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MICHELL TRUSSES AND LINES OF PRINCIPAL ACTION

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We study the problem of Michell trusses when the system of applied equilibrated forces is a vector measure with compact support. We introduce a class of stress tensors which can be written as a superposition of rank-one tensors carried by curves (lines of principal strains). Optimality conditions are given for such families showing in particular that optimal stress tensors are carried by mutually orthogonal families of curves. The method is illustrated on a specific example where uniqueness can be proved by studying an unusual system of hyperbolic PDEs. The questions we address here are of interest in elasticity theory, optimal designs, as well as in functional analysis.

Keywords: Michell trusses; lines of principal action; optimal structures.

AMS Subject Classification: 49

1. Introduction

A very old problem in optimal design consists of minimizing the total volume of a network of elastic bars (*truss*) while the resistance to a given load remains constant. As no assumption is made on the number of bars, this study belongs to the class of topological optimization problems. This is a problem of mechanical engineering known to have no solution in general. Indeed during the optimization process, the number of bars may increase to infinity leading thus to diffuse structures. The crucial contribution of Michell[11] in the 1900s was to formulate a generalized version (called *Michell problem*) in order to take into account all possible structures which may appear in the limit. In the generalized version, attention is focussed on the stress carried by the structure rather than on its geometry. Michell stated a duality principle and obtained optimality conditions on the stress and strain tensors: they share the same eigenvectors (principal directions) and their eigenvalues have the same sign, Moreover, Michell noticed that, in the two dimensions case, when the eigenvalues of the strain tensor have opposite sign and when the eigenvector fields are smooth enough to define stream lines (called "*lines of principal action*"), then these lines constitute a so-called *Hencky-net*. This is a family of orthogonal curves which represents the limit of the families of bars through the optimization process.

The construction of the lines of principal action associated to a general stress tensor field is a difficult mathematical problem with delicate regularity issues. In order to overcome this difficulty we propose in this paper an alternative strategy. We start by noticing that the stress in a network of elastic bars is concentrated along segments which constitute a finite family of curves. On the other hand, we know some limit structures in which the stress concentrates along infinitely many curves which are not straight lines. We therefore propose another optimal truss problem for which lines of principle action make sense even when the stress tensors are not regular. We search for optima in a class of structures smaller than the class of diffuse structures considered by Michell. In our setting, a truss is represented by a signed Radon measure γ on a set X of curves; classical trusses correspond to finitely supported γ concentrated on the subset of segment s. The positive part of γ corresponds to lines in tension whereas the negative one corresponds to lines in compression. To γ one associates a stress tensor $\sigma(\gamma)$ given explicitly by (3.5).

Our conjecture is that when the topological space X is rich enough, the reformulation of the optimal truss problem in terms of the unknown γ admits a solution. We prove in this paper that the infimum of this reformulated problem is the same as the infimum of the Michell problem. We also establish some optimality conditions.

The paper is organized as follows: in Sec. 2 we fix the notation and give a mathematical framework to the optimal truss problem. We recall the classical generalization in term of stress due to Michell and write it in a modern mathematical setting by using matrix-valued measures. We also describe the dual strain formulation. In Sec. 3, we introduce a space made of $C^{1,1}$ curves with a uniform bound on the curvature. This space is a locally compact metric space. In Sec. 3.2, the generalized optimization problem in terms of curves is stated. By duality arguments, it is proved in Sec. 3.3 that it has the same infimum as Michell problem. Optimality conditions are provided in Sec. 3.4. In Sec. 4 our approach is illustrated by two specific examples. In the second one uniqueness can be proven by studying an unusual system of hyperbolic PDEs. This section makes rigorous facts commonly accepted in the literature. We are not aware of any prior work providing these proofs. We conclude this work with a list of open problems, one of them being the existence of an optimal measure γ on the space of curves.

2. Michell Trusses

2.1. *Notation*

Vectors and matrices. Let us start fixing some basic notations: If a, b are two vectors in $\mathbb{R}^d$, we denote by $\langle a; b\rangle$ the standard scalar product between a and b and we set $|a|^2 = \langle a; a\rangle$. The segment $[a, b]$ is the convexhull of $\{a, b\}$. We denote by $\mathbb{R}^{d\times d}$ the set of $d \times d$ matrices, and by $\mathcal{S}^{d\times d}$ the subset of $\mathbb{R}^{d\times d}$ that consists of symmetric matrices. If $\xi = (\xi_{ij})_{i,j=1}^d$ and $\chi = (\chi_{ij})_{i,j=1}^d$, then

$$\xi^{\mathrm{T}} = (\xi_{ji})_{i,j=1}^d, \quad \langle \xi; \chi\rangle = \sum_{i,j=1}^{d} \xi_{ij}\chi_{ij}, \quad |\xi|^2 = \langle \xi; \xi\rangle$$

denote respectively, the transposed of ξ, the trace of $\xi\chi^{\mathrm{T}}$ and the square norm of ξ. For any matrix $\xi \in \mathcal{S}^{d\times d}$, we denote $\lambda_1(\xi), \dots, \lambda_d(\xi)$ its eigenvalues which are real numbers. We denote by I_d the $d \times d$ identity matrix and, for any $a, b \in \mathbb{R}^d$, by $a \otimes b$ the $d \times d$ rank-one matrix defined by $(a \otimes b)_{i,j} = a_i b_j$. We define $a \wedge b$ to be the skew-symmetric matrix $a \otimes b - b \otimes a$.

Continuous functions and measures. Let $\mathcal{E}$ be a locally compact metric space, $C(\mathcal{E})$ the set of continuous functions of $\mathcal{E}$ to $\mathbb{R}$ endowed with the topology of uniform convergence on compact subsets. We say that $f \in C_0(\mathcal{E})$ if for every $\epsilon > 0$ there exists a compact set $K \subset E$ such that $|f(x)| \le \epsilon$ on $\mathcal{E}\backslash K$. The elements of $C_0(\mathcal{E})$ are bounded functions and $C_0(\mathcal{E})$ is a closed subset of $C(\mathcal{E})$ on which the induced topology coincides with the uniform convergence on all $\mathcal{E}$.

We denote by $\mathcal{M}(\mathcal{E})$ (resp. $\mathcal{M}(\mathcal{E}; \mathbb{R}^d)$, $\mathcal{M}(\mathcal{E}; \mathbb{R}^{d\times d})$) the set of Borel signed-measures (resp. $\mathbb{R}^d$-valued vector measures, matrix-valued measures) on $\mathcal{E}$. The set of symmetric matrix valued measures is denoted $\mathcal{M}(\mathcal{E}; \mathcal{S}^{d\times d})$: it is the set of those measures $\mu \in \mathcal{M}(\mathcal{E}; \mathbb{R}^{d\times d})$ which satisfy $\mu_{i,j} = \mu_{j,i}$ for any i and j in $\{1, \dots, d\}$. The set of non-negative Borel measures on $\mathcal{E}$ is denoted by $\mathcal{M}^+(\mathcal{E})$. When $\mu \in \mathcal{M}(\mathcal{E})$, $\mu = \mu^+ - \mu^-$ is its Jordan decomposition where $\mu^\pm \in \mathcal{M}^+(\mathcal{E})$ and we have $|\mu| = \mu^+ + \mu^-$. For any element $M \in \mathcal{E}$, δ_M denotes the Dirac mass at M. When μ belongs to $\mathcal{M}(\mathcal{E}; \mathbb{R}^p)$, $u : \mathcal{E} \to \mathbb{R}^p$ is Borel-measurable and, for any i in $\{1, \dots, d\}$, $\int_{\mathcal{E}} |u_i| d|\mu_i|$ is finite, we write

$$\int \langle u; \mu\rangle := \sum_{i=1}^{p} \int u_i(x) d\mu_i(x).$$

We will essentially consider two cases for $\mathcal{E}$: in Sec. 3, $\mathcal{E}$ will be a subset X_Ω of $C^{1,1}$-curves in $\overline{\Omega} \subset \mathbb{R}^d$ but mostly $\mathcal{E}$ will be $\mathbb{R}^d$ (or an open subset $\Omega \subset \mathbb{R}^d$ or its closure).

Let S be a k-rectifiable subset of $\mathbb{R}^d$, $\mathcal{H}^k_{|S}$ will denote the k-dimensional Hausdorff measure on S. If $\mu \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ and satisfies for any i, j, $\int |x_i| d\mu_j < +\infty$, we may also define the "torque" of μ as the skew symmetric matrix:

$$\int_{\mathbb{R}^d} x \wedge \mu = \left(\int_{\mathbb{R}^d} x_i d\mu_j(x) - \int_{\mathbb{R}^d} x_j d\mu_i(x) \right)_{i,j=1}^{d}.$$

The vector-valued measure μ is said to be balanced if

$$\int_{\mathbb{R}^d} \mu = 0, \quad \int_{\mathbb{R}^d} x \wedge \mu = 0. \tag{2.1}$$

The set of such measures is denoted $\mathcal{M}_0(\mathbb{R}^d; \mathbb{R}^d)$ (resp. $\mathcal{M}_0(\overline{\Omega}; \mathbb{R}^d)$ if μ is supported in $\overline{\Omega}$). The forces $\mathbf{F}$ which are applied to the truss we want to optimize belong naturally to $\mathcal{M}_0(\Omega; \mathbb{R}^d)$.

For representing stresses we will very often use measures in the space $\mathcal{M}(\mathbb{R}^d; \mathcal{S}^{d\times d})$.

Eventually given h a positively one homogeneous function on $\mathcal{S}^{d\times d}$, we can associate a positive measure $h(\sigma)$ to any $\sigma \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d\times d})$ by setting $h(\sigma) := h\big(\frac{d\sigma}{d\mu}\big)\mu$, where μ is any measure such that each σ_{ij} is absolutely continuous with respect to μ. Indeed this definition does not depend on the choice of μ. In particular, $|\sigma|$ will stand for the measure associated with the Euclidean norm on $\mathbb{R}^d$.

Distributional divergence and strain. Assume that Ω is an open subset of $\mathbb{R}^d$ and consider $\lambda \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ and $f \in \mathcal{M}(\overline{\Omega})$. We say that $-\mathrm{div}(\lambda) = f$ holds in the distributional sense if

$$\int_{\overline{\Omega}} \langle \nabla\varphi; \lambda \rangle = \int_{\overline{\Omega}} \varphi df$$

for every compactly supported $\varphi \in C^1(\mathbb{R}^d)$. In other words, *the measures λ and f are viewed as measures on all* $\mathbb{R}^d$ supported on $\overline{\Omega}$. In particular, the test functions may not vanish on the boundary $\partial\Omega$ and so, this imposes a boundary condition on $\partial\Omega$.

To make clear this point which is important in the sequel of the paper, let us consider a simple example: assume that Ω is an open bounded set with Lipschitz boundary and let $\mathbf{n}_\Omega$ denote the unit outward normal to $\partial\Omega$. Assume that $f = \beta\mathcal{H}^d_{|\Omega} + \alpha\mathcal{H}^{d-1}_{|\partial\Omega}$ with $\beta \in L^1(\Omega)$ and $\alpha \in L^1(\partial\Omega)$. Then if $\lambda = \mathbf{a}\ \mathcal{H}^d_{|\Omega}$ with $\mathbf{a}$ a smooth vector field, then the equation $-\mathrm{div}(\lambda) = f$ on $\overline{\Omega}$ means nothing but

$$\begin{cases} -\mathrm{div}(\mathbf{a}) = \beta & \text{on } \Omega, \\ \langle \mathbf{a}; \mathbf{n}_\Omega \rangle = \alpha & \text{on } \partial\Omega, \end{cases}$$

where the first equation holds in the distributional sense on Ω and $\langle \mathbf{a}; \mathbf{n}_\Omega \rangle$ denotes the normal trace of $\mathbf{a}$.

When σ and $\mathbf{F}$ belong respectively to $\mathcal{M}(\overline{\Omega}; \mathcal{S}^{d\times d})$ and $\mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$, we say that $-\mathrm{div}\sigma = \mathbf{F}$ on $\overline{\Omega}$ if, for any i, the ith row σ_i of σ satisfies $-\mathrm{div}\sigma_i = F_i$.

Curves. In this paper we call curve any $C^{1,1}$-curve that is the image of a map $\mathbf{r} \in C^{1,1}(0, 1; \mathbb{R}^d)$ such that $\dot{\mathbf{r}}(s) \neq 0$. We will consider only simple curves that is for which $\mathbf{r}$ is injective. Without loss of generality we will assume that $|\dot{r}|$ is constant so that $|\dot{\mathbf{r}}| = \mathcal{H}^1(S)$ and $\ddot{\mathbf{r}}(s) = (\mathcal{H}^1(S))^2\,\mathbf{k}(s)$ holds for almost all s where $\mathbf{k}(t)$ denotes the the curvature vector of S at $\mathbf{r}(t)$. Such a curve is naturally oriented by the tangent unit vector at $M = \mathbf{r}(t)$ given by $\tau_M = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$. At those points where $\mathbf{k}(t) \neq 0$, we can write $\mathbf{k}(t) = \kappa(t)\mathbf{n}(t)$, where $\kappa(t)$ is the positive scalar curvature and $\mathbf{n}(t)$ the normal vector at $\mathbf{r}(t)$.

Singular curve stress. To any curve C, we associate the measure σ^C in $\mathcal{M}(\mathbb{R}^d; \mathcal{S}^{d\times d})$ defined by

$$\sigma^C := \tau \otimes \tau\,\mathcal{H}^1|_C. \tag{2.2}$$

If $C = \mathbf{r}([0,1])$ is simple with $|\dot{\mathbf{r}}(t)|$ constant, we have for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^{d\times d})$:

$$\int \langle \sigma^C; \xi\rangle = \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle \xi(\mathbf{r}(t)); \dot{\mathbf{r}}(t) \otimes \dot{\mathbf{r}}(t)\rangle dt. \tag{2.3}$$

Taking ξ to be the gradient of a test function $u \in C(\overline{\Omega}, \mathbb{R}^d)$, we deduce that

$$-\mathrm{div}\,\sigma^C = \delta_B\,\tau_B - \delta_A\,\tau_A + \mathbf{k}\,\mathcal{H}^1_{|C}, \tag{2.4}$$

where $A := \mathbf{r}(0)$, $B := \mathbf{r}(1)$ are the endpoints of C. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^d} \langle \sigma^C; \nabla u\rangle &= \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle \nabla u(\mathbf{r}(t)); \dot{\mathbf{r}}(t) \otimes \dot{\mathbf{r}}(t)\rangle dt \\
&= \frac{1}{\mathcal{H}^1(C)} \int_0^1 \left\langle \frac{d}{dt}(u \circ \mathbf{r})(t); \dot{\mathbf{r}}(t) \right\rangle dt \\
&= \left\langle u(\mathbf{r}(1)); \frac{\dot{\mathbf{r}}(1)}{|\dot{\mathbf{r}}(1)|} \right\rangle - \left\langle u(\mathbf{r}(0));\ \frac{\dot{\mathbf{r}}(0)}{|\dot{\mathbf{r}}(0)|} \right\rangle \\
&\quad - \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle u(\mathbf{r}(t)); \ddot{\mathbf{r}}(t)\rangle dt.
\end{aligned}$$

Remark 2.1. Note that when C is the segment $[A, B]$, Eq. (2.4) reduces to $-\mathrm{div}\sigma^{[A,B]} = (\delta_B - \delta_A)\frac{B-A}{|B-A|}$. Hence,

$$\int \langle \sigma^{[A,B]}; \nabla u\rangle = \Pi u(A, B) \tag{2.5}$$

if $[A, B]$ is contained in a convex open subset Ω of $\mathbb{R}^d$ and $u \in C^1(\overline{\Omega})$. Here, we have used the linear operator Π defined by $(\Pi u)(x, y) = \left\langle u(x) - u(y); \frac{x-y}{|x-y|}\right\rangle$.

2.2. *Bars and trusses*

In structural mechanics a *bar* (A, B) is a purely one-dimensional object. It inherits from its underlying three-dimensional nature a non-negative parameter S called *section*, a *volume* $V := S|B - A|$. It also inherits the ability to resist only to two opposite axial forces applied at the extremity points A and B. The *stress* σ produced by the applied forces is axial and concentrated along the segment $[A, B]$.

It takes the form $\lambda\tau \otimes \tau$ where τ is the unit vector $\tau := (B-A)/|B-A|$. The elastic energy stored in the bar during the loading is $S|B-A|f(\frac{\lambda}{S})$. Here f is the potential function of the material the bar is made of. For instance, if one considers a linear elastic material with Young modulus κ, the potential is $f(t) = \frac{\kappa}{2}t^2$. From now on, we assume that f is a convex and even function. It is essential here that the material have the same behavior in traction or in compression.

From the mathematical point of view, it is convenient to consider the applied forces $\mathbf{F}$ as the vector valued measure $\lambda\tau(\delta_B - \delta_A)$ and the stress σ as the matrix-valued measure $\lambda\sigma^{[A,B]}$ so that the equilibrium equation reads as $-\mathrm{div}\sigma = \mathbf{F}$ in the distributional sense.

A *truss* is a finite union of such bars (A_i, A_j) for $i \neq j$ in $\{1, 2, \ldots, l\}$. Its stress is

$$\sigma = \sum_{i,j=1}^{l} \lambda_{i,j}\sigma^{[A_i,A_j]}. \tag{2.6}$$

Submitted to a force distribution $\mathbf{F} = \sum_{i=1}^{k} F^i\delta_{M_i}$ the truss is in equilibrium if $-\mathrm{div}\,\sigma = \mathbf{F}$.

• *Truss stresses.* The possible stresses in trusses have already been described. There are those measures $\sigma \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d\times d})$ which can be written as a finite combination $\sigma = \sum_{i=1}^{l} \lambda_{i,j}\sigma^{[A_i,A_j]}$. The set of such measures is denoted $\Sigma^T(\overline{\Omega})$ and given a load $\mathbf{F}$ with finite support we denote by $\Sigma^T_{\mathbf{F}}(\overline{\Omega})$ the subset $\{\sigma \in \Sigma^T(\overline{\Omega}) \ : \ -\mathrm{div}\,\sigma = \mathbf{F}\}$.

The volume of the truss is $V = \sum_{i,j=1}^{l} S_{i,j}|A_j - A_i|$ and its energy is $E = \sum_{i,j=1}^{l} S_{i,j}|A_j - A_i| f(\frac{\lambda_{i,j}}{S_{i,j}})$. Noticing that $\sigma^{[A_j,A_i]} = \sigma^{[A_i,A_j]}$, we can impose (and that is what we do in the sequel) that the matrix $(\lambda_{i,j})$ is symmetric and has vanishing diagonal values ($\forall\ i,\ \lambda_{i,i} = 0$).

2.3. *Trusses with optimal rigidity-volume ratio*

In the theory of optimal design, one desires to engineer a structure with a given material at optimal cost. Optimality means for instance that the structure should be of least total volume among the structures that remain in equilibrium with a prescribed stored energy when subject to a prescribed system of forces F. Or, in an equivalent way that the structure should be of least stored energy among the structures with a prescribed volume.

Let us first optimize the $S_{i,j}$'s for a given geometry and given values $\lambda_{i,j}$. As f is even and convex, we have

$$\begin{aligned} E = \sum_{i,j=1}^{l} S_{i,j}|A_j - A_i|\, f\left(\frac{\lambda_{i,j}}{S_{i,j}}\right) &= \sum_{i,j=1}^{l} S_{i,j}|A_j - A_i|\, f\left(\left|\frac{\lambda_{i,j}}{S_{i,j}}\right|\right) \\ &\geq V\, f\left(\frac{\sum_{i,j=1}^{l} |A_j - A_i|\,|\lambda_{i,j}|}{V}\right). \end{aligned} \tag{2.7}$$

The optimal value for E is obtained when $S_{i,j} = C|\lambda_{i,j}|$ where $C = V^{-1}\sum_{i,j=1}^{l} |A_j - A_i||\lambda_{i,j}|$, and the stored energy is then simply $E = Vf(C)$: when one wants to minimize E for a fixed volume V, one has to minimize C, that is to minimize $\sum_{i,j=1}^{l} |A_j - A_i||\lambda_{i,j}|$. Owing to this remark, the problem of optimal design reads:

Find a set of points $\mathcal{A} = \{A_i\}_{i=1}^{l} \subset \mathbb{R}^d$ and a set of real numbers $\Lambda = \{\lambda_{ij}\}_{i,j=1}^{l} \subset \mathbb{R}$ which minimize

$$\inf \left\{ \mathcal{C}(\mathcal{A}, \Lambda) : \ -\mathrm{div}\, \sigma = \mathbf{F};\ \sigma = \sum_{i,j=1}^{l} \lambda_{i,j} \sigma^{[A_i, A_j]} \right\}, \tag{2.8}$$

where

$$\mathcal{C}(\mathcal{A}, \Lambda) = \sum_{i,j=1}^{l} |\lambda_{i,j}||A_j - A_i| = \int |\sigma|. \tag{2.9}$$

By rewriting the divergence condition in (2.8), the problem amounts to finding a decomposition of $\mathbf{F}$:

$$\mathbf{F} = \sum_{i,j=1}^{l} \lambda_{i,j} \left(\delta_{A_i} - \delta_{A_j}\right) \frac{A_i - A_j}{|A_i - A_j|} \tag{2.10}$$

for which $\mathcal{C}(\mathcal{A}, \Lambda)$ is minimal. When (2.10) holds, we say that the pair $(\mathcal{A}, \Lambda)$ is admissible for problem (2.8).

Frequently one desires to design the frame inside some given domain. We will assume that this domain is the closure $\overline{\Omega}$ of some convex open subset Ω of $\mathbb{R}^d$. We ask the load $\mathbf{F}$ and the stress σ to be supported on $\overline{\Omega}$ as Radon measures and the balance equation $-\mathrm{div}(\sigma) = \mathbf{F}$ to be satisfied in the sense of distributions on $\mathbb{R}^d$. When considering the problem in the form (2.8), this means that all the segments $[A_i, A_j]$ used in the decomposition of $\mathbf{F}$ should lie in $\overline{\Omega}$.

Existence of admissible trusses. A first natural question that comes in mind is to wonder if the admissible set $\Sigma_{\mathbf{F}}^{T}(\overline{\Omega})$ is nonempty. This amounts to finding a decomposition $\mathbf{F} = \sum_{q=1}^{k} F^q \delta_{M_q}$ as in (2.10). It is quite easy to check that what is usually called "the equilibrium of the system of forces" is a necessary condition: the net force and torque have to vanish $(\int \mathbf{F})_i = (\int x \wedge \mathbf{F})_{ij} = 0$ for all $i, j = 1, \ldots, d$.

It can be proven that these conditions are sufficient when the support of $\mathbf{F}$ is finite and Ω is a sufficiently large neighborhood of the support of $\mathbf{F}$. Example 2.2 shows that they may not be any admissible truss included in the convex hull of the support of $\mathbf{F}$.

A criterium for optimality. Let us now introduce a duality relation which is useful to characterize optimal trusses. To that aim, we consider any function u: $\mathbb{R}^d \to \mathbb{R}^d$ be such that, for any (x, y),

$$|\langle u(x) - u(y); x - y\rangle| \leq |x - y|^2 \tag{2.11}$$

and we assume $\mathbf{F} = \sum_{i,j=1}^{l} \lambda_{i,j}(\delta_{A_i} - \delta_{A_j})\frac{A_i - A_j}{|A_i - A_j|}$. We observe that, for any u satisfying (2.11) and any admissible pair $(\mathcal{A}, \Lambda)$ for problem (2.8), there holds

$$\int_{\mathbb{R}^d} \langle \mathbf{F}; u \rangle = \sum_{i,j=1}^{l} \lambda_{ij} \left\langle u(A_i) - u(A_j); \frac{A_i - A_j}{|A_i - A_j|} \right\rangle$$

$$\leq \sum_{i,j=1}^{l} |\lambda_{ij}||A_i - A_j| \tag{2.12}$$

$$\leq \mathcal{C}(\mathcal{A}, \Lambda). \tag{2.13}$$

Therefore, existence of u satisfying (2.11) and such equality hold in (2.12) and (2.13), yields that the configuration $(\mathcal{A}, \Lambda)$ is optimal.

Example 2.1. A very simple case of optimal truss can be established when equality holds in (2.12) and (2.13) for $u = u_0$, where u_0 denotes the identity map. We are then reduced to the identity

$$\sum_{i,j=1}^{l} \lambda_{ij}\ |A_j - A_i| = \int_{\mathbb{R}^d} \langle \mathbf{F}; u^o \rangle = \sum_{i,j=1}^{l} |\lambda_{ij}|\ |A_j - A_i|. \tag{2.14}$$

In other words any decomposition of $\mathbf{F}$ like in (2.10) will be trivially optimal provided the λ_{ij}'s are all non-negative. In particular, if there exists a center point C such that, for any $q \in \{1, \dots, k\}$

$$\langle F^q; M^q - C \rangle = |F^q||M^q - C| \tag{2.15}$$

(see Fig. 1), then the minimum value of $\mathcal{C}$ is

$$\min \mathcal{C}(\mathcal{A}, \Lambda) = \sum_{q=1}^{k} |F^q||M^q - C|, \tag{2.16}$$

which is achieved, for instance, by $(\mathcal{A}^{\mathrm{opt}}, \Lambda^{\mathrm{opt}})$ where

$$A^{\mathrm{opt}\,q} = M^q \quad (q = 1, \dots, k), \quad A^{\mathrm{opt}}_{q+1} = C, \tag{2.17}$$

and

$$\lambda^{\mathrm{opt}}_{i,j} = 0 \quad (i, j = 1, \dots, k+1), \quad \lambda^{\mathrm{opt}}_{i,(k+1)} = \lambda^{\mathrm{opt}}_{(k+1),i} = |F^i| \quad (i = 1, \dots, k). \tag{2.18}$$

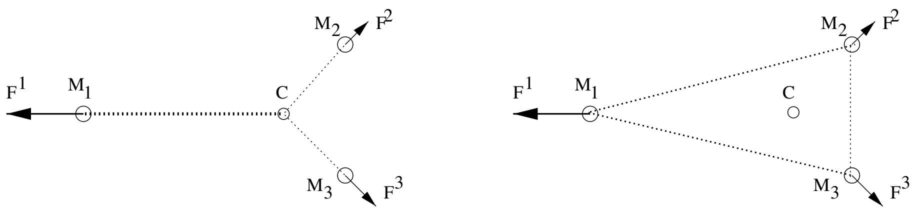

Fig. 1. Two simple optimal trusses for a particular system of central forces.

Note that the truss represented on the left-hand side of Fig. 1 is not the unique optimal truss. The truss represented on the right-hand side in one of the many others.

Example 2.2. The "bridge", a planar truss studied by Michell. In $\mathbb{R}^2$, let us consider the three points $A_1 := (-1, 0)$, $A_2 := (0, 0)$, $A_3 := (1, 0)$, the vector $\mathbf{e}_2 := (0, 1)$ and the equilibrated system of forces

$$\mathbf{F} := \mathbf{e}_2(\delta_{A_1} - 2\delta_{A_2} + \delta_{A_3}),$$

(see Fig. 2). We first note that, although the points of applications of $\mathbf{F}$ lie in the convex $\mathbb{R} \times \{0\}$ we cannot find any set $\{A_i\}_{i=1}^{l} \subset \mathbb{R} \times \{0\}$ and any symmetric matrix of real numbers $\{\lambda_{ij}\}_{i,j=1}^{l}$ such that the decomposition (2.10) holds.

If we set Ω to be the unit disk of $\mathbb{R}^2$, then the decomposition (2.10) with $A_4 = (0, 1)$, $\lambda_{2,4} = 1$, $\lambda_{1,2} = \lambda_{2,3} = 1/2$, $\lambda_{1,4} = \lambda_{4,3} = -\sqrt{2}/2$, $\{A_i\}_{i=1}^{l} \subset \Omega$ holds. But the cost is non-optimal. This decomposition is represented in Fig. 2 where the bars (the support of λ) are drawn in dotted lines when they are in traction (λ_{ij} is positive) and in plain lines when they are in compression (λ_{ij} is negative).

In fact Michell[11] himself noticed that an optimal truss does not always exist. In Sec. 4.2.2, we show that in the particular case of the "bridge", the optimal cost is obtained as the limit of a sequence σ_n (see Fig. 3 where they are represented successively σ_1, σ_2 and σ_5).

The limit of the sequence σ_n cannot, in any sense, correspond to a truss: the number of bars tends to infinity and the union of the upper bars converges to an arc of circle.

This last example shows the necessity of enlarging the class of structures in which the optimal design is to be searched.

2.4. *Stress formulation for Michell problem*

From (2.8) and (2.9), we learn that the optimal trusses problem can be written in terms of stress: given a load $\mathbf{F}$ with finite support, we minimize $\int |\sigma|$ where the

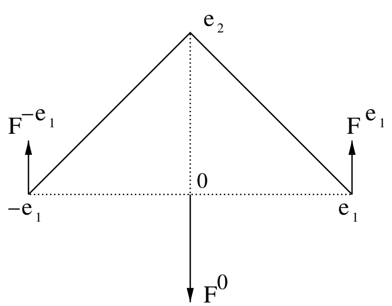

Fig. 2. An admissible truss for the "bridge".

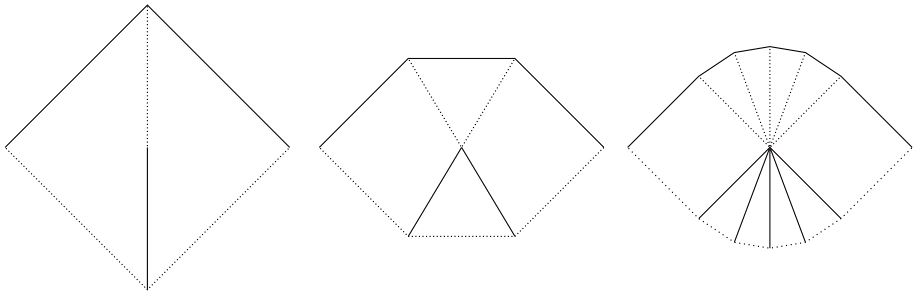

Fig. 3. A minimizing sequence of trusses.

stress measure $\sigma \in \Sigma^{\mathrm{T}}(\overline{\Omega})$ (see (2.6)) is subjected to the constraint $-\operatorname{div}\sigma = \mathbf{F}$. As this problem has in general no solution, it is standard to look for solutions σ in a larger class. Michell himself extended the problem to stresses with rank larger than one and to *diffuse* stress measures. Let us sketch this extension which in fact is a convexification procedure in a modern mathematical setting.

We extend the set of admissible stress measures to the set $\Sigma_{\mathbf{F}}(\overline{\Omega})$ of all measures supported in $\overline{\Omega}$, taking values in symmetric matrices $\mathcal{S}^{d\times d}$ and satisfying the constraint $-\operatorname{div}\sigma = \mathbf{F}$ in the distributional sense. Besides we notice that this extension allows considering more general loads: $\mathbf{F}$ is any vector measure compactly supported in $\overline{\Omega}$ and balanced in the sense of (2.1). The cost $\int|\sigma|$ appearing in (2.8) now has to be extended to non-rank one stresses: following Michell, we consider the largest convex potential $\rho^0(\chi)$ on $\mathcal{S}^{d\times d}$ which agrees with the Euclidean norm $|\chi|$ for rank one tensors. To compute ρ^0 we introduce for any $\xi \in \mathcal{S}^{d\times d}$

$$\rho(\xi) := \max_{b\in\mathbb{R}^d}\{|\langle \xi b; b\rangle| \colon |b| \le 1\} \ = \ \max_i |\lambda_i(\xi)|. \tag{2.19}$$

The convexified function ρ^0 is characterized in terms of its Fenchel conjugate:

$$(\rho^0)^*(\xi) = \sup_{b\in\mathbb{R}^d}\{\langle \xi; b\otimes b\rangle - |b\otimes b|\} = \begin{cases} 0 & \text{if } \ \rho(\xi)\le 1, \\ +\infty & \text{otherwise.}\end{cases}$$

Thus

$$\rho^0(\chi) = \max_{\xi\in\mathbb{R}^{d\times d}}\{\langle \xi;\chi\rangle \colon \rho(\xi)\le 1\} = \sum_{i=1}^d |\lambda_i(\chi)|. \tag{2.20}$$

Remark 2.2. The convex continuous functions ρ and ρ^0 belong to an important class of functions which depend on singular values of matrices. They enjoy the following property (see Ref. 2): for any symmetric matrices ξ and χ we have $\langle \xi;\chi\rangle \le \rho(\xi)\rho^0(\chi)$. The equality holds if and only if ξ and χ have a common basis of eigenvectors and for any i,

$$\lambda_i(\xi)\lambda_i(\chi) \ge 0 \quad \text{and} \quad \left(\lambda_i(\chi) = 0 \text{ or } |\lambda_i(\xi)| = \max_{1\le j\le d}|\lambda_j(\xi)|\right).$$

We notice that the function ρ^0 given by (2.20) is one homogeneous and therefore the measure $\rho^0(\sigma)$ is meaningful. The following duality relation holds (see Ref. 4)

$$\int \rho^0(\sigma) \;=\; \sup\left\{\int \langle \xi;\sigma\rangle \colon\; \xi \in C(\overline{\Omega};\mathcal{S}^{d\times d}),\;\rho(\xi)\le 1\right\}. \tag{2.21}$$

Since the measure $\rho^0(\sigma)$ coincides with $|\sigma|$ for rank one tensor measures σ, the problem

$$\inf\left\{\int \rho^0(\sigma)\colon\; \sigma \in \Sigma_{\mathbf{F}}(\overline{\Omega})\right\}. \tag{2.22}$$

is a natural extension of the original truss optimization problem. The functional $\sigma \to \int \rho^0(\sigma)$ is coercive. Due to (2.21), it is lower semicontinuous on the closed subset $\Sigma_{\mathbf{F}}(\overline{\Omega})$ of the space $\mathbf{M}(\overline{\Omega};\mathcal{S}^{d\times d})$ endowed with the weak-star topology. Existence of a minimizer is then ensured provided the set $\Sigma_{\mathbf{F}}(\overline{\Omega})$ is non-empty. This fact will be proved in Proposition 2.1.

2.5. *Strain formulation for Michell problem*

Let us now introduce a dual problem in a way similar to what we did in (2.13). For any function $u:\Omega \mapsto \mathbb{R}^d$, we denote

$$\|u\|^*_\Omega := \sup\left\{\frac{|\langle u(x)-u(y);x-y\rangle|}{|x-y|^2}\;,\; x\neq y\;,\;(x,y)\in\Omega^2\right\} \tag{2.23}$$

and we define $\mathcal{U}_1 = \mathcal{U}_1(\Omega)$ to be the set of continuous displacements $u:\overline{\Omega}\to\mathbb{R}^d$ such that $\|u\|^*_\Omega \le 1$. The subspace $\mathcal{U}_0$ of functions u satisfying $\|u\|^*_\Omega = 0$ coincides with the space of rigid displacements (i.e. the space of affine functions u with skew-symmetric matrix). In the sequel $e(u)$ denotes the symmetric part of the gradient of u in the sense of distributions on Ω: $e(u) = 1/2(\nabla u + (\nabla u)^{\mathrm{T}})$. We have the following result

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open connected subset and $u:\overline{\Omega}\to\mathbb{R}^d$.*

(i) *If $||u||^*_\Omega \le 1$, then u is continuous and differentiable everywhere, except maybe on a $(d-1)$-dimensional Hausdorff set. Moreover, $e(u)$ is an element of $L^\infty(\Omega;\mathcal{S}^{d\times d})$ and satisfies the inequality $\rho(e(u))\le 1$ a.e.*

(ii) *Let $C(\overline{\Omega};\mathbb{R}^d)$ be endowed with the topology of uniform convergence on compact subsets of $\overline{\Omega}$. Then $\mathcal{U}_1(\Omega)/\mathcal{U}_0$ is a convex compact subset of the quotient space $C(\overline{\Omega};\mathbb{R}^d)\,/\,\mathcal{U}_0$.*

(iii) *Assume that Ω is convex. Then the following equivalence holds*

$$\|u\|^*_\Omega \le 1 \Leftrightarrow u \in C(\overline{\Omega};\mathbb{R}^d),\quad e(u)\in L^\infty(\Omega;\mathcal{S}^{d\times d}),\quad \rho(e(u))\le 1 \text{ a.e. on } \Omega.$$

Moreover, every $u\in\mathcal{U}_1(\Omega)$ can be approximated by a smooth sequence $\{u_n\}\subset C^\infty$ such that $u_n \in \mathcal{U}_1(\Omega)$ and $u_n \to u$ uniformly.

Proof. The first assertion is a consequence of the fact that $\|u\|_\Omega^* \le 1$ implies the monotonicity of the map $u+I_d$ (and of $-u+I_d$ as well). Then u is almost everywhere differentiable and $e(u)$ is an element of $L^\infty(\Omega;\mathcal{S}^{d\times d})$ which satisfies $\rho(e(u)) \le 1$ at every point of differentiability (see Ref. 1).

Let ω be a connected relatively compact open subset of Ω. Then by Korn's inequality, we derive that, up to rigid displacements, $\mathcal{U}_1(\Omega)$ is a bounded subset of $W^{1,p}(\omega)$ for every $1 \le p < +\infty$ (see Ref. 12). Note, however, that this is not true for $p = \infty$ since $\mathcal{U}_1(\Omega)$ contains non-Lipschitz functions (see Ref. 8). Taking $p > d$, we deduce from Morrey's theorem that the restrictions to ω of functions in $\mathcal{U}_1(\Omega)$ are in a compact set of $\mathcal{U}_1(\omega)$ hence (ii).

The converse implication in (iii) can be checked as follows: by Korn's inequality u belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p > d$ and we may choose u to be the continuous representative in its class. For almost all $(x,y) \in \overline{\Omega}^2$, such a u is almost everywhere differentiable on $[x,y]$ and therefore the function $t \mapsto \langle u((1-t)x+ty)-u(x); \frac{y-x}{|y-x|^2}\rangle$, is 1-Lipschitz.

Let us now prove the last statement in (iii). Let $u \in \mathcal{U}_1(\Omega)$ and assume that $0 \in \Omega$. Take $t_k = 1 + 1/k$ and set $v_k(x) = t_k u(x/t_k)$ which clearly belongs to $\mathcal{U}_1(t_k\Omega)$. Then consider the mollified sequence $u_{n,k} = v_k \star \alpha_n$ where α_n is the usual convolution kernel. For n_k large enough, the restriction of $u_{n_k,k}$ to Ω belongs to $\mathcal{U}_1(\Omega)$, and $u_{n_k,k} \to u$ uniformly as $k \to \infty$. □

Michell himself contributed the essential insight that a dual problem to (2.22) is:

$$\sup_u \left\{ \int \langle u; \mathbf{F}\rangle : \ u \in \mathcal{U}_1(\Omega) \right\}. \tag{2.24}$$

Recall that $\mathbf{F}$ is always assumed to be compactly supported and equilibrated. By Lemma 2.1, we are maximizing a continuous linear form on a non-empty compact set. So problem (2.24) admits a solution. A more subtle use of duality arguments leads to

Proposition 2.1. *Let Ω be a convex open subset of $\mathbb{R}^d$ and let $\mathbf{F}$ be a balanced measure compactly supported in $\overline{\Omega}$. Then $\Sigma_{\mathbf{F}}(\overline{\Omega})$ is non-empty and*

$$\min \left\{ \int \rho^0(\sigma) : \ \sigma \in \Sigma_{\mathbf{F}}(\overline{\Omega}) \right\} = \ \max \left\{ \int \langle u; \mathbf{F}\rangle : \ u \in \mathcal{U}_1(\Omega) \right\}. \tag{2.25}$$

A pair $(u^o, \sigma^o) \in \mathcal{U}_1(\Omega) \times \Sigma_{\mathbf{F}}(\overline{\Omega})$ is optimal if and only if the following extremality relation is satisfied

$$\int \rho^0(\sigma^o) = \int \langle u^o; \mathbf{F}\rangle. \tag{2.26}$$

Proof. As seen before, problem (2.24) admits a solution and by the assertion (iii) of Lemma 2.1 $\alpha < +\infty$ where

$$\alpha = \max \left\{ \int \langle u; \mathbf{F}\rangle : \ u \in \mathcal{U}_1(\Omega) \right\} = \sup \left\{ \int \langle u; \mathbf{F}\rangle : \ u \in C^1(\overline{\Omega})\,,\ \rho(e(u) \le 1 \right\}.$$

Consider $C_0(\overline{\Omega}; \mathcal{S}^{d\times d})$ where $\mathcal{S}^{d\times d}$ is equipped with the norm $\|\xi\| := \rho(\xi)$ and let V be the (closed) subspace of $C_0(\overline{\Omega}; \mathcal{S}^{d\times d})$ defined by $V := \{e(u)\,,\, u \in C^1(\overline{\Omega}; \mathbb{R}^d)\}$. Then, as F vanishes on rigid displacements, we can define without ambiguity the linear form

$$L : \xi \in V \mapsto \int \langle u, \mathbf{F}\rangle \quad \text{whenever } \xi = e(u).$$

L is continuous and by the fact that $\alpha < +\infty$, its norm satisfies $\|L\| = \alpha$. By the Hahn–Banach Theorem, we may extend L to an element of $\big(C_0(\overline{\Omega}; \mathcal{S}^{d\times d})\big)^*$ with the same norm α. Therefore there exists a vector measure $\sigma \in \mathbf{M}(\overline{\Omega}; \mathcal{S}^{d\times d})$ such that $\int \langle e(u); \sigma\rangle = \int \langle u; \mathbf{F}\rangle$ for all $u \in C^1(\overline{\Omega}; \mathbb{R}^d)$. This yields $\sigma \in \Sigma_{\mathbf{F}}(\overline{\Omega})$ and the minimality of such σ follows since, by (2.21),

$$\alpha \ = \ \sup\left\{\int \langle \xi; \sigma\rangle :\ \xi \in C_0(\overline{\Omega}),\ \rho(\xi) \le 1\right\} = \int \rho^0(\sigma).$$

This proves (2.25) and the fact that any optimal pair satisfies (2.26). The converse is straightforward. □

2.6. *Lines of principal action*

The concept of lines of principal action is often evoked in the literature on Michell trusses. But is difficult to rigorously defined it since many stresses encountered are not regularity. Let us recall that concept assuming that we are dealing with displacements and stress tensors which are smooth enough. Assume for a moment that problem (2.24) admits a maximizer $u^o \in C^1(\overline{\Omega})$. Let σ^o be a minimizing stress measure for (2.22). By (2.26), integrating by parts and taking into account that $\sigma^o \in \Sigma_{\mathbf{F}}(\overline{\Omega})$ and $\rho(e(u^o)) \le 1$, we obtain

$$\int \rho^0(\sigma^o) = \int \langle \sigma^o; e(u^o)\rangle \ \le \int \rho(e(u^o))\rho^0(\sigma^o) \ \le \ \int \rho^0(\sigma^o).$$

Thus

$$\rho^0(\sigma^o) = \langle \sigma^o; e(u^o)\rangle \quad \text{and} \quad \rho(e(u^o)) = 1 \quad \ \sigma^o \text{ a.e.} \tag{2.27}$$

By diagonalization, there exist bounded real functions $\{\theta_i\}_{i=1}^d$ from Ω to $[-1, 1]$ and an orthonormal family $\{a^i\}_{i=1}^d$ of vector fields such that

$$e(u^o) = \sum_{i=1}^{d} \theta_i\ a^i \otimes a^i. \tag{2.28}$$

Thanks to Remark 2.2, we deduce from (2.27) and (2.28) the existence of signed measures $\lambda_1, \ldots, \lambda_d$ such that:

$$\sigma^o = \sum_{i=1}^{d} \lambda_i\ a^i \otimes a^i, \quad |\lambda_i| = \theta_i\,\lambda_i. \tag{2.29}$$

A standard approximation procedure shows that the localized optimality conditions (2.28) and (2.29) still hold under weaker assumptions. We may merely assume that $e(u^o)$ is a continuous function on $\mathbb{R}^d$ except possibly on a $|\sigma^o|$-negligible subset. For general $u^o \in \mathcal{U}_1(\Omega)$, it is possible to write a weak counterpart of the relations above. This could be done by introducing a suitable notion of tangent space to the stress measure tensor σ^o and by characterizing the density of the duality pairing measure $\langle \sigma^o; e(u^o)\rangle$ (see Ref. 3).

Now one may try to associate to the optimal pair (u^o, σ^o) the lines of the fields a_i which are called *lines of principal action* (or *lines of principal strain* or *lines of principal stress*).

Suppose for instance that each measure λ_i in (2.28) is absolutely continuous with a smooth density (denoted by the same symbol λ_i). The curve $s \to x(s,\omega)$ will be a line of principal action of σ^o if it lies in a region where some λ_i keeps a constant sign and

$$\begin{cases} \sigma^o(x(s,\omega))\dot{x}(s,\omega) = \lambda_i(x(s,\omega))\dot{x}(s,\omega) \\ x^i(0,\omega) = \omega, \end{cases} \tag{2.30}$$

Let $\mathcal{F}_+$ (respectively $\mathcal{F}_-$) be the lines of principal action whose tangents are eigenvectors corresponding to positive (resp. negative) eigenvalues of σ^o. Formally, we have the following property: *let $\omega, \overline{\omega} \in \Omega$ such that $x^i(\cdot,\omega)$ is a curve in $\mathcal{F}_+$ and $x^j(\cdot,\overline{\omega})$ is a curve in $\mathcal{F}_-$; then $x^i(\cdot,\omega)$ and $x^j(\cdot,\overline{\omega})$ are orthogonal where they intersect.* The families $\mathcal{F}_+$ and $\mathcal{F}_-$ are special orthogonal curvilinear coordinates, in the sense that they satisfy (2.28). When $d = 2$, $\mathcal{F}_+$ and $\mathcal{F}_-$ are the so-called *Hencky–Prandtl nets* (see Kohn and Strang in Ref. 13 for details).

The situation is still simpler in the case where σ^o is supported by a countable family of curves and can be written in the form

$$\sigma^o = \sum_{n=1}^{\infty} \theta_n^+ \, \sigma^{C_n^+} - \sum_{n=1}^{\infty} \theta_n^- \sigma^{C_n^-}, \tag{2.31}$$

where $\theta_n^{\pm}$ are positive densities and the $C_n^{\pm}$ are Lipschitz curves in $\overline{\Omega}$ and $\mathbf{t}_{C_n^{\pm}}$ is a unit tangent vector to $C_n^{\pm}$. In this case the λ_i in (2.28) are all zero except for i such that the unit vector a_i is tangent to the curve. Moreover $\mathcal{F}_+ = \{C_n^+\}$ and $\mathcal{F}_- = \{C_n^+\}$ and C_n^+ is orthogonal to C_n^- at any intersection point.

In this context, Michell made the following conjecture (see Refs. 11 and 9 p. 210 or Ref. 10 p. 71):

"*A frame given by a stress tensor σ^o is optimal if it can carry its given forces with stresses in its tension members equal to $\lambda_+ \geq 0$ and stresses in its compression members equal to $\lambda_- \leq 0$. They must be a virtual deformation of $u^o : \overline{\Omega} \to \mathbb{R}^d$ that satisfies the kinematics condition $\langle \sigma; e(u^o)\rangle = \rho(e(u^o))\rho^0(\sigma)$ and gives strain of 1 in its tension members and strain of -1 in its compression members. The deformation u^o must have no direct strain lying outside these limits in the sense that $\rho(e(u)) \leq 1$. The "bars" of the optimal structure are arranged along the lines of principal strain of u^o.*"

Note that the above conjecture would be false if we do not impose any restriction on the system of forces $\mathbf{F}$ and on the domain Ω. In fact rigorous optimality conditions for problems of the kind (2.22) and (2.24) have been obtained in a quite general case by using arguments of geometric measure theory (see Ref. 3). However, due to the particular structure of ρ^0, it appears from many examples, that the optimal stress measures σ^o inherits a very particular one dimensional geometric structure. This is roughly described by a system of curves like in (2.31).

The difficulty to deduce from an optimal stress measure the existence of such a system of curves leads us to adopt the reverse strategy: going back to the original truss optimization problem we directly enlarge the set of admissible curves. We search an optimal distribution of such curves from which we deduce the optimal stress measure. This new strategy is developed in the next section.

3. Reformulation of Michell Trusses Via Measures on the Set of Curves

In the original truss optimization problem (2.8), the set of curves which support the structure is a subset of the collection of all segments $[x, y]$. Here, $(x, y) \in \overline{\Omega} \times \overline{\Omega} \setminus \Delta$ and Δ is the diagonal. We propose to enlarge this class to a set called X_Ω. We describe in the following subsection two possible choices for X_Ω. As other choices could be also interesting, the duality result in Sec. 3.3 will be established assuming only general conditions on the class X_Ω.

3.1. *Some metric spaces involving curves*

We will consider simple oriented $C^{1,1}$-curves with an uniform upper bound on the scalar curvature. More precisely, given $l_0 > 0$ and $\kappa_0 \geq 0$ such that $l_0 \kappa_0 \leq 1$, we define

$$X_\Omega^{l_0,\kappa_0} := \left\{ C \subset \overline{\Omega} : C \text{ is a } C^{1,1}\text{-curve},\ 0 < \mathcal{H}^1(C) \leq l_0,\ \kappa(C) \leq \kappa_0 \right\} \tag{3.1}$$

$$X_\Omega^{\infty} := \left\{ C \subset \overline{\Omega} : C \text{ is a } C^{1,1}\text{-curve},\ 0 < \mathcal{H}^1(C) < \infty,\ \kappa(C)\, \mathcal{H}^1(C) \leq 1 \right\}. \tag{3.2}$$

We notice that for $\kappa_0 = 0$, our definition (3.2) coincides with the sets of segments $[x, y]$ contained in $\overline{\Omega}$ such that $0 < |x - y| \leq l_0$. On the other hand, the condition $\kappa(C)\, \mathcal{H}^1(C) \leq 1$ is imposed in (3.2) in order to ensure that all curves in our space are simple. More precisely the following lemma holds:

Lemma 3.1. *Let* $\mathbf{r} \in C^{1,1}(0, 1; \mathbb{R}^d)$ *such that* $|\dot{\mathbf{r}}(s)| = l$ *and* $|\ddot{r}(s)| \leq \kappa l^2$ *a.e. where* l, κ *are two positive constants. Then, for every* $s, t \in [0, 1]$, *there holds*

$$|\mathbf{r}(s) - \mathbf{r}(t)| \ \geq \frac{l|s - t|}{2} \quad \textit{whenever} \ \ \kappa l |s - t| \leq 1. \tag{3.3}$$

Proof. If $s, t \in [0, 1]$ and such that $0 < \kappa l(t - s) \leq 1$, then

$$|\mathbf{r}(t) - \mathbf{r}(s)| = \left| \dot{\mathbf{r}}(s)(t - s) + \int_s^t dv \int_s^v \ddot{\mathbf{r}}(\sigma) d\sigma \right| \geq l|s - t| - \frac{|s - t|^2}{2} \kappa l^2 \geq \frac{l|s - t|}{2}.$$

$\square$

The topology we next consider is the usual local Hausdorff convergence on the family $\mathcal{F}(\overline{\Omega})$ of closed subsets of $\overline{\Omega}$. This topology is induced by a distance we take to be the truncated Hausdorff metric

$$\mathrm{d_H}(C,D) := \min\{1, h(C,D)\}, \quad h(C,D) := \max\left\{\sup_{x\in C} d(x,D),\ \sup_{x\in D} d(x,C)\right\},$$

where $d(x,D)$ denotes the Euclidean distance from x to D. It turns out that $(\mathcal{F}(\overline{\Omega}), \mathrm{d_H})$ is a complete locally compact metric space. This space is compact if Ω is bounded (see for instance Chap II of Ref. 7). We have

Lemma 3.2. *The families $X_\Omega^{l_0,\kappa_0}$ and X_Ω^∞ are locally compact subspaces of $(\mathcal{F}(\overline{\Omega}), \mathrm{d_H})$ provided that $l_0\kappa_0 \le 1$.*

Proof. First we prove that X_Ω^∞ is locally compact. This amounts to showing that closed balls of the form $\mathcal{B} = \{C \in X_\Omega^\infty : \mathrm{d_H}(C, C_0) \le \eta\}$ satisfy the Bolzano–Weierstrass property. We may assume that $2\eta < \inf\{1, d_0\}$ where d_0 is the diameter of C_0.

Let C_h be a sequence in $\mathcal{B}$. Clearly, as $\eta < 1$, C_h remains in a fixed compact subset of $\mathbb{R}^d$. We consider a parametrization $\mathbf{r}_h : [0,1] \mapsto \mathbb{R}^d$ such that $|\dot{\mathbf{r}}_h| = l_h$ and $|\ddot{\mathbf{r}}_h| \le \kappa_h (l_h)^2$ where $l_h > 0$ and $l_h\kappa_h \le 1$. Then by (3.3), we infer that

$$|\mathbf{r}_h(s) - \mathbf{r}_h(t)| \ \ge\ \frac{l_h}{2}\,|s-t| \tag{3.4}$$

for every $0 < |s-t| \le 1$. Since the range of $\mathbf{r}_h$ remains in a fixed compact subset, (3.4) with $s = 0$, $t = 1$ gives that l_h is bounded and therefore $\mathbf{r}_h$ and $\dot{\mathbf{r}}_h$ are equi-Lipschitz. Possibly passing to a subsequence, we may assume that $\mathbf{r}_h \to \mathbf{r}$ uniformly, $l_h \to l$ and $\ddot{\mathbf{r}}_h \overset{*}{\rightharpoonup} \ddot{\mathbf{r}}$ where $\mathbf{r}$ is suitable element of $C^{1,1}$ such that $|\dot{\mathbf{r}}| = l$ and $|\ddot{\mathbf{r}}| \le \kappa l^2$. It is then clear that C_h does converge in the sense of the Hausdorff distance to $C := \mathbf{r}([0,1])$. In particular, $\mathrm{d_H}(C, C_0) \le d_0$ implies that C cannot be reduced to a single point. Thus $l > 0$ and C is an element of $\mathcal{B}$. We then conclude that $\mathcal{B}$ is compact.

Now, by using the same arguments, it is easy to check that $X_\Omega^{l_0,\kappa_0}$ as a closed subspace of X_Ω^∞ is locally compact as well. □

3.2. *Curves formulation of the truss problem*

Let us assume that, as in the previous subsection, we are given a locally compact metrizable space X_Ω whose elements are curves contained in $\overline{\Omega}$ *which we assume from now on to be convex.* Recall that we associate with each element $C \in X_\Omega$ the stress tensor σ^C given by (2.2).

We make the following assumptions:

(H1) *X_Ω is a set of $C^{1,1}$ simple curves with finite positive length, endowed with a metrizable locally compact topology.*

(H2) *The map $C \in X_\Omega \mapsto \sigma^C$ is continuous with respect to the weak-star topology on $\mathcal{M}(\overline{\Omega}; \mathcal{S}^{d\times d})$.*

(H3) X_Ω *contains all segments* $[x,y]$ *with* $x \neq y$ *and the map* $(x,y) \in \overline{\Omega} \times \overline{\Omega} \setminus \Delta \mapsto [x,y] \in X_\Omega$ *is continuous.*

Let us stress the fact that these formal assumptions are satisfied by the spaces X_Ω^∞ and $X_\Omega^{l_0,\kappa_0}$ considered in the previous subsection. We associate to any signed Radon measure γ on X_Ω the truss stress $\sigma(\gamma)$ defined by setting, for all test functions $\xi \in C^1(\overline{\Omega}, \mathbb{R}^{d\times d})$:

$$\int \langle \sigma(\gamma); \xi \rangle = \int_{\mathbf{X}_\Omega} \left(\int \langle \xi; \sigma^C \rangle \right) \gamma(dC). \tag{3.5}$$

Note that assumption (H2) implies the Borel measurability of $C \mapsto \int \langle \xi; \sigma^C \rangle$. As this function is bounded by $\mathcal{H}^1(C) \sup |\xi|$, the existence of the integral above is guaranteed provided γ satisfies the condition $\int \mathcal{H}^1(C) |\gamma|(dC) < \infty$. We will call *generalized Michell truss* such a signed measure γ. Its support is a family of curves which are candidate to be the lines of principal action. It consists of two subfamilies $\mathcal{F}_+$ and $\mathcal{F}_-$ corresponding to the supports of the positive and negative parts of γ.

In this setting we identify a truss of the kind (2.6) with the atomic measure $\gamma = \sum_{i,j=1}^{l} \lambda_{ij} \delta_{[A_i,A_j]}$. The energy of such a truss (see (2.9)) can be written as $\int_{X_\Omega} \mathcal{H}^1(C) |\gamma|(dC)$. Thus the natural extension of the truss optimization problem is

$$\inf \left\{ \int_{X_\Omega} \mathcal{H}^1(C) |\gamma|(dC) \colon \ \gamma \in \Gamma_{\mathbf{F}}(\Omega) \right\}. \tag{3.6}$$

Here $\Gamma_{\mathbf{F}}(\Omega) := \left\{ \gamma \in \mathcal{M}(X_\Omega) \colon \sigma(\gamma) \in \Sigma_{\mathbf{F}}(\overline{\Omega}) \right\}$. Under the above assumptions on the space X_Ω, we are going to establish that the infimum in (2.22) and (3.6) are the same. We observe first that:

$$\inf \left\{ \int_{X_\Omega} \mathcal{H}^1(C) |\gamma|(dC) \colon \ \gamma \in \Gamma_{\mathbf{F}}(\Omega) \right\} \geq \inf \left\{ \int \rho^0(\sigma) \colon \sigma \in \Sigma_{\mathbf{F}}(\overline{\Omega}) \right\}. \tag{3.7}$$

Indeed by Remark 2.2, for any $\xi \in C(\overline{\Omega}; \mathcal{S}^{d\times d})$ such that $\rho(\xi) \leq 1$, we have that $\mathcal{H}^1(C) \geq \int \langle \xi; \sigma^C \rangle$ for all curves $C \in X_\Omega$. Then by (2.21) and (3.5), we deduce that for every $\gamma \in \Gamma_{\mathbf{F}}(\Omega)$, there holds

$$\int_{X_\Omega} \mathcal{H}^1(C) |\gamma|(dC) \ \geq \ \int \rho^0(\sigma(\gamma)). \tag{3.8}$$

Establishing the converse of inequality (3.7) is a delicate problem we solve in the next subsection by means of a general duality argument.

Remark 3.1. The map defined in (H3) identifies $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$ (equipped with the Euclidean metric) with a subspace of X_Ω. This subspace is closed: indeed let $C_n = [x_n, y_n]$ converge to a curve C in X_Ω. Then, by (H2), $\sigma^{[x_n,y_n]}$ converges weakly-star to σ^C. Therefore (x_n, y_n) converges to a suitable $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $\sigma^C = \sigma^{[x,y]}$. Hence $C = [x,y]$ and, owing to (H1), $\mathcal{H}^1(C) > 0$ and so $x \neq y$.

3.3. *The general duality result*

Theorem 3.1. *Let X_Ω be a space of curves satisfying* (H1), (H2) and (H3). *Let $\mathbf{F} \in \mathcal{M}_0(\overline{\Omega})$ be compactly supported and define*

$$\Gamma_{\mathbf{F}}(\Omega) := \left\{\gamma \in \mathcal{M}(X_\Omega) \colon \sigma(\gamma) \in \Sigma_{\mathbf{F}}(\overline{\Omega})\right\}.$$

Then $\Gamma_{\mathbf{F}}(\Omega)$ is non-empty and the following equalities hold

$$\begin{aligned}\inf\left\{\int_{X_\Omega} \mathcal{H}^1(C)|\gamma|(dC) \colon \gamma \in \Gamma_{\mathbf{F}}(\Omega)\right\} &= \inf\left\{\int \rho^0(\sigma) \colon \sigma \in \Sigma_{\mathbf{F}}(\overline{\Omega})\right\} \\ &= \inf\left\{\int \rho^0(\sigma(\gamma)) \colon \gamma \in \Gamma_{\mathbf{F}}(\Omega)\right\} \\ &= \sup\left\{\int \langle \mathbf{F}; u\rangle \colon u \in \mathcal{U}_1(\Omega)\right\}. \qquad (3.9)\end{aligned}$$

Furthermore, the last supremum is achieved.

Remark 3.2. The existence of an optimal γ is ensured if there exists a minimizing sequence $\{\gamma_n\}$ such that $\sup_n \operatorname{var}(\gamma_n) < \infty$. Unfortunately we are unable to show this uniform bound for a reasonable choice of the space X_Ω. In other words, we cannot assert that the minimum in (3.9) is reached for a stress tensor of the form $\sigma(\gamma)$. It seems reasonable to conjecture that it will be the case if X_Ω contains enough curved curves. Notice that in the scalar case, an equivalent version of Theorem 3.1 holds (see Ref. 3) where the infimum is reached taking X_Ω to be the set of all segments.

Remark 3.3. Theorem 3.1 ensures that any load $\mathbf{F} \in \mathcal{M}_0(\overline{\Omega})$ can be equilibrated using a generalized truss tensor of the kind $\sigma(\gamma)$. In fact if $\mathbf{F}$ has a finite support, it is possible to show that the admissible set $\Gamma_{\mathbf{F}}(\overline{\Omega})$ contains atomic measures concentrated on finitely many bars provided Ω is large enough.

On the other hand, it is not restrictive to assume that the assumption (H3) holds only for segments $[x, y]$ whose length is below some given constant l_0. Indeed larger segments in $\overline{\Omega}$ can be decomposed in a finite union of smaller segments $[x_i, y_i]$ such that $|x_i - y_i| \le l_0$ and then $\sigma^{[x,y]} = \sum \sigma^{[x_i,y_i]}$ can be included in the family $\{\sigma(\gamma),\ \gamma \in X_\Omega\}$.

Remark 3.4. A question correlated to Remark 3.3 is to know if the infimum with respect to all admissible γ is the same if we restrict ourselves to those γ which are supported on a finite number of bars. The answer is yes in the case considered in Sec. 4.2 where the optimal γ is approximated using a sequence $\{\gamma_n\}$ whose support consists of n bars (see Sec. 4.2.2).

The proof of Theorem 3.1 is based on the following equicontinuity lemma:

Lemma 3.3. *Let $\{p_n\}$ be a sequence of continuous functions on $\overline{\Omega}\times\overline{\Omega}$ converging uniformly to 0 and let $\{u_n\}\subset C(\overline{\Omega};\mathbb{R}^d)$ such that*

$$|\langle u_n(x)-u_n(y);x-y\rangle+p_n(x,y)\,|x-y||\leq|x-y|^2\quad\forall\,(x,y)\in\overline{\Omega}\times\overline{\Omega}. \tag{3.10}$$

Then there a sequence of integers $\{i_k\}_{k\geq 1}$ and a sequence of rigid motions $\{r_{i_k}\}_{k\geq 1}$ such that the family $\{u_{i_k}-r_{i_k}\}_{k\geq 1}$ converges uniformly on every compact subset of $\overline{\Omega}$. Furthermore all cluster points w belong to $\mathcal{U}_1(\Omega)$.

Proof. We assume without loss of generality that $|p_n|\leq 1$ for all n.

Step 1. We first assume that Ω is bounded. Let $\{a_1,a_2,\ldots,a_{d+1}\}$ be a set of affinely independent points in Ω. We may assume without loss of generality that a_1 is the origin. First, we check that there exists a sequence of rigid motions $\{r_n\}$ such that the sequences $\{u_n(a_i)-r_n(a_i)\}$ are bounded for all i in $\{1,\ldots,d+1\}$. For instance, $x\to r_n^1(x)\equiv u_n(a_1)$ are rigid motion such that $\{u_n(a_1)-r_n^1(a_1)\}=\{0\}$. We use now an induction argument : let $p<d+1$ and assume that there exists a sequence of rigid motions $\{r_n^p\}$ and a real M such that $\|u_n(a_i)-r_n^p(a_i)\|<M$ for all n and all i in $\{1,\ldots,p\}$. Assumption (3.10) implies, for any i in $\{1,2,\ldots,p\}$ and for n large enough,

$$|\langle(u_n-r_n^p)(a_{p+1});t_i\rangle|\leq D_\Omega+M+1, \tag{3.11}$$

where D_Ω is the diameter of Ω and $t_i:=\frac{a_{p+1}-a_i}{\|a_{p+1}-a_i\|}$. As the points are affinely independent the map l_p defined by $l_p(w):=(\langle w;t_1\rangle,\langle w;t_2\rangle,\ldots,\langle w;t_p\rangle)$ is an invertible linear map from the vectorial space V_p associated to the points $(a_1,a_2,\ldots,a_{p+1})$ into $\mathbb{R}^p$. By (3.11), $(u_n-r_n^p)(a_{p+1})$ has a bounded projection on V_p. If $p=d$, we can conclude the proof of this first step. Otherwise, let a'_{p+1} be the orthogonal projection of a_{p+1} on V_{p-1}, the vectorial space spanned by $a_1,a_2,\ldots,a_p$. Set $\delta=|a'_{p+1}-a_{p+1}|>0$ and the unit vector $t=(a'_{p+1}-a_{p+1})/\delta$. let us consider the projection w_n^p of $(u_n-r_n^p)(a_{p+1})$ onto the orthogonal of V_p. Observe that for $i\leq p$, the vector $a'_{p+1}-a_i$ is orthogonal to t and w_n^p which are also orthogonal to each other. Define a rigid motion ρ_n^p by

$$\rho_n^p(x):=\frac{1}{\delta}(w_n^p\otimes t-t\otimes w_n^p)(x-a'_{p+1}).$$

Note that $\rho_n^p(a_i)=0$ for all $i\leq p$ while $\rho_n^p(a_{p+1})=w_n^p$. Setting $r_n^{p+1}:=r_n^p+\rho_n^p$, we obtain that the sequences $\{\|u_n(a_i)-r_n^{p+1}(a_i)\|\}_n$ are bounded for any i in $\{1,\ldots,p+1\}$. This complete the induction argument if we set $r_n:=r_n^{d+1}$.

Now let x_0 be any point in $\overline{\Omega}$. We can choose d points in the set $\{a_1,a_2,\ldots,a_{d+1}\}$ which are affinely independent of x_0. We assume without loss of generality that $\{a_1,a_2,\ldots,a_d,x_0\}$ is such a family. For $\varepsilon=\varepsilon(x_0)>0$ small enough $\{a_1,a_2,\ldots,a_d,x\}$ remains an affinely independent family for any x in a ball $B(x_0,\varepsilon)$. Setting now $t_i(x)=\frac{x-a_i}{\|x-a_i\|}$, the assumption (3.10) and the fact that there exists M such that, for any n and any $i \in \{1, \dots, d\}$, $\|(u_n - r_n)(a_i)\| < M$ imply that for $x \in B(x_0, \varepsilon)$,

$$|\langle (u_n - r_n)(x); t_i(x)\rangle| \leq D_\Omega + M + 1. \tag{3.12}$$

Defining the linear map ℓ_x by $\ell_x(w) := (\langle w; t_1(x)\rangle, \langle w; t_2(x)\rangle, \dots, \langle w; t_d(x)\rangle)$, we get

$$\|\ell_x((u_n - r_n)(x))\| \leq d(D_\Omega + M + 1).$$

The map ℓ_x is invertible for any $x \in K := \overline{B}(x_0, \varepsilon/2)$ and is a continuous function of x. As $C := \overline{B}(0, d(D_\Omega + M + 1)) \subset \mathbb{R}^d$ is compact, the set $\cup_{x\in K} \ell_x^{-1}(C)$ is bounded and by (3.12), $u_n - r_n$ is uniformly bounded on K. Since $\overline{\Omega} \subset \cup_{x\in\overline{\Omega}} B(x, \varepsilon(x)/2)$ and $\overline{\Omega}$ is compact, we obtain that $u_n - r_n$ is uniformly bounded on $\overline{\Omega}$.

Let us now prove that the sequence $\{u_n - r_n\}$ is equicontinuous on $\overline{\Omega}$. Assume on the contrary that there exists a real $a > 0$ and two sequences $\{x_n\}_n, \{y_n\}_n \subset \overline{\Omega}$ such that $d_n := |y_n - x_n|$ converges to 0 while

$$\|(u_n - r_n)(x_n) - (u_n - r_n)(y_n)\| > a. \tag{3.13}$$

Set $\tau_n := (y_n - x_n)/d_n$ and define

$$v_n(x) := (u_n - r_n)(x) - (u_n - r_n)(x_n), \quad x \in \Omega.$$

As Ω is convex, $\overline{\Omega}$ satisfies the interior cone condition. In particular, there exists $\delta_\Omega, K_\Omega > 0$ such that

$$\forall\, x \in \overline{\Omega}, \forall\, v \in \mathbb{R}^d, \quad \mathcal{C}(\delta_\Omega, x) \neq \emptyset, \quad \frac{|v|}{K_\Omega} \leq \sup_{k\in\mathcal{C}(\delta_\Omega,x)} |\langle v; k\rangle|. \tag{3.14}$$

Here,

$$\mathcal{C}(\delta_\Omega, x) := \{k \in \mathbb{R}^d : \ |k| = 1, [x, x + \delta_\Omega k] \subset \overline{\Omega}\}.$$

As $\mathcal{C}(\delta_\Omega, x)$ is a compact subset of $\mathbb{R}^d$, the supremum in (3.14) is attained for some vector $k_\Omega(x, v)$. We set

$$k_n = k_\Omega(x_n, v_n(y_n)), \quad z_n = x_n + \sqrt{d_n}\, k_n.$$

For n large enough, $\sqrt{d_n} < \delta_\Omega$ and so, $z_n \in \overline{\Omega}$. Choose ϵ positive such that $8K_\Omega\epsilon < a$. We apply (3.10) to $(x, y) = (y_n, z_n)$ for n large enough so that $|p_n(y_n, z_n)| \leq \epsilon$ to obtain that

$$|\langle v_n(z_n) - v_n(y_n); -d_n\tau_n + \sqrt{d_n}\, k_n\rangle| \leq |-d_n\tau_n + \sqrt{d_n}\, k_n|(|-d_n\tau_n + \sqrt{d_n}\, k_n| + \epsilon)$$

and so, using that $\{v_n\}$ is uniformly bounded, we have

$$|\langle v_n(z_n) - v_n(y_n); k_n\rangle| \leq 3\epsilon \tag{3.15}$$

for n large enough. Similarly, applying (3.10) to $(x, y) = (x_n, z_n)$, using that $v_n(x_n) = 0$, we obtain for n large enough,

$$|\langle v_n(z_n); k_n\rangle| \leq 3\epsilon. \tag{3.16}$$

By definition of k_n,

$$|v_n(y_n)| \leq K_\Omega |\langle v_n(y_n); k_n\rangle| \leq 6\epsilon K_\Omega < a. \tag{3.17}$$

To obtain the first inequality in (3.17), we have used the definition of k_n, whereas, the second inequality is a direct consequence of (3.15) and (3.16). Since (3.17) is at a variance with (3.13), we conclude that $\{u_n - r_n\}$ is equicontinuous on $\overline{\Omega}$. By Ascoli–Arzela theorem, $\{u_n - r_n\}$ is strongly relatively compact in $C(\overline{\Omega}, \mathbb{R}^d)$. One can readily check that every cluster point w of $\{u_n - r_n\}$ will satisfy $\|w\|_\Omega^* \leq 1$.

Step 2. To obtain the conclusions in step 1, we have used that Ω is a bounded set. Assume next that Ω is not bounded. For each integer $k \geq 1$ we define $\Omega_k := \Omega \cap B_k(0)$ where $B_k(0)$ is the open ball of radius k, centered at the origin. Note that Ω_k is a convex bounded set. For k large enough, $\Omega_k \neq \emptyset$. We assume without loss of generality that $\Omega_1 \neq \emptyset$. By step 1, we may find $\{u_n^1\}_n$, a subsequence of $\{u_n\}_n$ and $\{r_n^1\}_n$, a sequence of rigid motions such that $\{u_n^1 - r_n^1\}_n$ converges (uniformly) in $C(\overline{\Omega}_1)$ to some w^1. Suppose that we have inductively found sequences of rigid motions $\{r_n^1\}_n, \ldots, \{r_n^k\}_n$ and sequences $\{u_n^1\}_n, \ldots, \{u_n^k\}_n$ such that $\{u_n^{i-1}\}_n$ is a subsequence of $\{u_n^i\}_n$ for $i = 1, \ldots, k$. Here, we have set $u_n^0 := u_n$. Suppose that $\{u_n^i - r_n^i\}_n$ converges in $C(\overline{\Omega}_i)$ to some w^i and $w^{i-1} = w^i$ on $\overline{\Omega}_{i-1}$ for $i = 1, \ldots, k$. By step 1, there exists $\{u_{j_n}^k\}_n$ a subsequence of $\{u_n^k\}_n$ and a sequence of rigid motions, $\{\rho_n^{k+1}\}_n$ such that $\{u_n^{k+1} - \rho_n^{k+1}\}_n$ convergent in $C(\overline{\Omega}_{k+1})$ to some $\tilde{w}^{k+1}$. Here, we have set $u_n^{k+1} := u_{j_n}^k$. On Ω_k, we have the following uniform convergence:

$$\tilde{w}^{k+1} = \lim_{n\to\infty}\left(u_n^{k+1} - \rho_n^{k+1}\right) = \lim_{n\to\infty}\left(u_{j_n}^k - r_{j_n}^k + r_{j_n}^k - \rho_n^{k+1}\right) = w^k + \lim_{n\to\infty}\left(r_{j_n}^k - \rho_n^{k+1}\right).$$

Hence, the sequence $\{r_{j_n}^k - \rho_n^{k+1}\}_n$ converges uniformly on $\overline{\Omega}_k$. Since the $r_{j_n}^k - \rho_n^{k+1}$'s are rigid motions, this implies that the following convergence is uniform on every compact subset of $\mathbb{R}^d$:

$$\tilde{r}^k =: \lim_{n\to\infty}\left(r_{j_n}^k - \rho_n^{k+1}\right). \tag{3.18}$$

Set

$$r_n^{k+1} := \rho_n^{k+1} + \tilde{r}^k, \quad \text{and} \quad w^{k+1} := \tilde{w}^{k+1} - \tilde{r}^k.$$

Then w^k and w^{k+1} coincide on $\overline{\Omega}_k$ and we have the following uniform convergence on $\overline{\Omega}_{k+1}$:

$$w^{k+1} = \lim_{n\to\infty}\left(u_n^{k+1} - r_n^{k+1}\right).$$

This way, we have constructed inductively w^k for all integers $k \geq 1$. The function w defined by $w(x) = w^k(x)$ is well defined on $\mathbb{R}^d$. Furthermore, $\{u_n^k - r_n^k\}_n$ converges uniformly to w on $\overline{\Omega}_k$. Hence, we can find an increasing sequence $\{n_k\}_{k\geq 1}$ such that

$$\|u_{n_k}^k - r_{n_k}^k - w\|_{L^\infty(\overline{\Omega}_k)} < \frac{1}{k}. \tag{3.19}$$

Observe that $\{u_{n_k}^k\}_{k\geq 1}$ is a subsequence of $\{u_n\}_{n\geq 1}$ and $\{r_{n_k}^k\}_{k\geq 1}$ is a sequence of rigid motion. By (3.19), $\{u_{n_k}^k - r_{n_k}^k\}_{k\geq 1}$ converges uniformly to w on any compact subset of $\mathbb{R}^d$. Clearly, $w \in \mathcal{U}_1(\Omega)$. □

Proof of Theorem 3.1. We introduce, for every $p \in C_0(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)$, the following perturbation of dual problem (2.24):

$$h(p) := \inf \left\{ - \int \langle \mathbf{F}; u \rangle \ : \ u \in \mathcal{U}(p) \right\}.$$

Here,

$$\begin{aligned} \mathcal{U}(p) = \{ & u \in C(\overline{\Omega}; \mathbb{R}^d), |\langle u(x) - u(y); x - y \rangle + p(x,y)|x-y|| \\ & \leq |x-y|^2 \ \forall \ x \neq y \in \overline{\Omega} \}. \end{aligned}$$

The function h is convex and $-h(0)$ coincides with the supremum of (2.24). Let us assume for a moment that

$$h(0) > -\infty \quad \text{and} \quad h \text{ is lower semicontinuous at } 0. \tag{3.20}$$

Then the biconjugate h^{**} of h in the duality of $C_0(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)$ with $\mathcal{M}(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)$ satisfies:

$$h^{**}(0) = h(0) \ = - \inf h^*. \tag{3.21}$$

Let us evaluate $h^*(\mu)$ where μ is an element of $\mathcal{M}(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)$. We have:

$$h^*(\mu) = \sup\{L(u,p) \colon u \in C(\overline{\Omega}; \mathbb{R}^d) \ , \ p \in C_0(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)\},$$

where

$$L(u,p) := \begin{cases} \displaystyle\int_{\overline{\Omega} \times \overline{\Omega} \backslash \Delta} \langle \mu; p \rangle + \int_{\overline{\Omega}} \langle \mathbf{F}; u \rangle & \text{if } u \in \mathcal{U}(p), \\ -\infty & \text{otherwise,} \end{cases}$$

Set q to be defined on $\overline{\Omega} \times \overline{\Omega} \backslash \Delta$ by

$$q := \Pi u + p, \quad (\Pi u)(x,y) := \left\langle u(x) - u(y); \frac{x-y}{|x-y|} \right\rangle.$$

We may rewrite the Lagrangian $L(u,p)$ in terms of (u,q) as follows:

$$L(u,p) \ = \tilde{L}(u,q) \ = \ \begin{cases} \displaystyle\int_{\overline{\Omega} \times \overline{\Omega} \backslash \Delta} \langle \mu; q - \Pi u \rangle + \int \langle \mathbf{F}; u \rangle & \text{if } q \in \mathcal{Q} \\ -\infty & \text{otherwise.} \end{cases}$$

Here, $\mathcal{Q}$ is the set of $q \in C_0(\overline{\Omega} \times \overline{\Omega} \backslash \Delta)$ such that $|q(x,y)| \leq |x-y|$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega} \backslash \Delta$. We notice that the linear operator Π is continuous from $C_0(\overline{\Omega}; \mathbb{R}^d)$ to

$C_0(\overline{\Omega}\times\overline{\Omega}\backslash\Delta)$ (with a norm less than 2). Denote by Π^* the adjoint operator, so that $\Pi^*\mu = \mathbf{F}$ is equivalent to

$$\int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} \Pi^*\mu(x,y)\mu(dx,dy) = \int \langle \mathbf{F}; u\rangle \quad \forall\, u \in C(\overline{\Omega};\mathbb{R}^d).$$

We find that

$$\begin{aligned} h^*(\mu) &= \sup\{\ \tilde{L}(u,q) : u \in C(\overline{\Omega};\mathbb{R}^d),\ q \in C_0(\overline{\Omega}\times\overline{\Omega}\backslash\Delta)\} \\ &= \begin{cases} \displaystyle\int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} |x-y|\,|\mu|(dxdy) & \text{if } \Pi^*\mu = \mathbf{F}, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.22}$$

Let us show that

$$\inf\left\{\int \rho^0(\sigma(\gamma))\ :\ \sigma \in \Gamma_{\mathbf{F}}(\Omega)\right\}\ \le\ \inf h^*. \tag{3.23}$$

Since $h(0)$ is finite, the convex function h never assumes the value $-\infty$. We use this and (3.20) to conclude that $h^* \not\equiv \infty$. Let μ be a Radon measure on $\overline{\Omega}\times\overline{\Omega}\backslash\Delta$ such that $h^*(\mu) < +\infty$. Then as $\Pi^*(\mu) = \mathbf{F}$, we may then associate a measure γ on $\mathbf{X}_\Omega$ by setting

$$\int_{\mathbf{X}_\Omega} \langle \gamma; \Psi\rangle := \int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} \Psi([x,y])\,\mu(dxdy), \tag{3.24}$$

for every bounded continuous function Ψ on X_Ω. Notice that by (H3), the function $(x,y) \mapsto \Psi([x,y])$ is bounded continuous on $\overline{\Omega}\times\overline{\Omega}\backslash\Delta$. By (3.24), $\mathrm{var}(\gamma) \le \mathrm{var}(\mu)$ and recalling (3.8), we have:

$$\int \rho^0(\sigma(\gamma))\ \le\ \int_{\mathbf{X}_\Omega} \mathcal{H}^1(C)|\gamma|(dC) = \int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} |x-y||\mu|(dxdy). \tag{3.25}$$

We first use the definition of $\sigma(\gamma)$, (3.5), then we use (3.24) and eventually (2.5) to obtain for every $u \in C^1(\overline{\Omega})$,

$$\begin{aligned} \int \langle \sigma(\gamma); \nabla u\rangle &= \int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} \left(\int \langle \sigma^{[x,y]}; \nabla u\rangle\right)\mu(dx\,dy) \\ &= \int_{\overline{\Omega}\times\overline{\Omega}\backslash\Delta} \Pi u(x,y)\,\mu(dx\,dy) \\ &= \int \langle \mathbf{F}; u\rangle. \end{aligned}$$

We have used that $\Pi^*(\mu) = \mathbf{F}$ to obtain that last equality. Therefore the measure γ defined above, belongs to $\Gamma_{\mathbf{F}}(\Omega)$. This proves that $\Gamma_{\mathbf{F}}(\Omega) \neq \emptyset$. Furthermore, (3.22) and (3.25) yield (3.23). Then by taking into account (3.21), (3.22), we deduce that

$$\begin{aligned} \inf\left\{\int \rho^0(\sigma) : \sigma \in \Sigma_{\mathbf{F}}\right\} &\le \inf\left\{\int \rho^0(\sigma(\gamma)) : \sigma(\gamma) \in \Gamma_{\mathbf{F}}(\Omega)\right\} \\ &\le \inf h^* = -h(0) = \sup_u\left\{\int \langle u; \mathbf{F}\rangle\ :\ u \in \mathcal{U}_1\right\}. \end{aligned} \tag{3.26}$$

Since we already know by (2.25) that the reverse inequality holds, we obtain that all quantities above are finite and equal.

To finish the proof of Theorem 3.1, we still have to show (3.20) and that the supremum in (2.24) is achieved. Both facts are a consequence of Lemma 3.3. Let $p_n \in C_0(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$ such that $p_n \to 0$ uniformly. To prove the lower semicontinuity of h at 0, we may assume, after extracting a suitable subsequence, that $h(p_n)$ does converge to some $\alpha \in [-\infty, +\infty)$. Then we can choose $\alpha_n < h(p_n)$ and u_n so that (3.10) holds, $\alpha_n \to \alpha$ and $-\int \langle \mathbf{F}; u_n \rangle \leq \alpha_n$. By Lemma 3.3, there exists a sequence of rigid motion $\{\mathbf{r}_n\}$ such that, possibly passing to a subsequence, we have $u_n - r_n \to u$ uniformly on compact subsets where u belongs to $\mathcal{U}_1(\Omega)$. Since F is equilibrated and compactly supported, it follows that

$$\int \langle \mathbf{F}; u_n \rangle = \int \langle \mathbf{F}; u_n - r_n \rangle \to \int \langle \mathbf{F}, u \rangle.$$

Therefore $h(0) \leq -\int \langle \mathbf{F}, u \rangle \leq \alpha$. This proves the lower semicontinuity at 0. Applying the same argument with $p_n = 0$ for all n (thus $\alpha = h(0)$), we obtain the existence of $u \in \mathcal{U}_1(\Omega)$ such that $h(0) = -\int \langle \mathbf{F}; u \rangle$. Thus $h(0)$ is finite and the supremum of (2.24) is reached at u. □

3.4. *Optimality conditions*

Let u be an element of $\mathcal{U}_1(\Omega)$. It is shown in Lemma 3.4 below that, if C is a curve in space X_Ω^∞, then the restriction of u to C has a tangential component $u_\tau = u \cdot \mathbf{t}_C$. The function u_τ is Lipschitz with respect to the curvilinear abscissa s, whereas the orthogonal component u_ν is continuous. We then define the *tangential strain* $e^C(u)$ to be

$$e^C(u) := \dot{u}_\tau - \kappa(s)\, u_\nu, \tag{3.27}$$

where $\kappa(s)$ the curvature is a bounded measurable function. Then $e_C(u)$ is well-defined $\mathcal{H}^1$ a.e. along C, independent of the orientation of C and it coincides with $\langle e(u), \mathbf{t}_C \otimes \mathbf{t}_C \rangle$ at every point of differentiability of u.

Lemma 3.4. *Let $u \in \mathcal{U}_1(\Omega)$. Then for every $C^{1,1}$-curve C in $\overline{\Omega}$, the restriction to C of the tangential component u_τ is Lipschitz continuous and $e^C(u)$ given by* (3.27) *satisfies $|e^C(u)| \leq 1$ $\mathcal{H}^1$-a.e in C.*

Furthermore, there holds

$$\langle -\mathrm{div}\sigma^C; u \rangle = \int_C e^C(u)\, d\mathcal{H}^1.$$

In particular, we have $|\langle \mathrm{div}\sigma^C; u \rangle| \leq \mathcal{H}^1(C)$ with an equality if and only if $e^C(u) = \pm 1$ a.e. on C.

Proof. To prove that u_τ is Lipschitz on C, we observe that, if C is parametrized by the arclength as $\mathbf{r}(s)$ and if K is an upper bound for the curvature $|\ddot{\mathbf{r}}(s)|$, then the tangent vector $\tau(s) = \dot{\mathbf{r}}(s)$ is such that, for every $s \in [s_1, s_2]$:

$$\left|\tau(s) - \frac{\mathbf{r}(s_2) - \mathbf{r}(s_1)}{s_2 - s_1}\right| \;\le\; \frac{K}{2}\,|s_2 - s_1|.$$

Therefore $\left|\frac{\mathbf{r}(s_2)-\mathbf{r}(s_1)}{s_2-s_1}\right| \le 1 + \frac{K}{2}$ and taking into account the fact that $u \in \mathcal{U}_1(\Omega)$, we obtain

$$\begin{aligned}
|\langle u(\mathbf{r}(s_2)); \tau(s_2)\rangle - \langle u(\mathbf{r}(s_1)); \tau(s_1)\rangle| &\le \left|\left\langle u(\mathbf{r}(s_2) - u(\mathbf{r}(s_1)); \frac{\mathbf{r}(s_2) - \mathbf{r}(s_1)}{s_2 - s_1}\right\rangle\right| \\
&\quad + K\|u\|_{L^\infty}|s_2 - s_1| \\
&\le (1 + K\|u\|_{L^\infty(C)})|s_2 - s_1|. \qquad (3.28)
\end{aligned}$$

Let $\{u_n)\} \subset \mathcal{U}_1(\Omega)$ be the smooth approximation sequence defined in Lemma 2.1. Then by (3.28), the sequence of scalar functions $u_n \cdot \mathbf{t}_C$ is equi-Lipschitz on C and converges weakly to u_τ in $W^{1,\infty}(C)$. Therefore, by (3.27), $e^C(u_n)$ as an element of $L^\infty(C, \mathcal{H}^1)$ converges weakly-star to $e^C(u)$. In particular, as u_n belongs to $\mathcal{U}_1(\Omega)$, we have that $|e^C(u_n)| = |\langle \nabla u_n, \mathbf{t}_C \otimes \mathbf{t}_C\rangle| \le 1$ and, passing to the limit, we deduce that $|e^C(u)| \le 1$ $\mathcal{H}^1$-a.e in C. Furthermore, by the uniform convergence of u_n, we have

$$\langle -\operatorname{div}\sigma^C; u\rangle = \lim_{n\to\infty} \langle -\operatorname{div}\sigma^C; u_n\rangle = \lim_{n\to\infty} \int_C e^C(u_n)\, d\mathcal{H}^1 = \int_C e^C(u)\, d\mathcal{H}^1.$$

The last statement of the theorem follows easily from the fact that $e^C(u) \le 1$ a.e. □

Theorem 3.2. (i) *A pair $(u, \gamma) \in \mathcal{U}_1(\Omega) \times \Gamma_{\mathbf{F}}(\Omega)$ is optimal for* (3.9) *if and only if the following equalities hold*

$$\begin{aligned}
&e^C(u) = 1 \quad \mathcal{H}^1\text{-a.e. for all } C \in \operatorname{spt}\gamma^+, \quad e^C(u) = -1 \\
&\qquad \mathcal{H}^1\text{-a.e. for all } C \in \operatorname{spt}\gamma^- \qquad (3.29)
\end{aligned}$$

(ii) *Let u_0 be a particular maximizer in* (3.9) *and let x_0 belong to an open subset where u_0 is of class C^1. Then every curve C in the support of an optimal γ passing through x_0 is such that $\mathbf{t}_C(x_0)$ is an eigenvector of $e(u)(x_0)$ associated with the eigenvalue* 1 *if $C \in \operatorname{spt}\gamma^+$ and -1 if $C \in \operatorname{spt}\gamma^-$. In particular, two curves respectively in $\operatorname{spt}\gamma^+$ and $\operatorname{spt}\gamma^-$ passing through x_0 are orthogonal.*

Proof. Owing to Theorem 3.1, a pair $(u, \gamma) \in \mathcal{U}_1(\Omega) \times \Gamma_{\mathbf{F}}(\Omega)$ with $\sigma(\gamma) \in \Sigma_{\mathbf{F}}(\overline{\Omega})$ is optimal if and only if the following equality holds

$$\langle \mathbf{F}, u\rangle = \int_{X_\Omega} \mathcal{H}^1(C)\,|\gamma|(dC). \qquad (3.30)$$

By Lemma 3.4, exploiting (3.5), we have:

$$\langle \mathbf{F}, u\rangle = \langle -\operatorname{div}\sigma(\gamma); u\rangle = \int_{X_\Omega} \left(\int_C e^C(u)\, d\mathcal{H}^1\right) \gamma(dC).$$

Therefore, (3.30) can be rewritten as

$$\int_{X_\Omega}\left(\int_C (e^C(u)-1)\,d\mathcal{H}^1\right)\gamma^+(dC)+\int_{X_\Omega}\left(\int_C (e^C(u)+1)\,d\mathcal{H}^1\right)\gamma^-(dC)=0$$

The assertion (i) follows since $u\in\mathcal{U}_1(\Omega)$ and thus satisfies $|e^C(u)|\le 1$ a.e.

To prove the assertion (ii), it is enough to note that $e^C(u_0)$ is continuous on a neighborhood V of x_0 and therefore constant equal to 1 (resp. -1) in $C\cap V$ where C is any curve is the support of γ^+ (resp. γ^-). Then, if such a curve passes through x_0, we have

$$\langle e(u_0)(x_0);\mathbf{t}_C\otimes\mathbf{t}_C(x_0)\rangle = e^C(u_0)(x_0)=1\quad(\text{resp. }-1),$$

whereas $\rho(e(u_0))(x_0)\le 1$ and $\rho^0(\mathbf{t}_C\otimes\mathbf{t}_C)(x_0)=1$. By Remark 2.2, we infer that $\mathbf{t}_C(x_0)$ is an eigenvector of $e(u_0)(x_0)$ associated with eigenvalue 1 (resp. -1). The conclusion is then straightforward. □

A straightforward application of Theorem 3.2 allows us to recover the optimality criterium obtained in Example 2.1 (where u^o is the identity map). We have

Corollary 3.1. (Optimal trusses in tension or compression) *Let* $\mathbf{F}\in\mathcal{M}_0(\overline{\Omega})$. *Then* $u_0^+(x):=x$ (*resp.* $u_0^-(x):=-x$) *is a solution of* (2.24) *if and only if the admissible set* $\Gamma_{\mathbf{F}}(\Omega)$ *contains a non-negative* (*resp. nonpositive*) *element* γ_0. *In this case* $\sigma(\gamma_0)$ *is a minimizer of* (2.22).

Proof. As $e_C(u_0^+)=1$ and $e_C(u_0^-)=-1$ for every curve C, we need only to apply the optimality conditions (3.29). □

4. Examples of Optimal Structures

4.1. *A structure with all lines of action in tension*

In Example 2.1 we already described optimal trusses with all bars in tension. Let us reformulate one of these examples via measures on the set of curves $\mathbf{X}_{\mathbb{R}^2}$.

In $\mathbb{R}^2$ we consider the points $A(0,-1)$, $B(-\sqrt{3}/2,1/2)$, $C(\sqrt{3}/2,1/2)$ and the balanced system of forces

$$\mathbf{F}=\delta_A A+\delta_B B+\delta_C C.$$

For each $t\in(0,1)$ and $M\in\mathbb{R}^2$ we set $M_t=tM$. We consider the measure $\gamma_t=\gamma_t^1+\gamma_t^2$ on $X_{\mathbb{R}^2}$, where

$$\gamma_t^1=\frac{1}{\sqrt{3}}(\delta_{[A_t,B_t]}+\delta_{[B_t,C_t]}+\delta_{[C_t,A_t]}),\quad \gamma_t^2=\delta_{[A_t,A]}+\delta_{[B_t,B]}+\delta_{[C_t,C]}.$$

We set

$$\gamma_0^1=0=\gamma_1^2,\quad \gamma_0^2=\delta_{[O,A]}+\delta_{[O,B]}+\delta_{[O,C]},\quad \gamma_1^1=\frac{1}{\sqrt{3}}(\delta_{[A,B]}+\delta_{[B,C]}+\delta_{[C,A]}).$$

It is easy to check that, for any $t \in [0,1]$,

$$\begin{aligned}\operatorname{div}(\sigma(\gamma_t)) &= \operatorname{div}\left(\frac{1}{\sqrt{3}}\left(\sigma^{[A_t,B_t]} + \sigma^{[B_t,C_t]} + \sigma^{[C_t,A_t]}\right) + \sigma^{[A_t,A]} + \sigma^{[B_t,B]} + \sigma^{[C_t,C]}\right)\\ &= -F.\end{aligned}$$

Hence $\sigma(\gamma_t)$ belongs to $\Sigma_{\mathbf{F}}(\mathbb{R}^2)$ and, as γ_t is a non-negative measure, Corollary 3.1 ensures that $\sigma(\gamma_t)$ is a minimizer of (2.22). This shows that there are uncountably many optimal Michell trusses in $\Sigma_{\mathbf{F}}(\mathbb{R}^2)$.

There are also uncountably many optimal structures which are not trusses (in the sense that these structures are not one-dimensional). Let us describe them using our formulation: let p be any probability measure on $[0,1]$ and set $\gamma_p := \int_{[0,1]} \gamma_t \, p(dt)$. Corollary 3.1 still apply to the measure γ_p. For this optimal structure, we have $\sigma(\gamma_p)) = \int_{[0,1]} \sigma(\gamma_t) \, p(dt)$. Note that the Hausdorff dimension of this measure can be any real in $[1,2]$ (it is two if, for instance, $p(dt)$ is chosen to be the Lebesgue measure on $[0,1]$).

4.2. *A structure with some lines of action in tension and others in compression*

In $\mathbb{R}^2$ we consider the points $A = (\sqrt{2}, 0)$, $B = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $C = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and the symmetrical points $D = -A$, $E = -B$, $F = -C$. Denoting $e_1 = (1,0)$ and $e_2 = (0,1)$, we consider the equilibrated system of forces

$$\mathbf{F} = \frac{\sqrt{2}}{2}(\alpha + \beta)\ (\delta_A - 2\delta_0 + \delta_D)\ \mathbf{e}_2\ + \frac{\sqrt{2}}{2}(\beta - \alpha)\ (\delta_A - \delta_D)\ \mathbf{e}_1, \tag{4.1}$$

where $\alpha, \beta > 0$ are two positive parameters. In the particular case $\alpha = \beta$, Michell[11] provided a picture of an optimal stress minimizing (2.25). He laid down arguments which guide us to write an analytic description of the optimal structure in $\mathbb{R}^2$ for the system of forces (4.1).

4.2.1. *Existence of optimal structure*

For any $\theta \in [0, 2\pi)$, we consider the radial segment $C_\theta := \big[0, \big(\cos(\theta), \sin(\theta)\big)\big]$. A straightforward computation shows that the Hausdorff distance between two such segments C_θ and $C_{\overline{\theta}}$ is lower than $|\theta - \overline{\theta}|$ and so, $\theta \to C_\theta := \Theta(\theta)$ is Lipschitz. Therefore, it makes sense to define measures on $\mathbf{X} := \mathbf{X}_{\mathbb{R}^2}$ as the push forward of measures on $[0, 2\pi)$. For instance, we can define γ^{up} and γ^{low} on $\mathbf{X}$ by

$$\forall\, \varphi \in C(\mathbf{X}), \quad \int_{\mathbf{X}} \varphi d\gamma^{\mathrm{up}} = \int_{\pi/4}^{3\pi/4} \varphi(C_\theta) d\theta, \quad \int_{\mathbf{X}} \varphi d\gamma^{\mathrm{low}} = \int_{5\pi/4}^{7\pi/4} \varphi(C_\theta) d\theta. \tag{4.2}$$

Let us denote $\widehat{BC}$ the curve $\{(\cos(\theta), \sin(\theta));\ \theta \in [\pi/4, 3\pi/4]\}$ and $\widehat{EF}$ the symmetrical curve $\{(\cos(\theta), \sin(\theta));\ \theta \in [5\pi/4, 7\pi/4]\}$.

We define $\gamma = \gamma^+ - \gamma^-$ where

$$\gamma^+ := \alpha\gamma^{\mathrm{up}} + \beta(\delta_{[D,E]} + \delta_{\widehat{EF}} + \delta_{[F,A]}), \quad \gamma^- := \beta\gamma^{\mathrm{low}} + \alpha(\delta_{[A,B]} + \delta_{\widehat{BC}} + \delta_{[C,D]}).$$

Recall that, when $\mathbf{r} : \ [0,\ell]$ is a parametrization by the arclength of a $C^{1,1}$-curve C, with endpoints $M = \mathbf{r}(0)$, $N = \mathbf{r}(\ell)$, then the tangent vector $t_C = \dot{\mathbf{r}}$ and the curvature vector $k_C = \ddot{\mathbf{r}}$ satisfy the relation

$$-\mathrm{div}(t_C \otimes t_C\, \mathcal{H}^1|_C) = -k_C\, \mathcal{H}^1|_C - t_C\delta_M + t_C\delta_N$$

in the sense of distributions. This enables us to compute $\mathrm{div}(\sigma(\delta_{\widehat{EF}}))$ and $\mathrm{div}(\sigma(\delta_{\widehat{BC}}))$. As the support of the remaining part of γ contains only segments, it is straightforward to check that

$$-\mathrm{div}(\sigma(\gamma)) = \mathbf{F}.$$

At any point $x \in \mathbb{R}^2\setminus\{0\}$ let us introduce the polar coordinates $(\rho(x), \theta(x)) \in (0, +\infty) \times [0, 2\pi)$. Thus $x = \rho(x)(\cos(\theta(x)), \sin(\theta(x)))$. We divide $\mathbb{R}^2\setminus\{0\}$ in four angular sectors

$$\mathcal{S}^{\mathrm{up}} := \left\{x;\ \theta(x) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right)\right\}, \quad \mathcal{S}^{\mathrm{left}} := \left\{x; \theta(x) \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right)\right\},$$

$$\mathcal{S}^{\mathrm{low}} := \left\{x;\ \theta(x) \in \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right)\right\}, \quad \mathcal{S}^{\mathrm{right}} := \left\{x;\ \theta(x) \in \left[\frac{7\pi}{4}, 2\pi\right) \cup \left[0, \frac{\pi}{4}\right)\right\}.$$

We introduce the matrices

$$\tilde{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A(\theta) := \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix},$$

and define $u_o \in \mathcal{U}_1(\mathbb{R}^2)$ by $u_o(0) = 0$ and

$$u_o(x) = \begin{cases} x + (\pi - 2\theta(x))\, J \cdot x, & \text{if } x \in \mathcal{S}^{\mathrm{up}}, \\ -\tilde{J} \cdot x - \frac{\pi}{2} J \cdot x, & \text{if } x \in \mathcal{S}^{\mathrm{left}}, \\ -x + (2\theta(x) - 3\pi)\, J \cdot x, & \text{if } x \in \mathcal{S}^{\mathrm{low}}, \\ \tilde{J} \cdot x + \frac{\pi}{2} J \cdot x, & \text{if } x \in \mathcal{S}^{\mathrm{right}}. \end{cases} \tag{4.3}$$

It is easy to check that u_o is continuous on $\mathbb{R}^2$ while ∇u_o and so $e(u_o)$ are continuous on $\mathbb{R}^2\setminus\{0\}$. The explicit computation of $e(u_o)$ reads

$$e(u_o)(x) = \begin{cases} A(\theta(x)), & \text{if } x \in \mathcal{S}^{\mathrm{up}}, \\ -\tilde{J}, & \text{if } x \in \mathcal{S}^{\mathrm{left}}, \\ -A(\theta(x)), & \text{if } x \in \mathcal{S}^{\mathrm{low}}, \\ \tilde{J}, & \text{if } x \in \mathcal{S}^{\mathrm{right}}. \end{cases} \tag{4.4}$$

Recalling that for any curve $C \in \mathbf{X}$, $e^C(u_o)$ coincides with $\langle e(u_o)\,;\, t_C \otimes t_C\rangle$, it is straightforward to check for any curve $C \in \mathrm{spt}(\gamma^+)$ that $e^C(u_o) = 1$ and for any curve $C \in \mathrm{spt}(\gamma^-)$ that $e^C(u_o) = -1$. Therefore Theorem 3.2 states that the pair (u_o, γ) is optimal.

Moreover, as $u_o(0) = 0$ and $u_o(A) = u_o(D) = \sqrt{2}(1 + \pi/2)e_2$, the value of the optimal cost simply reads

$$\int_{\mathbb{R}^2} \langle u_o; \mathbf{F} \rangle = (2 + \pi)(\alpha + \beta).$$

Remark 4.1. There are several vector fields $\mathbf{a} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|\mathbf{a}| = 1$ and

$$e(u_o) = \mathbf{a} \otimes \mathbf{a} - \mathbf{a}^\perp \otimes \mathbf{a}^\perp. \tag{4.5}$$

Despite the fact that $e(u_o)$ is continuous on $\mathbb{R}^2 \backslash 0$, it is not possible to find a vector field $\mathbf{a}$ which is continuous on the whole set $\mathbb{R}^2 \backslash 0$ such that (4.5) holds.

4.2.2. *Approximation by trusses*

The optimal structure γ described in the previous subsection is made of four bars $[F, A]$, $[A, B]$, $[C, D]$, $[D, E]$, two arcs of circle $\overset{\frown}{EF}$, $\overset{\frown}{BC}$ and two parts which correspond to two-dimensional measures $\sigma(\gamma^{\mathrm{up}})$ and $\sigma(\gamma^{\mathrm{low}})$. In no way can this structure be considered as a truss. In this subsection, we show that it can be considered as the limit of a sequence of trusses. We construct a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of signed measures on $\mathbf{X}$ such that $\sigma(\gamma_n)$ converges to $\sigma(\gamma)$ and is a minimizing sequence for both (2.22) and (3.6). The existence of such a sequence supports, in this particular case, the conjecture that (3.6) is a relaxation of (2.8).

We already quickly described the sequence $\{\gamma_n\}$ in Fig. 3. Let us be more precise. Let $n > 0$ be an integer and $k \in \{0, \ldots, n\}$. We denote M_k the point $M_k := (\cos((\frac{1}{4} + \frac{k}{2n})\pi), \sin((\frac{1}{4} + \frac{k}{2n})\pi)$. We set

$$t_n = 1/\cos\left(\frac{\pi}{4n}\right), \quad s_n = 2\tan\left(\frac{\pi}{4n}\right),$$

and we introduce the signed-measure γ_{up}^n on $\mathbf{X}$ by setting

$$\gamma_{\mathrm{up}}^n = s_n \left(\sum_{k=1}^{n-1} \delta_{[0, M_k]} + \frac{1}{2}\delta_{[0,B]} + \frac{1}{2}\delta_{[0,C]} \right) - t_n \left(\sum_{k=1}^{n} \delta_{[M_{k-1}, M_k]} - \delta_{[A,B]} - \delta_{[C,D]} \right).$$

Considering $\widetilde{\gamma}_{\mathrm{up}}^n$, the measure obtained from γ_{up}^n by a symmetry with respect to the first axis, we set $\gamma^n = \alpha\gamma_{\mathrm{up}}^n - \beta\widetilde{\gamma}_{\mathrm{up}}^n$. Direct computations reveal that $-\mathrm{div}(\sigma(\gamma^n)) = \mathbf{F}$ and

$$\int \rho^0(\sigma^n) = 2(\alpha + \beta) \left(1 + 4n \left(\sin\frac{\pi}{4n} \right)^2 \left(\sin\frac{\pi}{2n} \right)^{-1} \right). \tag{4.6}$$

Letting n tends to $+\infty$ we obtain that

$$\lim_{n \to +\infty} \int \rho^0(\sigma^n) = (\alpha + \beta)(2 + \pi).$$

Hence the cost converges to the optimal cost obtained in the previous subsection.

4.2.3. *Uniqueness of solutions; a special system of hyperbolic PDEs*

Michell, when studying the case $\alpha = \beta$, conjectured that the minimizer is unique. This fact seems to be well accepted in the literature but we are not aware of any work where the proof of uniqueness has been provided. The proof we propose is based on the analysis of an unusual system of hyperbolic PDEs.

In Sec. 4.2.1 we exhibited an optimal stress measure $\sigma(\gamma)$ and an associated displacement field u_o. We use the same notation and introduce a change of variables which make easier the description of $e(u_o)$. For any (s,t) in $\widetilde{\mathcal{S}} := \mathbb{R}\times(0,+\infty)$, we set

$$T(s,t) := \begin{cases} \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{4}+s-t, \dfrac{\pi}{4}+s+t\right), & \text{if } s < -\dfrac{\pi}{4}, \\ t\,(\sin(s),\cos(s)), & \text{if } s \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right], \\ \dfrac{\sqrt{2}}{2}\left(-\dfrac{\pi}{4}+s+t, -\dfrac{\pi}{4}-s+t\right), & \text{if } s > \dfrac{\pi}{4}. \end{cases}$$

This is a piecewise diffeomorphism from $\widetilde{\mathcal{S}}$ onto $\mathcal{S} := \mathcal{S}^{\mathrm{left}} \cup \mathcal{S}^{\mathrm{up}} \cup \mathcal{S}^{\mathrm{right}}$. Note that T maps the set $\widetilde{\mathcal{S}}^{\mathrm{up}} := [-\frac{\pi}{4}, \frac{\pi}{4}] \times (0,+\infty)$ onto $\mathcal{S}^{\mathrm{up}}$ and that, for any $(s,t) \in \widetilde{\mathcal{S}}^{\mathrm{up}}$, we simply have $\theta(T(s,t)) = \frac{\pi}{2} - s$. The Jacobian j of this change of variable is given by $j(s,t) = t$, if $(s,t) \in \widetilde{\mathcal{S}}^{\mathrm{up}}$, $j(s,t) = 1$ otherwise. We associate to any measure ξ on $\mathcal{S}$, the measure $T^{-1\sharp}\xi$ on $\widetilde{\mathcal{S}}$ defined by

$$\forall\, \varphi \in C_c^o(\widetilde{\mathcal{S}}), \qquad \int_{\widetilde{\mathcal{S}}} \varphi\, d(T^{-1\sharp}\xi) = \int_{\mathcal{S}} \varphi \circ T^{-1}\, d\xi.$$

We also introduce the orthonormal basis $(\widetilde{e}_s, \widetilde{e}_t)$:

$$\widetilde{e}_s(s,t) = \frac{1}{j(s,t)}\frac{\partial T}{\partial s}(s,t), \quad \widetilde{e}_t(s,t) = \frac{\partial T}{\partial t}(s,t).$$

Denoting $e_s := \widetilde{e}_s \circ T^{-1}$ and $e_t := \widetilde{e}_t \circ T^{-1}$, it is straightforward to check that, on $\mathcal{S}$,

$$e(u_o) = -e_s \otimes e_s + e_t \otimes e_t.$$

We want to prove that any optimal stress measure σ^o coincides with $\sigma(\gamma)$. Both measures belong to $\Sigma_{\mathbf{F}}$, have finite energy and, since $e(u_0)$ is continuous, they satisfy the localized optimality conditions (2.28), (2.29) (where in (2.29) the orthonormal basis $\{a^i\}$ becomes $\{e_s, e_t\}$). Thus the restriction to $\mathcal{S}$ of the difference $\sigma := \sigma^o - \sigma(\gamma)$ is divergence free and takes the form

$$\sigma = -\lambda_1\, e_s \otimes e_s + \lambda_2\, e_t \otimes e_t,$$

where λ_1 and λ_2 are two (signed) measures on $\mathcal{S}$ with finite variation. In order to prove that λ_1 and λ_2 vanish, let us consider a test function $\varphi \in C_c^\infty(\widetilde{\mathcal{S}})$ and set $v = \varphi \circ T^{-1}\, e_s$ and $w = \varphi \circ T^{-1}\, e_t$. A direct computation gives

$$\nabla v = \left(\frac{1}{j}\frac{\partial \varphi}{\partial s}\right) \circ T^{-1}\, e_s \otimes e_s + \frac{\partial \varphi}{\partial t} \circ T^{-1}\, e_s \otimes e_t - (g\varphi) \circ T^{-1}\, e_t \otimes e_s,$$

where g is the function defined on $\widetilde{\mathcal{S}}$ by $g(s,t) := 1/j\, \partial j/\partial t = 1/t\, \chi_{|\widetilde{\mathcal{S}}^{\mathrm{up}}}$. The function g is discontinuous for $s \in \{-\pi/4, \pi/4\}$ and so is ∇v; but it is easy to check that, if ρ_ϵ is a standard mollifier, the following uniform convergence hold:

$$\langle \nabla v * \rho_\epsilon \,;\, e_s \otimes e_s \rangle \longrightarrow \left(\frac{1}{j}\frac{\partial \varphi}{\partial s}\right) \circ T^{-1}, \quad \langle \nabla v * \rho_\epsilon \,;\, e_t \otimes e_t \rangle \longrightarrow 0.$$

Hence

$$\begin{aligned} 0 &= \int_{\mathcal{S}} \langle v * \rho_\epsilon \,;\, -\mathrm{div}(\sigma) \rangle = \int_{\mathcal{S}} \langle \nabla v * \rho_\epsilon \,;\, \sigma \rangle \\ &= \int_{\mathcal{S}} \langle \nabla v * \rho_\epsilon \,;\, e_s \otimes e_s \rangle d\lambda_1 + \int_{\mathcal{S}} \langle \nabla v * \rho_\epsilon \,;\, e_t \otimes e_t \rangle d\lambda_2, \end{aligned}$$

and passing to the limit

$$0 = \int_{\mathcal{S}} \langle \nabla v \,;\, e_s \otimes e_s \rangle \, d\lambda_1 = \int_{\mathcal{S}} \left(\frac{1}{j}\frac{\partial \varphi}{\partial s}\right) \circ T^{-1} \, d\lambda_1 = \int_{\widetilde{\mathcal{S}}} \frac{1}{j}\frac{\partial \varphi}{\partial s} \, d(T^{-1\sharp}\lambda_1).$$

As this is true for any $\varphi \in C_c^\infty(\widetilde{\mathcal{S}})$, we deduce that there exists a measure ν_1 on $(0, +\infty)$ such that $T^{-1\sharp}\lambda_1(ds, dt) = ds \otimes \nu_1(dt)$. Then $\int_{\mathcal{S}} d|\lambda_1| = \int_{\widetilde{\mathcal{S}}} d|T^{-1\sharp}\lambda_1| < +\infty$ implies $\nu_1 = 0$ and so $\lambda_1 = 0$.

Now, using this first result, and performing similar computations for the test function w, we get

$$\nabla w = \left(\frac{1}{j}\frac{\partial \varphi}{\partial s}\right) \circ T^{-1} \, e_t \otimes e_s + \frac{\partial \varphi}{\partial t} \circ T^{-1} \, e_t \otimes e_t + (g\varphi) \circ T^{-1} \, e_s \otimes e_s$$

and

$$0 = \int_{\mathcal{S}} \langle \nabla w \,;\, e_t \otimes e_t \rangle \, d\lambda_2 = \int_{\widetilde{\mathcal{S}}} \frac{\partial \varphi}{\partial t} \, d(T^{-1\sharp}\lambda_2).$$

We deduce that there exists a measure ν_2 on $\mathbb{R}$ such that $T^{-1\sharp}\lambda_2(ds, dt) = \nu_2(ds) \otimes dt$. Again, the finite total variation of λ_2 implies that ν_2 and so λ_2 vanish.

To conclude, we observe that σ vanishes on $\mathcal{S}$ thus on $\mathbb{R} \times \mathbb{R}^+ \backslash \{0\}$. The same holds true on $\mathbb{R} \times \mathbb{R}^- \backslash 0$ (this by using a symmetrical change of variables). Thus σ is concentrated at 0 which is incompatible with the divergence free condition unless $\sigma = 0$. □

5. Open Problems

The aim of this paper was to give a rigorous fundament to the notion of Michell's lines. These lines are designed to carry the constraints and support the optimal structure; in our framework they are exactly the elements of the support of an optimal (signed) measure γ. The lines in tension correspond to the positive part γ^+ whereas the lines in compression are carried by the negative part γ^-. We strongly believe that our approach could be a useful tool to investigate the properties of optimal structures. However, it is still necessary to prove the existence of a minimizing measure γ for the infimum problem in (3.9). Hereafter we propose some open problems or conjectures.

Problem 5.1. (Existence) *Assume that the load F is supported by a finite number of points. Then we conjecture that an optimal γ exists and that this γ is supported on a subset of $X_\Omega^{l_0,\kappa_0}$ for κ_0 large enough* (recall that $X_\Omega^{l_0,\kappa_0}$ introduced in Sec. 2.2 consists of all smooth curves whose length and curvature are upper bounded respectively by l_0 and κ_0).

A long standing conjecture concerns the relation between the support Y of the load F and the region $\mathcal{R}$ in which an optimal structure lies. As noted in Example 2.2, $\mathcal{R}$ is not included in the convex hull of Y.

Problem 5.2. (Boundedness of optimal structure) *Assume that the design region is all $\mathbb{R}^d$ and that Y is bounded. Then it seems reasonable to conjecture that $\mathcal{R}$ is bounded. Can we more precisely estimate the value R_0 such that the value of*

$$\inf_{\sigma \in \Sigma_{\mathbf{F}}(\overline{B}_R)} \int_{\overline{B}_R} \rho^0[\sigma] \tag{5.1}$$

is independent on R for $R \geq R_0$? Furthermore, in the more ambitious attempt to describe the domain $\mathcal{R}$ in the case d $= 2$, can we characterize a set of finitely many extreme curves surrounding $\mathcal{R}$, i.e. such that $\mathcal{R}$ is contained in the convex hull of these lines?

In contrast with the scalar case of Monge transport problem, almost nothing is known about the regularity of optimal pairs (u^o, σ^o). In the usual case of application, the load $\mathbf{F}$ is concentrated and we expect that the optimal stress measure σ exhibits concentrations as it is shown in Sec. 3. For what concerns the optimal displacement u^o which should be everywhere defined in the design (in general it is not unique), we expect that it is regular at least in the complementary of the lower dimensional subsets where σ^o is concentrated.

Problem 5.3. (Regularity of optimal deformations) *Assume that $\mathbf{F}$ is an equilibrated system of forces with finitely many points of application $\mathbb{M} = \{M_1, M_2, \ldots, M_k\} \subset \Omega \subset \mathbb{R}^d$. Let u^o be a maximizer of $u \to \int < \mathbf{F}; u >$ over $\mathcal{U}_1(\Omega)$. Can we predict in terms of $\mathbb{M}$ the location of the subset where u^o is not differentiable? or of the subset where the symmetric tensor $e(u^o)$ is discontinuous?*

The last question concerns, in the case $d = 2$, the subset of the optimal structure $\mathcal{R}$ where the strain tensor $e(u_0)$ has two eigenvalues 1 and -1. This sub-region $\mathcal{R}'$ plays an important role as it is the one where the Michell's lines make a Hencky–Prandl net: recall that, by the assertion (ii) in Theorem 3.2, two curves respectively in sp γ^+ and sp γ^+ passing trough a point $x \in \mathcal{R}'$ are orthogonal.

Problem 5.4. *Assume that $\mathbf{F}$ is an equilibrated system of forces with finitely many points of application $\mathbb{M} = \{M_1, M_2, \ldots, M_k\} \subset \mathbb{R}^2$. How can we characterize the set $\mathcal{R}'$ associated with a maximizer u^o of $u \to \int \langle \mathbf{F}; u\rangle$ over $\mathcal{U}_1(\Omega)$? Can we state that the curvature of Michell's lines vanishes away from $\mathcal{R}'$, i.e. that $\kappa(s) = 0$ a.e. on $C \backslash \mathcal{R}'$ fo all curves C in the support of an optimal measure γ?*

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References

1. G. Alberti and L. Ambrosio, A geometric approach to monotone functions in $\mathbb{R}^d$, *Math. Z.* **230** (1999) 259–316.
2. J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization* (Springer-Verlag, 2000).
3. G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge–Kantorovich equation, *J. Eur. Math. Soc.* **3** (2001) 139–168.
4. G. Bouchitté and M. Valadier, Integral representation of convex functionals on a space of measure, *J. Funct. Anal.* **80** (1988) 398–420.
5. G. Bouchitté, G. Buttazzo and P. Seppecher, Shape optimization solutions via Monge–Kantorovich equation, *C. R. Acad. Sci. Paris* **324** (1997) 1185–1191.
6. G. Bouchitté, G. Buttazzo and P. Seppecher, Energies with respect to a measure and applications to low dimensional structures, *Calc. Var.* **5** (1997) 37–54.
7. C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lect. Notes Math., Vol. 580 (Springer, 1977).
8. F. Demengel, Déplacements à déformations bornées et champs de contrainte mesures, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12** (1985) 243–318.
9. G. A. Hegemier and W. Prager, On Michell trusses, *Int. J. Mech. Sci.* **11** (1969) 209–215.
10. W. S. Hemp, *Optimum Structures* (Oxford Univ. Press, 1973).
11. A. G. Michell, The limits of economy of material in framed-structures, *Phil. Mag. S. 6* **8** (1904) 589–597.
12. J. Malek, J. Necas, M. Rokyta and M. Ruzicka, *Weak and Measured-Valued Solutions to Evolutionary PDEs*, Applied Mathematics and Mathematical Computation, Vol. 13 (Chapman & Hall, 1996).
13. G. Strang and R. Kohn, Hencky–Prandtl nets and constrained Michell trusses, *Comp. Meth. Appl. Mech. Engrg.* **36** (1983) 207–222.