot d}$ of the skew-symmetric matrix $M_i \wedge \mathbf{F}_i$ is the vector $f_i M_i$. This proves that

$$\sum_{i=1}^k \mathbf{F}_i^h = \frac{1}{a}\sum_{i=1}^k f_i M_i = \frac{1}{a}\sum_{i=1}^k (M_i \wedge \mathbf{F}_i)_{\cdot d} = \vec{0}$$

and so,

$$\mathbf{F}^h = \sum_{i=1}^k \mathbf{F}_i^h \delta_{\mathbf{M}_i} = \sum_{i=1}^k \mathbf{F}_i^h (\delta_{\mathbf{M}_i} - \delta_{\vec{0}}) = \sum_{i=1}^k \frac{f_i ||M_i||}{a} (\delta_{\mathbf{M}_i} - \delta_{\vec{0}}) \frac{M_i}{||M_i||}. \tag{2.4}$$

By (2.4) $\mathbf{F}^h$ is a linear combination of beams and so, it is a system of forces in equilibrium. Thus, $\mathbf{F}^o = \mathbf{F} - \mathbf{F}^h$ is also in equilibrium as the difference of two systems of forces in equilibrium. In particular, $\sum_{i=1}^k \mathbf{F}_i^o = \vec{0}$, which allows us to write that

$$\mathbf{F}^o = \sum_{i=1}^k \mathbf{F}_i^o \delta_{\mathbf{M}_i} = \sum_{i=1}^k \mathbf{F}_i^o (\delta_{\mathbf{M}_i} - \delta_{\bar{O}}). \tag{2.5}$$

We use (2.5) and the fact that $\mathbf{F}_i^o = -\frac{f_i}{a}(M_i - \bar{O})$ to conclude that

$$\mathbf{F}^o = \sum_{i=1}^k -\frac{f_i}{a}\sqrt{||M_i||^2 + a^2}\frac{M_i - \bar{O}}{||M_i - \bar{O}||}(\delta_{\mathbf{M}_i} - \delta_{\bar{O}}). \tag{2.6}$$

Note that (2.4) and (2.6) give a decomposition of $\mathbf{F}$ into beams whose points of applications are in $I\!\!M \cup \{\bar{O}\}$. Set

$$A_{k+1} = \bar{O}, \quad A_i = M_i, \quad \lambda_{ik} = \lambda_{ki} = \frac{f_i |M_i|}{a} \quad \lambda_{i(k+1)} = \lambda_{(k+1)i} = -\frac{f_i}{a}\sqrt{|M_i|^2 + a^2}, \tag{2.7}$$

$i = 1, \cdots, k$. The first inequality in (2.2) is immediat for any $a \leq diam(I\!\!M)$, while the second inequality holds for $a = diam(I\!\!M)$.

We have that

$$\sum_{1 \leq i < j \leq k+1} |\lambda_{ij}||A_i - A_j| \leq \Lambda(a) := \frac{2}{|a|}\Big(\sum_{i=1}^k |f_i|\ |M_i|^2\Big) + a\sum_{i=1}^k |f_i| \tag{2.8}$$

When $a = diam(I\!\!M)$, (2.8) yields (2.3). QED

Remark 2.2 *Note that (2.3) can be improved by chosing a in order to optimize $\Lambda(a)$: It is easy to see that the minimum value of $\Lambda(a)$ is*

$$\Lambda(\bar{a}) = \sqrt{\sum_{i=1}^k 2|f_i|\ |M_i|^2}\sqrt{\sum_{i=1}^k |f_i|}, \qquad \bar{a} = \sqrt{\frac{\sum_{i=1}^k 2|f_i|\ |M_i|^2}{\sum_{i=1}^k |f_i|}}.$$

Remark 2.3 Assume that $\bar{M}_1, \cdots, \bar{M}_r, \in I\!R^{d-1}$, that $\bar{\mathbf{F}}_1, \cdots, \bar{\mathbf{F}}_r, \in I\!R^d$ and that $\bar{\mathbf{F}} = \sum_{i=1}^r \bar{\mathbf{F}}_i \delta_{\bar{M}_i}$ is a system of forces in equilibrium in $I\!R^d$. We divide the set of forces into two classes by assuming that there exists $n \in (1, r)$ such that $\bar{\mathbf{F}}_1, \cdots, \bar{\mathbf{F}}_n \notin I\!R^{d-1}$ *and* $\bar{\mathbf{F}}_{n+1}, \cdots, \bar{\mathbf{F}}_r \in I\!R^{d-1}$. Note that

$$(\bar{\mathbf{F}}_i \wedge \bar{M}_i)_{\alpha j} = 0, \qquad (i = n+1, \cdots, r) \tag{2.9}$$

if $\alpha = d$ *or* $j = d$, *and*

$$(\bar{\mathbf{F}}_i \wedge \bar{M}_i)_{\alpha j} = 0, \qquad (i = 1, \cdots, n) \tag{2.10}$$

if $\alpha, j \neq d$ *or* $\alpha = j = d$. We decompose $\bar{\mathbf{F}}_i$ *into* $\bar{\mathbf{F}}_i^h + \bar{f}_i^v \mathbf{e}_d$, such that $\bar{\mathbf{F}}_i^h \in I\!R^{d-1}$, so that $\bar{f}_i^v = 0$ *for* $i = n+1, \cdots, k$. Set

$$\mathbf{H} = \sum_{i=1}^n \bar{\mathbf{F}}_i^h \delta_{\bar{M}_i} + \sum_{i=n+1}^r \bar{\mathbf{F}}_i \delta_{\bar{M}_i}$$

and

$$\overline{I\!M} = \{\bar{M}_1, \cdots, \bar{M}_k\}.$$

Observe that the points of applications of the forces of $\mathbf{H}$ *are contained in* $\overline{I\!M}$.

Clearly $\sum_{i=1}^n \bar{\mathbf{F}}_i^h + \sum_{i=n+1}^k \bar{\mathbf{F}}_i = \vec{0}$. This, together with *(2.9)* and *(2.10)* implies that $\mathbf{H}$ is in equilibrium. Set

$$\mathbf{F}_\perp = \bar{\mathbf{F}} - \mathbf{H} = \sum_{i=1}^r \bar{f}_i^v \mathbf{e}_d \delta_{\bar{M}_i}.$$

Note that

$$|\bar{\mathbf{F}}_i|^2 = |\mathbf{H}_i|^2 + |\bar{f}_i^v|^2 \tag{2.11}$$

for all $i = 1, \cdots, k$.

Because $\mathbf{H}$ *and* $\bar{\mathbf{F}}$ are in equilibrium, we have that $\mathbf{F}_\perp$ is in equilibrium. By *proposition 2.1 (ii),* there exist $m \leq r+1$ points of application $A_{r+1}, \cdots, A_{r+m} \in I\!R^d$, and a $m \times m$ symmetric matrix $\{\lambda_{ij}\}_{i,j=r+1}^{r+m}$ such that

$$\mathbf{F}_\perp = \sum_{r+1<i<j\leq r+m} \lambda_{ij} \mathbf{t}_{A_i A_j} (\delta_{A_i} - \delta_{A_j}),$$

$$dist(A_i, \overline{I\!M}) \leq diam(\overline{I\!M}), \tag{2.12}$$

$$\sum_{r+1\leq i<j\leq r+m} |\lambda_{ij}| \leq 3 \sum_{i=1}^n |\bar{f}_i^v| \leq 3 \sum_{i=1}^n |\bar{\mathbf{F}}_i|, \tag{2.13}$$

and

$$\begin{aligned} \sum_{r+1\leq i<j\leq r+m} |\lambda_{ij}| \ |A_i - A_j| &\leq diam(\overline{I\!M}) \sum_{i=1}^n |f_i^v| \ (2|M_i| + 1) \\ &\leq diam(\overline{I\!M}) \sum_{i=1}^n |\bar{\mathbf{F}}_i| \ (2|M_i| + 1). \end{aligned} \tag{2.14}$$

In (2.14) we have used the fact that from (2.11), $|\bar{\mathbf{F}}_i| \leq |\bar{f}_i^v|$.

Theorem 2.4 *Assume that* $\mathbf{F}$ *is given by (2.1). Then* $\mathbf{F}$ *is a linear combination of finitely many beams: meaning that there exist* l *points of application* $A_1, \cdots, A_l \in I\!R^d$, *and a* $l \times l$ *symmetric matrix* $\{\lambda_{ij}\}_{i,j=1}^l$, *of null diagonal such that*

$$\mathbf{F} = \sum_{1<i<j\leq l} \lambda_{ij} \mathbf{t}_{A_iA_j} (\delta_{A_i} - \delta_{A_j}).$$

Furthermore, there exists a constant C_d *depending only on* d *such that these beams can be chosen to satisfy*

$$dist(A_i, I\!M) \leq C_d diam(I\!M), \qquad \sum_{1<i<j\leq l} |\lambda_{ij}| \ \leq C_d diam(I\!M) \sum_{i=1}^k |\mathbf{F}_i|, \tag{2.15}$$

and

$$\sum_{1<i<j\leq l} |\lambda_{ij}| \ |A_i - A_j| \leq C_d diam(I\!M) \sum_{i=1}^k |\mathbf{F}_i| \ (|M_i| + 1). \tag{2.16}$$

Proof: The idea of the proof is to decompose $\mathbf{F}$ into the sum of equilibrated systems of forces

$$\mathbf{F} = \mathbf{G} + \bar{\mathbf{F}}. \tag{2.17}$$

Here, $\mathbf{G}$ as an explicite linear combination of beams, whose volumes are controlled by $\mathbf{F}$ and $\bar{\mathbf{F}}$ is an equilibrated system of forces in $I\!R^d$, whose points of applications are all in $I\!R^{d-1}$. We use remark 2.3 to write that

$$\bar{\mathbf{F}} = \mathbf{F}_\perp + \mathbf{H}, \tag{2.18}$$

where $\mathbf{H}$ is an equilibrated system of forces in $I\!R^{d-1}$ and $\mathbf{F}_\perp$ is an equilibrated system of forces perpendicular to $I\!R^{d-1}$, whose points of applications are all in $I\!R^{d-1}$. Thanks to proposition 2.1 (i), we can proceed with the proof of the theorem assuming that it holds for $\mathbf{H}$. We use (2.17) and (2.18) to show that it suffices to prove the theorem for $\mathbf{F}_\perp$. That task was done in proposition 2.1 (ii).

Up to a rotation and a translation, we may assume that one of the points of application of $\mathbf{F}$ is the origin and so, is in $I\!R^{d-1}$; reordering the M_i if necessary, we may also assume that there exists $p \in \{0, \cdots, k-1\}$ such that

$$M_{p+1}, \cdots, M_k \in I\!R^{d-1}, \quad M_k = \vec{O}, \quad \text{and} \quad M_1, \cdots, M_p \not\in I\!R^{d-1}.$$

Step 1. This step consists in reducing the proof of the theorem from general systems of forces in $I\!R^d$ into the case where the points of application of the forces are in a hyperplan. This step will be skipt in case $p = 0$ and so, we assume that $p \neq 0$.

We refer to

$$< \mathbf{F}_i; \mathbf{e}_d > \mathbf{e}_d = f_i^v \mathbf{e}_d$$

as the "*vertical*" component of $\mathbf{F}_i$ and to

$$\mathbf{F}_i^h = \mathbf{F}_i - < \mathbf{F}_i; \mathbf{e}_d > \mathbf{e}_d \in I\!R^{d-1}$$

as its "*horizontal*" component.

Here, we only deal with the forces $\mathbf{F}_1, \cdots, \mathbf{F}_p$. If the angle between $\mathbf{F}_i$ and $\mathbf{e}_d$ is less than $\pi/4$ then the straight line passing through M_i and parellel to $\mathbf{F}_i$ intersects $I\!R^{d-1}$ at a point $\bar{M}_i$ not too far from M_i. To be more precised, if $|f_i^v| \geq |\mathbf{F}_i^h|$, then

$$|\bar{M}_i - M_i| \leq \sqrt{2}|(M_i)_d| \leq \sqrt{2}|M_i|, \tag{2.19}$$

where $(M_i)_d$ is the vertical component of M_i :

Reordering if necessary, we may assume that

$$|f_i^v| \geq |\mathbf{F}_i^h|, \qquad (i = 1, \cdots, n_1)$$

and

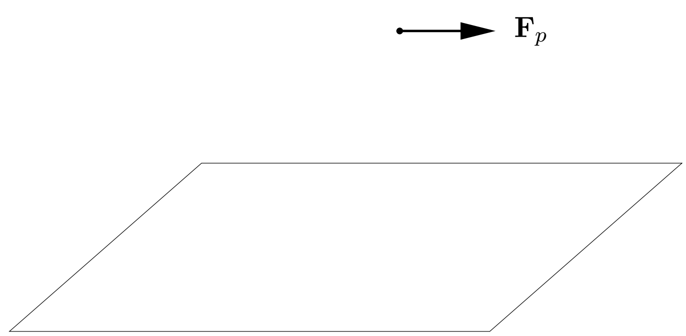
$$|f_i^v| < |\mathbf{F}_i^h|, \qquad (i = n_1 + 1, \cdots, p).$$

When $i = n_1 + 1, \cdots, p$, the straight line passing through M_i and parellel to $\mathbf{F}_i$ intersects $\mathbb{R}^{d-1}$ at a point "too far" from M_i (see figure 2). We set

$$\mathbf{F}_i = \mathbf{F}_i' + \mathbf{F}_i'', \qquad \mathbf{F}_i' = \mathbf{F}_i/2 + |\mathbf{F}_i^h|\mathbf{e}_d, \quad \mathbf{F}_i'' = \mathbf{F}_i/2 - |\mathbf{F}_i^h|\mathbf{e}_d.$$

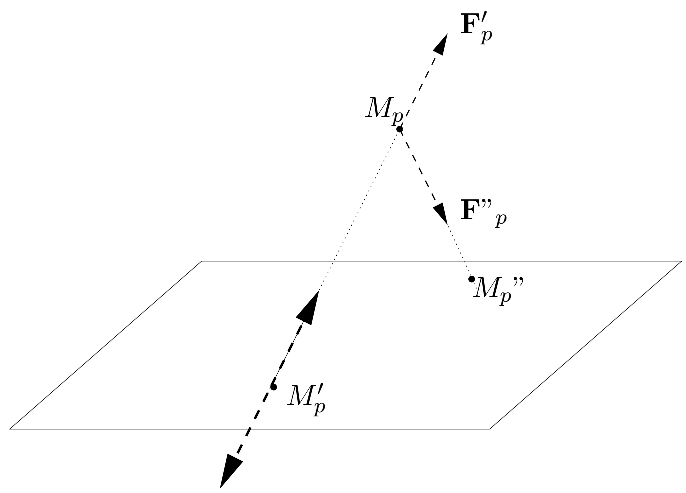
The straight line passing through M_i and parellel to $\mathbf{F}_i'$, (respectively $\mathbf{F}_i''$), intersects $\mathbb{R}^{d-1}$ at the point (see figure 3)

$$M_i' = M_i - \frac{\mathbf{F}_i/2 + |\mathbf{F}_i^h|\mathbf{e}_d}{f_i^v/2 + |\mathbf{F}_i^h|}(M_i)_d \qquad \text{(respectively } \ M_i'' = M_i - \frac{\mathbf{F}_i/2 - |\mathbf{F}_i^h|\mathbf{e}_d}{(\mathbf{F}_i)_d/2 - |\mathbf{F}_i^h|}(M_i)_d).$$



Worst case: $\mathbf{F}_i$ is horizontal

Figure 2:



Introducing new points of applications in $I\!R^{d-1}$

Figure 3:

Since by assumption $|f_i^v| < |\mathbf{F}_i^h|$ we conclude that

$$|\mathbf{F}_i|/2 \leq |\mathbf{F}'_i|, \ \ |\mathbf{F}''_i| \leq 3/2|\mathbf{F}_i|, \tag{2.20}$$

and so,

$$|M'_i - M_i|, \ \ |M''_i - M_i| \leq 6|M_i|. \tag{2.21}$$

We set

$$\mathbf{G} = \sum_{i=1}^{n_1} \mathbf{F}_i(\delta_{M_i} - \delta_{\bar{M}_i}) + \sum_{i=n_1+1}^{p} \mathbf{F}'_i(\delta_{M_i} - \delta_{M'_i}) + \mathbf{F}''_i(\delta_{M_i} - \delta_{M''_i}) \tag{2.22}$$

and $q = 3p - n_1 \leq 3(k-1)$. Note that the points of application of $\mathbf{G}$ consists of q points $\{A_1, \cdots, A_q\}$ (allowing repetitions), contained in the following union

$$\{M_1, \cdots, M_p\} \cup \{\bar{M}_1, \cdots, \bar{M}_{n_1}\} \cup \{M'_{n_1+1}, \cdots, M'_p\} \cup \{M''_{n_1+1}, \cdots, M''_p\}.$$

We use (2.19), (2.21) and the fact that $\vec{O} \in I\!M$ to conclude that

$$dist(A_i, I\!M) \leq 6|M_i| \leq 6\, diam(I\!M). \tag{2.23}$$

One can easily read off that (2.22) that there exists a $q \times q$ symmetric matrix $\{\lambda_{i,j}\}_{i,j=1}^q$ such that

$$\mathbf{G} = \sum_{1 \leq i < j \leq q} \lambda_{ij}(\delta_{A_i} - \delta_{A_j}) \frac{A_i - A_j}{|A_i - A_j|}. \tag{2.24}$$

Althought we don't write here the explicite expression of λ_{ij}, we use (2.20) to obtain that

$$\sum_{1 \leq i < j \leq q} |\lambda_{ij}| \leq \sum_{i=1}^{n_1} |\mathbf{F}_i| + \sum_{i=n_1+1}^{p} (|\mathbf{F}'_i| + |\mathbf{F}''_i|) \leq 3 \sum_{i=1}^{p} |\mathbf{F}_i|. \tag{2.25}$$

Also, (2.19– 2.21) imply that

$$\sum_{1\le i<j\le q} |\lambda_{ij}|\ |A_i - A_j| \le \sum_{i=1}^{n_1} |\mathbf{F}_i|\ |M_i - \bar{M}_i| + \sum_{i=n_1+1}^{n} \Big(|\mathbf{F}'_i|\ |M_i - M'_i| + |\mathbf{F}''_i|\ |M_i - M''_i| \Big)$$
$$\le 18 \sum_{i=1}^{k} |\mathbf{F}_i|\ |M_i| \tag{2.26}$$

By (2.24), $\mathbf{G}$ is a system of forces in equilibrium and so, as the difference of two systems in equilibrium,

$$\bar{\mathbf{F}} = \mathbf{F} - \mathbf{G} = \sum_{i=1}^{n_1} \mathbf{F}_i \delta_{\bar{M}_i} + \sum_{i=n_1+1}^{p} \Big(\mathbf{F}'_i \delta_{M'_i} + \mathbf{F}''_i \delta_{M''_i} \Big) + \sum_{i=p+1}^{k} \mathbf{F}_i \delta_{M_i}$$

is also a system of forces in equilibrium.

Set $r = k+p-n_1 \le 2k-1$. Note that the points of application of $\bar{\mathbf{F}} - \mathbf{G}$ are in the set $\{\bar{M}_1, \cdots, \bar{M}_r\}$ of $I\!R^{d-1}$ which is:

$$\{\bar{M}_1, \cdots, \bar{M}_{n_1}\} \cup \{M'_{n_1+1}, \cdots, M'_p\} \cup \{M_{p+1}, \cdots, M_k\} \cup \{M''_{n_1+1}, \cdots, M''_p\}.$$

We have implicitely used the ordering

$$\bar{M}_{n_1+1} = M'_{n_1+1}, \cdots, \bar{M}_p = M'_p, \qquad \bar{M}_{p+1} = M_{p+1}, \cdots, \bar{M}_k = M_k$$

and

$$\bar{M}_{k+1} = M''_{n_1+1}, \cdots, \bar{M}_{k+p-n_1} = M''_p.$$

Using (2.19) and (2.21) we conclude that not only

$$|M_i| \le 6 diam(I\!M), \quad diam(\overline{I\!M}) \le 14 diam(I\!M), \tag{2.27}$$

but also,

$$|\bar{M}_i| \le 7|M_i| \tag{2.28}$$

for $i = 1, \cdots, k$ and

$$|\bar{M}_i| \le 7|M_{i+p-n_1}|, \tag{2.29}$$

for $i = k+1, \cdots, r$. and

We use that $\bar{F}_i = \mathbf{F}_i$ for $i \in \{1, \cdots, n_1\} \cup \{p+1, \cdots, k\}$ and (2.20) to obtain that

$$|\bar{\mathbf{F}}_i| \le |\mathbf{F}_i|, \quad i \in \{1, \cdots, n_1\} \cup \{p+1, \cdots, k\} \tag{2.30}$$
$$|\bar{\mathbf{F}}_i| \le 3/2|\mathbf{F}_i|, \quad i \in \{n_1+1, \cdots, p\} \cup \{k+1, \cdots, r\}. \tag{2.31}$$

Step 2. We have reduced the decomposition of systems of forces problem to the case where the points of application of the forces are all in $I\!R^{d-1}$. It remains to prove the theorem in the case where the forces are all horizontal or all vertical. Indeed, as in remark 2.3 we write

$$\bar{\mathbf{F}} = \mathbf{F}_\perp + \mathbf{H}, \tag{2.32}$$

where $\mathbf{H}$ is an equilibrated system of forces in $I\!R^{d-1}$ whose points of applications are in $\overline{I\!M}$ and

$$\mathbf{F}_\perp = \sum_{i=1}^{r} f_i^v \mathbf{e}_d \delta_{\bar{M}_i}$$

is an equilibrated system of forces in $I\!R^d$, is perpendicular to $I\!R^{d-1}$ with its points of applications are in $I\!R^{d-1}$. We write that

$$\mathbf{F}_\perp = \sum_{r+1<i<j\leq r+m} \lambda_{ij} \mathbf{t}_{A_iA_j}(\delta_{A_i} - \delta_{A_j}), \tag{2.33}$$

where $\{\lambda_{ij}\}_{i,j=r+1}^{r+m}$ is a symmetric matrix and $A_{r+1}, \cdots, A_{r+m} \in I\!R^d$ satisfy (2.13) and (2.14). These, together with (2.28–2.29) and (2.30–2.31) imply that

$$\sum_{r+1<i<j\leq r+m} |\lambda_{ij}| \leq C'_d \sum_{i=1}^k |\mathbf{F}_i| \tag{2.34}$$

and

$$\sum_{r+1<i<j\leq r+m} |\lambda_{ij}|\, |A_i - A_j| \leq C'_d \sum_{i=1}^k |\mathbf{F}_i|(|M_i| + 1), \tag{2.35}$$

for a constant C'_d depending only on d. Since $\mathbf{H}$ is in equilibrium in $I\!R^{d-1}$ and has its points of application in $\overline{I\!M}$, the induction argument ensures that there are s points $A_{1+r+m}, \cdots, A_{s+r+m}$ and a $s \times s$ symmetric matrix $\{\lambda_{ij}\}_{i,j=1+r+m}^{s+r+m}$ such that

$$\mathbf{H} = \sum_{1+r+m\leq i<j\leq s+r+m} \lambda_{ij}(\delta_{A_i} - \delta_{A_j})\mathbf{t}_{A_iA_j}, \tag{2.36}$$

$$dist(A_i, \overline{I\!M}) \leq C'_{d-1} diam(\overline{I\!M}) \leq C_{d-1} diam(I\!M), \tag{2.37}$$

$$\sum_{1+r+m\leq i<j\leq s+r+m} |\lambda_{ij}| \leq C_{d-1} \sum_{i=1}^r |\mathbf{H}_i| \leq C'_{d-1} \sum_{i=1}^r |\bar{\mathbf{F}}_i| \leq C_{d-1} \sum_{i=1}^k |\mathbf{F}_i|, \tag{2.38}$$

$$\begin{aligned} \sum_{1+r+m\leq i<j\leq s+r+m} |\lambda_{ij}||A_i - A_j| &\leq C''_{d-1} diam(\overline{I\!M}) \sum_{i=1}^r |\mathbf{H}_i|(|\bar{M}_i| + 1) \\ &\leq C'_{d-1} diam(I\!M) \sum_{i=1}^r |\bar{\mathbf{F}}_i|(|M_i| + 1) \\ &\leq C_{d-1} diam(I\!M) \sum_{i=1}^k |\mathbf{F}_i|(|M_i| + 1). \end{aligned} \tag{2.39}$$

In (2.38) and (2.39) we used (2.11) and (2.20) to obtain that $|\mathbf{H}_i| \leq |\bar{\mathbf{F}}_i|$. We have used that $r \leq 2k$. We have used (2.28) and (2.29) to bound $|M_i|$.

Observe that by (2.22) and (2.32) we have that

$$\mathbf{F} = \mathbf{G} + \mathbf{F}_\perp + \mathbf{H}$$

This, together with (2.24), (2.33) and (2.36) gives that

$$\mathbf{F} = \sum_{1\leq i<j\leq l} \lambda_{ij} \mathbf{t}_{A_iA_j}(\delta_{A_i} - \delta_{A_j}). \tag{2.40}$$

We use (2.27) and (2.37) to obtain that

$$dist(A_i, I\!\!M) \leq 14 diam(I\!\!M), \qquad (i = 1, \cdots, l). \tag{2.41}$$

We use (2.25–2.26), (2.34–2.35) and (2.38–2.39) to deduce that (2.15) and (2.16) hold a constant C_d that depends only on d.

This concludes the proof of the theorem. QED