Hamiltonian ODE's in the Wasserstein space of probability measures

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Abstract

In this paper we consider a Hamiltonian H on $\mathscr{P}_2(\mathbf{R}^{2d})$, the set of probability measures with finite quadratic moments on the phase space $\mathbf{R}^{2d} = \mathbf{R}^d \times \mathbf{R}^d$, which is a metric space when endowed with the Wasserstein distance W_2. We study the initial value problem $d\mu_t/dt + \nabla \cdot (\mathbb{J}_d \mathbf{v}_t \mu_t) = 0$, where $\mathbb{J}_d$ is the canonical symplectic matrix, μ_0 is prescribed, $\mathbf{v}_t$ is a tangent vector to $\mathscr{P}_2(\mathbf{R}^{2d})$ at μ_t, and belongs to $\partial H(\mu_t)$, the subdifferential of H at μ_t. Two methods for constructing solutions of the evolutive system are provided. The first one concerns only the case where μ_0 is absolutely continuous. It ensures that μ_t remains absolutely continuous and $\mathbf{v}_t = \nabla H(\mu_t)$ is the element of minimal norm in $\partial H(\mu_t)$. The second method handles any initial measure μ_0. If we furthermore assume that H is λ–convex, proper and lower semicontinuous on $\mathscr{P}_2(\mathbf{R}^{2d})$, we prove that the Hamiltonian is preserved along any solution of our evolutive system: $H(\mu_t) = H(\mu_0)$. © 2000 Wiley Periodicals, Inc.

1 Introduction

In the last few years there has been a considerable interest in the theory of gradient flows in the Wasserstein space $\mathscr{P}_2(\mathbf{R}^D)$ of probability measures with finite quadratic moments in $\mathbf{R}^D$, starting from the fundamental papers [35], [43], with several applications ranging from rates of convergence to equilibrium to the proof of functional and geometric inequalities. In particular, in [4] (see also [13]), a systematic theory of these gradient flows is built, providing existence and uniqueness results, contraction estimates and error estimates for the implicit Euler scheme.

In this paper, motivated by a work in progress by Gangbo & Pacini [31], we propose a rigorous theory concerning evolution problems in $\mathscr{P}_2(\mathbf{R}^D)$ of Hamiltonian type. Here typically $D = 2d$ and the measures we are dealing with are defined in the phase space. As shown in Section 8, our study covers a large class of systems which have recently generated a lot of interest, including the Vlasov-Poisson in one space dimension [9] [47], the Vlasov-Monge-Ampère [12] [18] and the semigeostrophic systems [10] [16] [17] [19] [18] [23] [20] [21] [22] [40].

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We note that a general theory of Hamiltonian ODE's for non-smooth Hamiltonian H, in particular when H is only convex, seems to be completely understood only in finite-dimensional spaces, and even in these spaces the uniqueness question has been settled only in very recent times, see Remark 6.5. In infinite-dimensional Hilbert spaces very little appears to be known at the level of existence of solutions, and nothing is known at the level of uniqueness.

Besides its comprehensive character, another nice feature of our theory is its ability to handle singular initial data and singular solutions. This class of solutions is natural, for instance, to include solutions (e.g. those generated by classical non-kinetic solutions) with one or finitely many velocities, see [47] for a first result in this direction. At the same time, there is the possibility to handle discrete and continuous models with the same formalism, and to show stability results (the first one in this direction, for two specific models, is [18]).

We recall that $\mathscr{P}_2(\mathbf{R}^D)$ is canonically endowed with the Wasserstein distance W_2, defined as follows:

$$W_2^2(\mu,\nu) := \min_{\gamma}\left\{\int_{\mathbf{R}^D\times\mathbf{R}^D}|x-y|^2 d\gamma(x,y) :\ \gamma\in\Gamma(\mu,\nu)\right\}. \tag{1.1}$$

Here $\Gamma(\mu,\nu)$ is the set of Borel probabilty measures on $\mathbf{R}^D\times\mathbf{R}^D$ which have μ and ν as their marginals. The Riemannian structure of $\mathscr{P}_2(\mathbf{R}^D)$, introduced at a formal level in [43] and later fully developed in [4], will be intensively exploited in this work. Notice that, as soon as $\mathscr{P}_2(\mathbf{R}^D)$ is endowed with a differentiable structure, the theory of ODE's in the finite-dimensional space $\mathbf{R}^D$ naturally extends to a theory of ODE's in the infinite-dimensional space $\mathscr{P}_2(\mathbf{R}^D)$: it suffices to consider the isometry $I : z\to\delta_z$, where δ_z stands for the Dirac mass at z.

In particular, we consider the case when $D = 2d$ and we are given a lower semicontinuous Hamiltonian $H : \mathscr{P}_2(\mathbf{R}^{2d})\to\mathbf{R}$. As we will be mostly considering semiconvex Hamiltonians, in the sense of displacement convexity [38], mimicking some classical concepts of convex analysis we introduce in Definition 3.2 the subdifferential $\partial H(\mu)$ and denote by $\nabla H(\mu)$ its element with minimal $L^2(\mu;\mathbf{R}^{2d})$ norm (well defined whenever $\partial H(\mu)\neq\emptyset$).

The problem we study in Section 6 is: given an initial measure $\bar\mu\in\mathscr{P}_2(\mathbf{R}^{2d})$, find a path $t\to\mu_t\in\mathscr{P}_2(\mathbf{R}^{2d})$ such that

$$\begin{cases} \dfrac{d}{dt}\mu_t + \nabla\cdot(J\nabla H(\mu_t)\mu_t) = 0, & t\in(0,T) \\ \mu_0 = \bar\mu \end{cases} \tag{1.2}$$

and $\|\nabla H(\mu_t)\|_{L^2(\mu_t)}\in L^1(0,T)$. Here, J is a $(2d)\times(2d)$ symplectic matrix.

Using a suitable "chain rule" in the Wasserstein space first introduced in [4], we prove in Theorem 5.2 that H is constant among all solutions μ_t of (1.2), provided H is λ–convex (or λ–concave) for some real number λ. The proof of this fact requires neither regularity assumptions on the velocity field $J\nabla H(\mu_t)$ nor the absolute continuity of μ_t.

Existence of solutions can be established in (1.2) if one imposes a growth condition on the gradient, as

(H1) *the existence of constants* $C_o \in (0,+\infty)$, $R_o \in (0,+\infty]$ *that for all* $\mu \in \mathscr{P}_a^2(\mathbf{R}^{2d})$ *with* $W_2(\mu,\bar{\mu}) < R_o$ *we have* $\mu \in D(H)$, $\partial H(\mu) \neq \emptyset$ *and* $|\nabla H(\mu)(z)| \leq C_o(1+|z|)$ *for* μ*–almost every* $z \in \mathbf{R}^{2d}$

and a "continuity property" of the gradient as

(H2) *If* $\mu = \rho \mathscr{L}^{2d}$, $\mu_n = \rho_n \mathscr{L}^{2d} \in \mathscr{P}_2^a(\mathbf{R}^{2d})$, $\sup_n W_2(\mu_n,\bar{\mu}) < R_o$ *and* $\mu_n \to \mu$ *narrowly, then there exist a subsequence* $n(k)$ *and functions* $\mathbf{w}_k, \mathbf{w} : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ *such that* $\mathbf{w}_k = \nabla H(\mu_{n(k)})$ $\mu_{n(k)}$*-a.e.,* $\mathbf{w} = \nabla H(\mu)$ μ*-a.e. and* $\mathbf{w}_k \to \mathbf{w}$ $\mathscr{L}^{2d}$*–a.e. in* $\mathbf{R}^{2d}$ *as* $k \to +\infty$.

Here we are denoting by $\mathscr{P}_2^a(\mathbf{R}^{2d})$ the elements of $\mathscr{P}_2(\mathbf{R}^{2d})$ that are absolutely continuous with respect to $\mathscr{L}^{2d}$. The requirements of bounds and continuity on the gradient naturally appear also in the finite dimensional theory, in order to obtain bounds on the discrete solutions of the ODE and to pass to the limit.

In Theorem 6.6 we show that a minor variant of the algorithms used in [10], [12], [17] in connection with specific models, establishes existence of a solution μ_t in (1.2) up to some time $T = T(C_o,R_o)$ ($T = +\infty$ whenever $R_o = +\infty$), when $\mu_0 = \rho_0 \mathscr{L}^{2d}$ is absolutely continuous with respect to $\mathscr{L}^{2d}$ and (H1) and (H2) hold. A good feature of this algorithm is that it preserves the absolute continuity condition, so that $\mu_t = \rho_t \mathscr{L}^{2d}$, and provides the "entropy" inequalities

$$\int_{\mathbf{R}^{2d}} S(\rho_t)\,dz \leq \int_{\mathbf{R}^{2d}} S(\rho_0)\,dz \qquad t \in [0,T], \text{ with } S \text{ convex.}$$

Unlike the theory of gradient flows, where the selection of the gradient among all subdifferentials is ensured on any solution by energy reasons (see [4]), in our case it is not clear why in general this selection should be the natural one, even though it provides the tangency condition and it is more likely to provide bounds, by the minimality of the gradient. Therefore, we consider also a weaker version of (1.2), which works for arbitrary initial measures $\bar{\mu}$: find a path $t \to \mu_t \in \mathscr{P}_2(\mathbf{R}^{2d})$ and vector fields $\mathbf{v}_t \in L^2(\mu_t;\mathbf{R}^{2d})$ such that

$$\begin{cases} \dfrac{d}{dt}\mu_t + \nabla \cdot (J\mathbf{v}_t\mu_t) = 0, \qquad \mu_0 = \bar{\mu}, \qquad\qquad t \in (0,T) \\ \mathbf{v}_t \in T_{\mu_t}\mathscr{P}_2(\mathbf{R}^{2d}) \cap \partial H(\mu_t) \quad \text{for a.e. } t. \end{cases} \tag{1.3}$$

Here $T_{\mu_t}\mathscr{P}_2(\mathbf{R}^{2d})$ is the tangent space to $\mathscr{P}_2(\mathbf{R}^{2d})$ at μ, according to Otto's calculus [4], defined as the $L^2(\mu;\mathbf{R}^{2d})$ closure of the gradients of $C_c^\infty(\mathbf{R}^{2d})$ maps. Even in this case we are able to show that H is constant along solutions of (1.3), provided H is λ–convex (or λ–concave) for some $\lambda \in \mathbf{R}$.

For the system in (1.3), we weaken (H1) and (H2) and only assume that

(H1') *the existence of constants* $C_o \in [0,+\infty)$, $R_o \in (0,+\infty]$ *such that for all* $\mu \in \mathscr{P}_2(\mathbf{R}^{2d})$ *with* $W_2(\mu,\bar{\mu}) < R_o$ *we have* $\mu \in D(H)$, $\partial H(\mu) \neq \emptyset$ *and* $\|\nabla H(\mu)\|_{L^2(\mu)} \leq C_o$

and

(H2') *If* $\sup_n W_2(\mu_n, \bar{\mu}) < R_o$ *and* $\mu_n \to \mu$ *narrowly, then the limit points of convex combinations of* $\{\nabla H(\mu_n)\mu_n\}_{n=1}^{\infty}$ *for the weak***-topology are representable as* $\mathbf{w}\mu$ *for some* $\mathbf{w} \in \partial H(\mu) \cap T_\mu \mathscr{P}_2(\mathbf{R}^{2d})$.

In Section 7 a second algorithm, based on linear interpolation of transport maps, provides existence of solutions to (1.3). We refer to Theorem 7.4 for a complete statement of the results we obtain. In particular, when $\bar{\mu} = \delta_{(\bar{x},\bar{v})}$, defining h on $\mathbf{R}^{2d}$ by $h(x,v) = H(\delta_{(x,v)})$, the algorithm used in this section coincides with a natural finite-dimensional algorithm yielding in the limit the volume-preserving flow associated to the ODE (see Remark 6.5 for a more precise discussion):

$$
\begin{cases} \mathbb{J}_d(\dot{x}(t), \dot{v}(t)) \in & \partial h(x(t), v(t)), \qquad t \in (0,T) \\ (x(0), v(0)) = & (\bar{x}, \bar{v}). \end{cases} \tag{1.4}
$$

Note that proving existence of (1.3) is harder, compared to proving existence for the symplified system

$$
\begin{cases} \dfrac{d}{dt}\mu_t + \nabla \cdot (J\mathbf{v}_t \mu_t) = 0, \qquad \mu_0 = \bar{\mu}, \qquad t \in (0,T) \\ \mathbf{v}_t \in \partial H(\mu_t) \quad \text{for a.e. } t, \end{cases} \tag{1.5}
$$

where we drop the constraint that $\mathbf{v}_t \in T_{\mu_t}\mathscr{P}_2(\mathbf{R}^{2d})$, and so $\mathbf{v}_t$ may be not tangent to $\mathscr{P}_2(\mathbf{R}^{2d})$. The system in (1.5) does not make geometrical sense, except in special cases such as when μ_t is concentrated on finitely many points (in this case $L^2(\mu_t; \mathbf{R}^{2d}) = T_{\mu_t}\mathscr{P}_2(\mathbf{R}^{2d})$). On the technical side, the lack of the tangency condition seems to prevent the possibility of proving constancy of the Hamiltonian along solutions of (1.5).

Finally, we add more motivations for the terminology "Hamiltonian" adopted for the systems (1.2) and (1.3) (particularly when J is the canonical symplectic matrix). A first justification is given in [31], where $\mathbb{J}_d \nabla H(\mu)$ is shown to be the "symplectic gradient" induced by a suitable skew-symmetric 2-form (see the more detailed discussion made right after Definition 5.1). Moreover, in the recent work [18] the authors consider Hamiltonians on $\mathbf{R}^{2nd}$ of the form

$$
(x_1, v_1; \cdots; x_n, v_n) \to H_n(x_1, v_1; \cdots; x_n, v_n) = -\frac{1}{2} W_2^2 \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, v_i)}, \frac{1}{n} \sum_{i=1}^{n} \delta_{(a_i^n, b_i^n)} \right),
$$

where $(a_1^n, b_1^n), \cdots, (a_n^n, b_n^n) \in \mathbf{R}^{2d}$ are prescribed. They study the classical finite-dimensional Hamiltonian systems

$$
\begin{cases} x_i^n(t) = n\nabla_{v_i} H_n(x_1^n(t), v_1^n(t); \cdots; x_n^n(t), v_n^n(t)) & t \in (0,T) \\ v_i^n(t) = -n\nabla_{x_i} H_n(x_1^n(t), v_1^n(t); \cdots; x_n^n(t), v_n^n(t)) & t \in (0,T) \\ (x_i^n(0), v_i^n(0)) \quad \text{prescribed} & i = 1, \cdots, n. \end{cases} \tag{1.6}
$$

Defining

$$\mu_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{(x_i^n(t),v_i^n(t))},$$

it is readily checked that the paths $t \longrightarrow \mu_t^n \in \mathscr{P}_2(\mathbf{R}^{2d})$ satisfy (1.3) with H_n in place of H. In [18], it is proven that if the initial conditions $(x_i^n(0), v_i^n(0))$ are suitably chosen and $\nu^n = 1/n\sum_{i=1}^n \delta_{(a_i^n,b_i^n)}$ tends to ν as n tends to $+\infty$, then up to a subsequence which is independent of the time variable t, the measures $\{\mu_t^n\}_{n=1}^\infty$ narrowly converge as $n \to +\infty$ to measures $\{\mu_t\}_{t\in[0,T]}$ satisfying (1.2) for the Hamiltonian $H(\mu) = -1/2W_2^2(\mu,\nu)$.

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2 Basic notation and terminology

In this section we fix our basic notation and terminology on measure theory and Hamiltonian systems.

- The effective domain of a function $H : A \to (-\infty, +\infty]$ is the set $D(H)$ of all $a \in A$ such that $H(a) < +\infty$. We say that H is proper if $D(H) \neq \emptyset$.

- Let d, D be integers. We denote by $\mathbb{I}_D$ the identity matrix on $\mathbf{R}^D$ and we denote by $\mathbb{J}_d$ the sympletic $(2d)\times(2d)$ matrix

$$\mathbb{J}_d = \begin{pmatrix} 0 & \mathbb{I}_d \\ -\mathbb{I}_d & 0 \end{pmatrix}.$$

When $d = 1$, this is the clockwise rotation of angle $\pi/2$. We denote by **id** the identity map on $\mathbf{R}^D$ or $\mathbf{R}^{2d}$.

- If $r > 0$ and $z \in \mathbf{R}^D$, $B_r(z)$ denotes the ball in $\mathbf{R}^D$ of center z and radius r. If $B \subset \mathbf{R}^D$ we denote by B^c the complement of B.

- Assume that μ is a nonnegative Borel measure on a topological space X and that ν is a nonnegative Borel measure on a topological space Y. We say that a Borel map $\mathbf{t} : X \to Y$ transports μ onto ν, and we write $\mathbf{t}_\#\mu = \nu$, if $\nu[B] = \mu[\mathbf{t}^{-1}(B)]$ for all Borel sets $B \subset Y$. We sometimes say that $\mathbf{t}$ pushes μ to ν. We denote by $\mathscr{T}(\mu,\nu)$ the set of all $\mathbf{t}$ such that $\mathbf{t}_\#\mu = \nu$.

If γ is a nonnegative Borel measure on $X\times Y$ then its projection $\mathrm{proj}_X\gamma$ is a nonnegative Borel measure on X and its projection $\mathrm{proj}_Y\gamma$ is a nonnegative Borel measure on Y; they are defined by

$$\mathrm{proj}_X\gamma[A] = \gamma[A\times Y], \qquad \mathrm{proj}_Y\gamma[B] = \gamma[X\times B].$$

A measure γ on $X\times Y$ is said to have μ and ν as its marginals if $\mu = \mathrm{proj}_X\gamma$ and $\nu = \mathrm{proj}_Y\gamma$. We write that $\gamma \in \Gamma(\mu,\nu)$ and call γ a transport plan between μ and ν.

- When $X = Y = \mathbf{M}$, any minimizer γ_o in (1.1) is called an optimal transport plan between μ and ν. We write $\gamma_o \in \Gamma_o(\mu,\nu)$.

- We denote by $\mathscr{P}(\mathbf{R}^D)$ the set of Borel probability measures on $\mathbf{R}^D$. The D–dimensional Lebesgue measure on $\mathbf{R}^D$ is denoted by $\mathscr{L}^D$. The 2-moment of $\mu \in \mathscr{P}(\mathbf{R}^D)$ with respect to the origin is defined by

$$M_2(\mu) = \int_{\mathbf{R}^D} |x|^2 d\mu(x).$$

Notice that $W_2^2(\mu, \delta_0) = M_2(\mu)$. We will be dealing in particular with

$$\mathscr{P}_2(\mathbf{R}^D) := \left\{\mu \in \mathscr{P}(\mathbf{R}^D) :\ M_2(\mu) < +\infty\right\}$$

and its subspace $\mathscr{P}_2^a(\mathbf{R}^D)$, made of absolutely continuous measures with respect to $\mathscr{L}^D$.

- If $\mu \in \mathscr{P}_2(\mathbf{R}^D)$ and $v_1, \ldots, v_k \in L^2(\mathbf{R}^D, \mu)$, we write $\mathbf{v} = (v_1, \ldots, v_k) \in L^2(\mathbf{R}^D, \mu; \mathbf{R}^k)$ or simply $\mathbf{v} \in L^2(\mu; \mathbf{R}^k)$.

- Assume that μ, ν are Borel probability measures on $\mathbf{M} = \mathbf{R}^D$ with $M_2(\mu), M_2(\nu) < +\infty$ and μ absolutely continuous with respect to $\mathscr{L}^D$. Then there exists a unique minimizer γ_o in (1.1), characterized by the fact that $\gamma_o = (\mathbf{id} \times \mathbf{t}_\mu^\nu)_\# \mu$ for some map $\mathbf{t}_\mu^\nu : \mathbf{R}^D \to \mathbf{R}^D$ which coincides μ–a.e. with the gradient of a convex function. Therefore, the map $\mathbf{t}_\mu^\nu$ is the unique minimizer of

$$\mathbf{t} \to \int_{\mathbf{R}^D} |z - \mathbf{t}(z)|^2 d\mu(z)$$

over $\mathscr{T}(\mu, \nu)$.

- If $h \in C^1(\mathbf{R}^{2d})$, the Hamiltonian vector field associated to h is $X_h = \mathbb{J}\nabla h$. When $X \in C^1(\mathbf{R}^{2d}, \mathbf{R}^{2d})$, the flow of X is the map $\Phi : [a,b] \times \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ defined by

$$\begin{cases} \dot\Phi(t,z) = & X(t, \Phi(t,z)) \quad t \in [a,b],\ z \in \mathbf{R}^{2d} \\ \Phi(0,z) = & z, \quad z \in \mathbf{R}^{2d}. \end{cases} \tag{2.1}$$

The flow Φ is unique, and the growth condition

$$|X(t,z)| \le C(t)(1 + |z|) \qquad \text{with} \qquad C \in L^1(a,b)$$

ensures its existence.

- If $\mu_o = \delta_z$ and we set $\mu_t = \Phi(t, \cdot)_\# \mu_o = \delta_{\Phi(t,z)}$, then μ_t satisfy the continuity equation

$$\frac{d}{dt}\mu_t + \nabla \cdot (X\mu_t) = 0 \tag{2.2}$$

in the sense of distributions. When $X = X_h$ for a Hamiltonian h, (2.1) is called a Hamiltonian system.

In this work, we consider the infinite-dimensional version of (2.1 –2.2), where δ_z is replaced by a measure $\mu \in \mathscr{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$ and X_h is replaced by the Hamiltonian vector field X_H of a Hamiltonian $H : \mathscr{P}_2(\mathbf{R}^d \times \mathbf{R}^d) \to (-\infty, +\infty]$. When $d = 1$, that vector field is defined to be the clockwise "rotation" by the angle $\pi/2$, on the tangent space at μ of $\mathscr{P}_2(\mathbf{R}^{2d})$ of the the gradient of H.

3 The differentiable structure of the Wasserstein space $\mathscr{P}_2(\mathbf{R}^D)$

In this section we introduce the differentiable an Riemannian structure of $\mathscr{P}_2(\mathbf{R}^D)$ following essentially the approach developed in [4] (see also [11] [43], two seminal papers on this subject).

We recall first that $\big(\mathscr{P}_2(\mathbf{R}^D), W_2\big)$ is a complete and separable space, not locally compact. We refer to Proposition 7.1.5 and Remark 7.1.9 in [4] for more comments . However, bounded sets in $\mathscr{P}_2(\mathbf{R}^D)$ are (sequentially) relatively compact with respect to the so-called narrow convergence, i.e. weak convergence in the duality with $C_b(\mathbf{R}^D)$, the space of continuous and bounded functions in $\mathbf{R}^D$. Actually a sequence $\{\mu_n\}_{n=1}^\infty$ converges to μ in $\mathscr{P}_2(\mathbf{R}^D)$ if and only if μ_n narrowly converge to μ and $M_2(\mu_n) \to M_2(\mu)$ as $n \to +\infty$. The lack of compactness in $\mathscr{P}_2(\mathbf{R}^D)$ is precisely due to the fact that narrow convergence does not always imply convergence of second moments.

To derive the differentiable structure from the metric structure, we start from the following fact, proved in Theorem 8.3.1 of [4]: if $\mu_t \in \mathscr{P}_2(\mathbf{R}^D)$ solve the continuity equation

$$\frac{d}{dt}\mu_t + \nabla\cdot(\mathbf{w}_t\mu_t) = 0 \tag{3.1}$$

in the sense of distributions in $(a,b)\times\mathbf{R}^D$, for some time-dependent velocity field $\mathbf{w}_t$ with $\|\mathbf{w}_t\|_{L^2(\mu_t)} \in L^1(a,b)$, then

$$W_2(\mu_s,\mu_t) \le \int_s^t \|\mathbf{w}_\tau\|_{L^2(\mu_\tau;\mathbf{R}^D)}\,d\tau \qquad \forall a\le s\le t\le b. \tag{3.2}$$

As a consequence we obtain that if the maps $t\mapsto\mu_t$ is absolutely continuous from $[a,b]$ to $\mathscr{P}_2(\mathbf{R}^D)$. Conversely, it was proved in the same theorem in [4] that for any absolutely continuous curve $t\mapsto\mu_t$, there is always a unique, up to negligible sets in time, velocity field $\mathbf{v}_t$ for which both the continuity equation and, asymptotically, equality holds in (3.2):

$$\lim_{h\to 0}\frac{1}{|h|}W_2(\mu_{t+h},\mu_t) = \|\mathbf{v}_t\|_{L^2(\mu_t)} \quad \text{for a.e. } t. \tag{3.3}$$

In Proposition 8.4.5 of [4], this minimality property of $\mathbf{v}_t$ is proved to be equivalent to the fact that $\mathbf{v}_t$ belongs to the $L^2(\mu_t;\mathbf{R}^D)$ closure of $\{\nabla\varphi : \varphi\in C_c^\infty(\mathbf{R}^D)\}$. Hence, we may view $\mathbf{v}_t$ as the "tangent" velocity field to μ_t and define the tangent space to $\mathscr{P}_2(\mathbf{R}^D)$ at μ, as follows:

$$T_\mu\mathscr{P}_2(\mathbf{R}^D) = \overline{\{\nabla\varphi:\ \varphi\in C_c^\infty(\mathbf{R}^D)\}}^{L^2(\mu;\mathbf{R}^D)}. \tag{3.4}$$

Notice also that a simple duality argument gives (see Lemma 8.4.2 of [4])

$$\big[T_\mu\mathscr{P}_2(\mathbf{R}^D)\big]^\perp = \big\{\mathbf{w}\in L^2(\mu;\mathbf{R}^D):\ \nabla\cdot(\mathbf{w}\mu)=0\big\}. \tag{3.5}$$

In the following we shall denote by $\pi_\mu : L^2(\mu;\mathbf{R}^D) \to T_\mu\mathscr{P}_2(\mathbf{R}^D)$ the canonical orthogonal projection.

Summing up, the previous results can be rephrased as follows:

Theorem 3.1 (Due to [4])**.** *The class of absolutely continuous curves $\mu_t : [a,b] \to \mathscr{P}_2(\mathbf{R}^D)$ coincides with the class of solutions of the continuity equation for some velocity field $\mathbf{w}_t$ with $\|\mathbf{w}_t\|_{L^2(\mu_t;\mathbf{R}^D)} \in L^1(a,b)$.*
For any absolutely continuous curve $\mu_t : [a,b] \to \mathscr{P}_2(\mathbf{R}^D)$ there exist $\mathbf{v}_t \in L^2(\mu_t;\mathbf{R}^D)$ for which both the continuity equation and (3.3) *hold. Given a solution of the continuity equation* (3.1)*, equality holds in* (3.2) *if and only if $\mathbf{w}_t \in T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)$ for a.e. t.*
Finally, the map $t \mapsto \mathbf{v}_t \in L^2(\mu_t;\mathbf{R}^D)$ is uniquely determined up to $\mathscr{L}^1$–negligible sets.

It is proven in (8.4.6) in [4] that the above tangent velocity vector $\mathbf{v}_t$, is identified for almost every t by the following property :

$$\lim_{h\to 0}\left(x, \frac{y-x}{h}\right)_{\#}\gamma_h = (\mathbf{id}, \mathbf{v}_t)_{\#}\mu_t \qquad \text{in } \mathscr{P}_2(\mathbf{R}^D \times \mathbf{R}^D) \tag{3.6}$$

for any choice of $\gamma_h \in \Gamma_o(\mu_t, \mu_{t+h})$. Essentially this property says that optimal plans between μ_{t+h} and μ_t asymptotically behave as the plans induced by the transport maps $(\mathbf{id} + h\mathbf{v}_t)_{\#}\mu_t$. In the case when $\mu_t \in \mathscr{P}_2^a(\mathbf{R}^D)$, where optimal plans are unique and induced by maps, (3.6) reduces to

$$\frac{\mathbf{t}_h - \mathbf{id}}{h} \to \mathbf{v}_t \qquad \text{in } L^2(\mu_t;\mathbf{R}^D) \text{ as } h \to 0, \tag{3.7}$$

where $\mathbf{t}_h$ are the optimal transport maps between μ_t and μ_{t+h}.

Several notions of differential can be defined, according to this differentiable structure. We state here the one more relevant for our purposes, motivated by the fact that we will be dealing with convex Hamiltonians (for concave ones, one should instead use a superdifferential).

Definition 3.2 (Fréchet subdifferential)**.** *Let $H : \mathscr{P}_2(\mathbf{R}^D) \to (-\infty, +\infty]$ be a proper, lower semicontinuous function and let $\mu \in D(H)$. We say that $\mathbf{w} \in L^2(\mu, \mathbf{R}^D)$ belongs to the Fréchet subdifferential $\partial H(\mu)$ if*

$$H(\nu) \geq H(\mu) + \sup_{\gamma\in\Gamma_o(\mu,\nu)} \int_{\mathbf{R}^D\times\mathbf{R}^D} \langle \mathbf{w}(x), y - x\rangle \, d\gamma(x,y) + o(W_2(\mu,\nu))$$

as $\nu \to \mu$.

Definition 3.2 is a particular case of Definition 10.3.1 [4] (with the replacement of a sup with an inf, see also Proposition 4.2), where the elements of the subdifferential are plans, and so, are measures in the product $\mathbf{R}^D \times \mathbf{R}^D$, instead of maps on $\mathbf{R}^D$. If $\gamma \in \Gamma_o(\mu,\nu)$, recall that its barycentric projection $\bar{\gamma}$ is characterized by $\bar{\gamma}\mu = (\pi_1)_\#(y\gamma)$ or, equivalently, by

$$\int_{\mathbf{R}^D} \varphi(x)\bar{\gamma}(x)d\mu(x) = \int_{\mathbf{R}^D\times\mathbf{R}^D} \varphi(x)yd\gamma(x,y) \qquad \forall\varphi \in C_b(\mathbf{R}^D). \tag{3.8}$$

Hence, we can rephrase the condition $\mathbf{w} \in \partial H(\mu)$ as

$$H(\nu) \geq H(\mu) + \sup_{\gamma\in\Gamma_o(\mu,\nu)} \int_{\mathbf{R}^D\times\mathbf{R}^D} \langle \mathbf{w}(x), \bar{\gamma}(x) - x\rangle d\mu(x) + o(W_2(\mu,\nu)). \tag{3.9}$$

Notice that, whenever $\mu \in \mathscr{P}_2^a(\mathbf{R}^D)$, there is only an optimal plan induced by $\mathbf{t}_\mu^\nu$ and $\bar{\gamma} = \mathbf{t}_\mu^\nu$.

It has been proved in Theorem 12.4.4 of [4] that

$$\bar{\gamma} - \mathbf{id} \in T_\mu\mathscr{P}_2(\mathbf{R}^D) \qquad \forall\nu \in \mathscr{P}_2(\mathbf{R}^D),\ \forall\gamma \in \Gamma_o(\mu,\nu). \tag{3.10}$$

By (3.9) and (3.10) we infer that $\mathbf{w} \in \partial H(\mu)$ iff $\pi_\mu\mathbf{w} \in \partial H(\mu)$. Notice that $\partial H(\mu)$ is a closed and convex subset of $L^2(\mu;\mathbf{R}^D)$. Therefore, as it is customary in subdifferential analysis, we shall denote by $\nabla H(\mu)$ the element of $\partial H(\mu)$, of minimal $L^2(\mu;\mathbf{R}^D)$–norm. The previous comments show in particular that, by the minimality of its norm, $\nabla H(\mu) = \pi_\mu \nabla H(\mu)$ belongs to $\partial H(\mu) \cap T_\mu\mathscr{P}_2(\mathbf{R}^D)$.

In the following lemma we state a well-known continuity property of optimal plans or maps. Its proof, which is by now standard in the Monge-Kantorovich theory, can be found for instance in Proposition 7.1.3 [4]. We reproduce part of it for the reader's convenience.

Lemma 3.3 (Continuity of optimal plans and maps)**.** *Assume that $\{\mu_n\}_{n=1}^\infty$, $\{\nu_n\}_{n=1}^\infty$ are bounded sequences in $\mathscr{P}_2(\mathbf{R}^D)$ narrowly converging respectively to μ and ν. Assume that $\Gamma_o(\mu,\nu)$ contains a unique plan γ. Then (i)*

$$\lim_{n\to+\infty} \int_{\mathbf{R}^D\times\mathbf{R}^D} g(x,y)d\gamma_n(x,y) = \int_{\mathbf{R}^D\times\mathbf{R}^D} gd\gamma \tag{3.11}$$

for any choice of $\gamma_n \in \Gamma_o(\mu_n,\nu_n)$ and for any continuous function $g : \mathbf{R}^D\times\mathbf{R}^D \to \mathbf{R}$ satisfying

$$\lim_{|(x,y)|\to+\infty} \frac{|g|(x,y)}{|x|^2+|y|^2} = 0. \tag{3.12}$$

(ii) Assume furthermore that $\mu_n, \mu \in \mathscr{P}_2^a(\mathbf{R}^D)$ and that there exists a closed ball B, of finite radius, containing the supports of ν_n and ν. Then there exist Lipschitz, convex functions $u_n, u : \mathbf{R}^D \to \mathbf{R}\cup\{+\infty\}$ such that $\nabla u_n = \mathbf{t}_{\mu_n}^{\nu_n}$ μ_n-a.e. in $\mathbf{R}^D$ and $\nabla u = \mathbf{t}_\mu^\nu$ μ-a.e. in $\mathbf{R}^D$. In addition, there exists a subsequence $\{n_k\}_{k=1}^\infty$ of integers such that

$$\nabla u_{n_k} \to \nabla u \quad \mathscr{L}^D\text{–a.e. in } \mathbf{R}^D. \tag{3.13}$$

Proof. An argument which is by now standard and can be found in [30] characterizes the elements $\Gamma_o(\mu_n,\nu_n)$ to be the elements of $\Gamma(\mu_n,\nu_n)$ whose supports, suppγ_n, are cyclically monotone. More precisely, $\gamma_n \in \Gamma_o(\mu_n,\nu_n)$ if and only if

$\gamma_n \in \Gamma(\mu_n, \nu_n)$ and there exist convex, lower semicontinuous functions, $u_n : \mathbf{R}^D \to \mathbf{R} \cup \{+\infty\}$, such that

$$\operatorname{supp}\gamma_n \subset \partial u_n. \tag{3.14}$$

If $v_n = u_n^*$ is the Fenchel-Moreau transform of u_n and B is any closed set containing the support of μ_n, then

$$u_n(x) = \inf_{y \in B}\{\frac{1}{2}\langle x; y\rangle - v_n(y)\} \qquad x \in \mathbf{R}^D. \tag{3.15}$$

Using the fact that $\gamma_n \in \Gamma_o(\mu_n, \nu_n)$ and $\{\mu_n\}_{n=1}^\infty$, $\{\nu_n\}_{n=1}^\infty$ are bounded in $\mathscr{P}_2(\mathbf{R}^D)$, we obtain that

$$\sup_n \int_{\mathbf{R}^D \times \mathbf{R}^D} (|x|^2 + |y|^2) d\gamma_n(x,y) = \sup_n\{M_2(\mu_n) + M_2(\nu_n)\} < +\infty. \tag{3.16}$$

By (3.16), $\{\gamma_n\}_{n=1}^\infty$ is precompact for the narrow topology. Assume $\{\gamma_{n_k}\}_{k=1}^\infty$ is a narrowly convergent subsequence whose limit is $\bar\gamma$. Using again (3.16), it is clear that $\bar\gamma \in \Gamma(\mu, \nu)$ and (3.11) holds if we substitute $\{\gamma_n\}_{n=1}^\infty$ by $\{\gamma_{n_k}\}_{k=1}^\infty$. By Proposition 7.1.3 of [4], every point in $\operatorname{supp}\bar\gamma$ is a limit of points in $\operatorname{supp}\gamma_{n_k}$ and so, $\operatorname{supp}\bar\gamma$ is cyclically monotone. This implies $\bar\gamma \in \Gamma_o(\mu, \nu) = \{\gamma\}$. Since the limit $\bar\gamma$ is independent of the subsequence $\{\gamma_{n_k}\}_{k=1}^\infty$, we have proven that $\{\gamma_n\}_{n=1}^\infty$ narrowly converges to γ and (3.11) holds. This proves (i).

Let $\mathbf{id}$ be the identity map on $\mathbf{R}^D$ and assume now that $\mu_n, \mu \in \mathscr{P}_2^a(\mathbf{R}^D)$, so that

$$\gamma_n \in \Gamma_o(\mu_n, \nu_n) = \{\mathbf{id} \times \mathbf{t}_{\mu_n}^{\nu_n}\} \quad \text{and} \quad \gamma \in \Gamma_o(\mu, \nu) = \{\mathbf{id} \times \mathbf{t}_\mu^\nu\}. \tag{3.17}$$

Since convex functions are differentiable $\mathscr{L}^D$–almost everywhere, (3.14) and the first equality in (3.17) imply that $\mathbf{t}_{\mu_n}^{\nu_n} = \nabla u_n$ μ_n-a.e. in $\mathbf{R}^D$. Let us furthermore assume that there exists a closed ball B, of finite radius, containing the supports of ν_n and ν. Enlarging B if necessary, we may without loss of generality that B contains the origin and so, by (3.15), u_n is Lipschitz with a Lipschitz constant bounded above by the radius of B. We may substitute u_n by $u_n - u_n(0)$ without altering the validity of the above reasonings. Therefore, in the sequel, we may assume without loss of generality that $u_n(0) = 0$. Ascoli-Arzela lemma ensures the existence of a subsequence $\{u_{n_k}\}_{k=1}^\infty$ which is locally uniformly convergent. Its limit u is necessary convex, with a Lipschitz constant bounded above by the diameter of B.

Now, let us show the convergence of the transport maps. Passing to the limit as $n \to \infty$ in the suddifferential inequality

$$u_n(x') \geq u_n(x) + \langle \nabla u_n(x); x' - x\rangle$$

we immediately obtain that, at any differentiability point of all maps u_n, any limit point of $\{\nabla u_n(x)\}_{n=1}^\infty$ belongs to the subdifferential $\partial u(x)$. It follows that ∇u_n converge to ∇u wherever all gradients (including ∇u) are defined, hence $\mathscr{L}^D$–a.e. in $\mathbf{R}^D$. In particular, recalling (3.14) and the fact that every point in $\operatorname{supp}\gamma$ is a limit of points in $\mathrm{supp}\gamma_{n_k}$, we conclude that $\mathrm{supp}\gamma \subset \partial u$. This, together with the second inequality in (3.17) implies that $\mathbf{t}_\mu^\nu = \nabla u$ μ–almost everywhere on $\mathbf{R}^D$. QED.

4 Convex analysis on $\mathscr{P}_2(\mathbf{R}^D)$

Let $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbf{R}^D)$ and let $\gamma \in \Gamma_o(\mu_0, \mu_1)$ be an optimal transport plan. Let $\pi_1 : \mathbf{R}^D \times \mathbf{R}^D : (z, w) \to z$ and $\pi_2 : \mathbf{R}^D \times \mathbf{R}^D : (z, w) \to w$ be the first and second projections of $\mathbf{R}^D \times \mathbf{R}^D$ onto $\mathbf{R}^D$. As suggested in [38], the interpolation $(1-t)\pi_1 + t\pi_2$ between maps can be used to interpolate between the measures μ_0 and μ_1 as follows:

$$\mu_t = \Big((1-t)\pi_1 + t\pi_2\Big)_{\#}\gamma. \tag{4.1}$$

The proof of the well known fact that $t \to \mu_t$ is a geodesic in $\mathscr{P}_2(\mathbf{R}^D)$ of constant speed, i.e. $W_2(\mu_s, \mu_t) = |t-s| W_2(\mu_0, \mu_1)$ for all $s, t \in [0,1]$, can be found in Theorem 7.2.2 of [4]; furthermore, any constant speed geodesic has this representation for a suitable optimal γ. As it is customary in Riemannian geometry, the identification of constant speed geodesics with segments allows the introduction of various notions of convexity for functions (see Chapter 9 of [4] and [34]).

Definition 4.1 (λ–convexity)**.** *Let $H : \mathscr{P}_2(\mathbf{R}^D) \to (-\infty, +\infty]$ be proper and let $\lambda \in \mathbf{R}$. We say that H is λ–convex if for every $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbf{R}^D)$ and every optimal transport plan $\gamma \in \Gamma_o(\mu_0, \mu_1)$ we have*

$$H(\mu_t) \le (1-t)H(\mu_0) + tH(\mu_1) - \frac{\lambda}{2} t(1-t) W_2^2(\mu_0, \mu_1) \quad \forall t \in [0,1]. \tag{4.2}$$

Here $\mu_t = ((1-t)\pi_1 + t\pi_2)_{\#}\gamma$, where π_1 and π_2 are the above projections.

For a real-valued map, λ-convexity means that the second distributional derivative of $t \to H(\mu_t)$ is larger than $\lambda \mathscr{L}^1$. In general, the inequality above is equivalent to saying that $t \to H(\mu_t)$ is $\lambda W_2^2(\mu_0, \mu_1)$–convex. In particular, 0–convexity corresponds to the notion of displacement convexity introduced in [38]. Finally, notice that this notion of convexity is slightly stronger than the one introduced in [4], where the inequality above is imposed only on some optimal transport plan.

Proposition 4.2 (Characterization of subdifferentials of λ–convex functions)**.** *Let $H : \mathscr{P}_2(\mathbf{R}^D) \to (-\infty, +\infty]$ be lower semicontinous and λ–convex for some $\lambda \in \mathbf{R}$ and let $\mu \in D(H)$. Then, any of the following two conditions is equivalent to $\mathbf{w} \in \partial H(\mu)$:*

(i)

$$H(\nu) \ge H(\mu) + \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbf{R}^D} \langle \mathbf{w}(x); \bar{\gamma}(x) - x \rangle d\mu(x) + o(W_2(\mu, \nu)); \tag{4.3}$$

(ii) *for all* $\nu \in \mathscr{P}_2(\mathbf{R}^{2d})$ *we have*

$$H(\nu) \ge H(\mu) + \sup_{\gamma\in\Gamma_o(\mu,\nu)} \int_{\mathbf{R}^D} \langle \mathbf{w}(x); \overline{\gamma}(x) - x\rangle \, d\mu(x) + \frac{\lambda}{2} W_2^2(\mu,\nu). \tag{4.4}$$

Proof. It is clear that $\mathbf{w} \in \partial H(\mu)$ implies (i), and that (ii) implies $\mathbf{w} \in \partial H(\mu)$. So, it remains to show that (i) implies (ii). To this aim, fix $\nu \in \mathscr{P}_2(\mathbf{R}^{2d})$, $\gamma \in \Gamma_o(\mu,\nu)$ and define the constant speed geodesic $\{\mu_t\}_{t\in[0,1]}$, between μ and ν as in (4.1). Then, we know that for $t<1$ there is a unique optimal plan between μ and μ_t, induced by $\gamma_t = (\pi_1,(1-t)\pi_1 + t\pi_2)_\# \gamma$ (see Lemma 7.2.1 of [4]), so that (4.3) and the identity $\overline{\gamma_t} - \mathbf{id} = t(\overline{\gamma} - \mathbf{id})$ give

$$\liminf_{t\downarrow 0} \frac{H(\mu_t) - H(\mu)}{t} \ge \int_{\mathbf{R}^D} \langle \mathbf{w}(x); \overline{\gamma}(x) - x\rangle \, d\mu(x).$$

Then, by applying (4.2) we get

$$H(\nu) - H(\mu) \ge \int_{\mathbf{R}^D} \langle \mathbf{w}(x); \overline{\gamma}(x) - x\rangle \, d\mu(x) + \frac{\lambda}{2} W_2^2(\mu,\nu).$$

QED.

It is not difficult to show that the infimum in (i) and the supremum (ii) are achieved. As shown in Chapter 10 of [4], the “inf” definition of subdifferential in (i) ensures the weak closure properties of the graph of the subdifferential. Again, in the case when $\mu \in \mathscr{P}_2^a(\mathbf{R}^D)$, the previous formula reduces to

$$H(\nu) \ge H(\mu) + \int_{\mathbf{R}^D} \langle \mathbf{w}(x); \mathbf{t}_\mu^\nu(x) - x\rangle \, d\mu + \frac{\lambda}{2} W_2^2(\mu,\nu) \qquad \forall \nu \in \mathscr{P}_2(\mathbf{R}^D).$$

The typical Hamiltonian we consider in this paper is the negative squared Wasserstein distance. Some of its properties, established in Proposition 9.3.12 and Theorem 10.4.12 of [4], are summarized in the following proposition.

Proposition 4.3 (Convexity of the negative Wasserstein distance)**.** *Let* $\nu \in \mathscr{P}_2(\mathbf{R}^D)$ *and define*

$$H(\mu) = -\frac{1}{2} W_2^2(\mu,\nu) \qquad \mu \in \mathscr{P}_2(\mathbf{R}^D). \tag{4.5}$$

Then H *is* (-1)*-convex. Furthermore, if* $\mu \in \mathscr{P}_2(\mathbf{R}^D)$*,*

$$\partial H(\mu) \cap T_\mu \mathscr{P}_2(\mathbf{R}^D) = \{\overline{\gamma} - \mathbf{id} : \ \gamma \in \Gamma_o(\mu,\nu)\} \tag{4.6}$$

and therefore $\nabla H(\mu)$ *is the minimizer in*

$$\min\left\{ \int_{\mathbf{R}^D} |\overline{\gamma} - \mathbf{id}|^2 \, d\mu : \ \gamma \in \Gamma_o(\mu,\nu) \right\}. \tag{4.7}$$

Here $\overline{\gamma}$ *is the barycentric projection of* γ*, as defined in* (3.8)*. In particular,*

$$\partial H(\mu) \cap T_\mu \mathscr{P}_2(\mathbf{R}^D) = \left\{\mathbf{t}_\mu^\nu - \mathbf{id}\right\} \qquad \forall \mu \in \mathscr{P}_2^a(\mathbf{R}^D). \tag{4.8}$$

Notice that $W_2^2(\cdot,\nu)$ is, on the other hand, trivially convex with respect to the conventional linear structure of $\mathscr{P}_2(\mathbf{R}^D)$, as $t\gamma_1+(1-t)\gamma_2\in\Gamma(t\mu_1+(1-t)\mu_2,\nu)$ whenever $\gamma_1\in\Gamma(\mu_1,\nu)$ and $\mu_2\in\Gamma(\mu_2,\nu)$. Also, as shown in Example 9.1.5 of [4], for each $\lambda\in\mathbf{R}$, $W^2(\cdot,\nu)$ *fails to be* λ-convex along geodesics.

5 Basic properties of solutions of Hamiltonian ODE's

We now have all the necessary ingredients for the definition of Hamiltonian flow in $\mathscr{P}_2(\mathbf{R}^{2d})$. In order to cover more examples (see Section 8) we consider also the case when the space is $\mathscr{P}_2(\mathbf{R}^D)$ and $J:\mathbf{R}^D\to\mathbf{R}^D$ is a linear map satisfying $Jv\perp v$ for all $v\in\mathbf{R}^D$ (this framework includes the canonical case $D=2d$ and $J=\mathbb{J}_d$).

Definition 5.1. *Let $H:\mathscr{P}_2(\mathbf{R}^D)\to(-\infty,+\infty]$ be a proper, lower semicontinuous function. We say that an absolutely continuous curve $\mu_t:[0,T]\to D(H)$ is a Hamiltonian ODE relative to H, starting from $\bar\mu\in\mathscr{P}_2(\mathbf{R}^D)$, if there exist $\mathbf{v}_t\in L^2(\mu_t;\mathbf{R}^D)$ with $\|\mathbf{v}_t\|_{L^2(\mu_t)}\in L^1(0,T)$, such that*

$$
\begin{cases}
\dfrac{d}{dt}\mu_t+\nabla\cdot(J\mathbf{v}_t\mu_t)=0, \qquad \mu_0=\bar\mu, \qquad\qquad t\in(0,T)\\[2ex]
\mathbf{v}_t\in T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)\cap\partial H(\mu_t) \quad \text{for a.e. } t.
\end{cases}
\tag{5.1}
$$

The terminology "Hamiltonian ODE" is fully justified in the case $D=2d$, $J=\mathbb{J}_d$ in a work in progress by Gangbo and Pacini [31]. There, they prove that $\mathbb{J}_d$ induces a nondegenerate bilinear skew-symmetric closed 2–form Ω as follows. Denoting by $T^*\mathscr{P}_2(\mathbf{R}^{2d})$ the subbundle defined by

$$
T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d}):=\Big\{\pi_\mu(\mathbb{J}_d\mathbf{v}):\ \mathbf{v}\in T_\mu\mathscr{P}_2(\mathbf{R}^{2d})\Big\},
$$

they define $\Omega_\mu:T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d})\times T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d})\to\mathbf{R}$ as follows: if $\bar{\mathbf{v}}_1=\pi_\mu(\mathbb{J}_d\mathbf{v}_1)$, $\bar{\mathbf{v}}_2=\pi_\mu(\mathbb{J}_d\mathbf{v}_2)\in T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d})$, with $\mathbf{v}_1$, $\mathbf{v}_2\in T_\mu\mathscr{P}_2(\mathbf{R}^{2d})$, they set

$$
\Omega_\mu(\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2)=\int_{\mathbf{R}^{2d}}\langle\mathbb{J}_d\mathbf{v}_1;\mathbf{v}_2\rangle\,d\mu \qquad \mu\in\mathscr{P}_2(\mathbf{R}^{2d}).
$$

It is easy to check that Ω_μ is well defined (i.e. it does not depend on the choice of the vectors $\mathbf{v}_i$ such that $\bar{\mathbf{v}}_i=\pi_\mu(J\mathbf{v}_i)$), skew-symmetric and nondegenerate.

For any $\mu\in\mathscr{P}_2(\mathbf{R}^{2d})$ where ∇H exists, the Hamiltonian vector field $X_H\in T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d})$ is classically defined by the identity

$$
\Omega_\mu(X_H(\mu),\bar{\mathbf{v}})=\int_{\mathbf{R}^{2d}}\langle\nabla H(\mu);\bar{\mathbf{v}})\rangle=dH(\bar{\mathbf{v}}) \qquad \forall\bar{\mathbf{v}}\in T^*_\mu\mathscr{P}_2(\mathbf{R}^{2d}).
$$

In other words, $\Omega_\mu(X_H(\mu),\cdot)=dH(\cdot)$. The system (5.1) with $\mathbf{v}_t=\nabla H(\mu_t)$ is then easily seen to be equivalent to the condition that the tangent velocity vector $\pi_{\mu_t}(\mathbb{J}_d\mathbf{v}_t)$ to μ_t is $X_H(\mu_t)$ or equivalently, $\dot\mu_t=X_H(\mu_t)$. More generally, one could define a "Hamiltonian subdifferential" by considering the vectors $\pi(\mathbb{J}_d\mathbf{v})$ with $\mathbf{v}\in\partial H(\mu)\cap T_\mu\mathscr{P}_2(\mathbf{R}^{2d})$.

The integrability condition $\|\mathbf{v}_t\|_{L^2(\mu_t)} \in L^1(0,T)$ ensures that the continuity equation makes sense in the sense of distributions; furthermore (see for instance Lemma 8.1.2 in [4]), possibly redefining μ_t in a negligible set of times, we can assume that $t \mapsto \mu_t$ is narrowly continuous in $[0,T]$. We shall always make tacitly this continuity assumption in the sequel.

In the construction of solutions to Hamiltonian ODE's by approximation, one finds that the subdifferential inclusion $\mathbf{v}_t \in \partial H(\mu_t)$ (and therefore the continuity equation with velocity field $J\mathbf{v}_t$) has good stability properties (see for instance Lemma 10.1.3 and Lemma 10.3.8 of [4], or Remark 6.5). The tangency condition, on the other hand, is not stable in general; however this condition is crucial to show that $t \mapsto H(\mu_t)$ is constant for Hamiltonian ODE's. In the proof of this fact we follow the "Wasserstein chain rule" in §10.1.2 and Proposition 10.3.18 of [4], whose proof (based on a subdifferentiability argument) we reproduce for the reader's convenience.

Theorem 5.2. *Let H be as in Definition 5.1, and let μ_t be a Hamiltonian ODE, with $\|\mathbf{v}_t\|_{L^2(\mu_t)} \in L^\infty(0,T)$. If H is λ–convex for some $\lambda \in \mathbf{R}$ then $t \mapsto H(\mu_t)$ is constant.*

Proof. We first prove that $t \mapsto H(\mu_t)$ is a Lipschitz function. Let C be the L^∞ norm of $\|\mathbf{v}_t\|_{L^2(\mu_t)}$ and notice that (3.2) gives that the Lipschitz constant of $t \mapsto \mu_t$ is less than C. We denote by $\mathbf{w}_t$ the tangent velocity field to μ_t and notice that, as $J\mathbf{v}_t$ is an admissible velocity field for μ_t, we have that $\mathbf{w}_t - J\mathbf{v}_t$ is orthogonal to $T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)$ for a.e. t.

Let now $D \subset (0,T)$ be the set of points where both $\mathbf{v}_t \in \partial H(\mu_t)$ and $\|\mathbf{v}_t\|_{L^2(\mu_t)} \leq C$ hold. Let $t \in D$, $s \in [0,T]$ and notice that by Proposition 4.2

$$\begin{aligned} H(\mu_t) - H(\mu_s) &\leq \inf_{\gamma \in \Gamma_o(\mu_t,\mu_s)} \int_{\mathbf{R}^D \times \mathbf{R}^D} -\langle \mathbf{v}_t(x); y-x \rangle \, d\gamma - \frac{\lambda}{2} W_2^2(\mu_t,\mu_s) \\ &\leq C^2|t-s| + \frac{C^2\lambda^-}{2}(t-s)^2 \\ &\leq C^2\left(1+\frac{T\lambda^-}{2}\right)|t-s|. \end{aligned}$$

As H is lower semicontinuous, by approximation the same inequality holds when $s,t \in [0,T]$. Reversing the rôles of s and t we obtain that the Lipschitz constant of $t \mapsto H(\mu_t)$ is less than $C^2(1+T\lambda^-/2)$.

It remains to show that the derivative of $t \mapsto H(\mu_t)$ is equal to 0. Fix $t \in (0,T)$ where this derivative exists, (3.6) holds, $\mathbf{v}_t \in \partial H(\mu_t) \cap T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)$ and $\mathbf{w}_t - J\mathbf{v}_t$ is orthogonal to $T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)$. We have then the existence of optimal plans $\gamma_h \in \Gamma_o(\mu_t,\mu_{t+h})$ satisfying

$$\frac{C^2\lambda^-}{2}h^2 + H(\mu_{t+h}) - H(\mu_t) \geq \int_{\mathbf{R}^D \times \mathbf{R}^D} \langle \mathbf{v}_t(x); y-x \rangle d\gamma_h.$$

Next, we define $\eta_h = (x,(y-x)/h)_\#\gamma_h$ to obtain

$$H(\mu_{t+h}) - H(\mu_t) \geq h\int_{\mathbf{R}^D\times\mathbf{R}^D} \langle \mathbf{v}_t(x); y\rangle d\eta_h + o(h)$$

and use (3.6) to obtain[1]

$$\begin{aligned} H(\mu_{t+h}) - H(\mu_t) &\geq h\int_{\mathbf{R}^D\times\mathbf{R}^D} \langle \mathbf{v}_t(x); y\rangle d(\mathbf{id},\mathbf{w}_t)_\#\mu_t + o(h) \\ &= h\int_{\mathbf{R}^{2d}} \langle \mathbf{v}_t(x); \mathbf{w}_t(x)\rangle d\mu_t + o(h) \\ &= h\int_{\mathbf{R}^{2d}} \langle \mathbf{v}_t(x); J\mathbf{v}_t(x)\rangle d\mu_t + o(h) = o(h). \end{aligned}$$

Since $s \mapsto H(\mu_s)$ is differentiable at $s = t$, this can happen only if the derivative is 0. QED.

6 Existence of Hamiltonian flows: regular initial data

Before stating our main existence theorem, we state a technical lemma concerning the approximation of tangent vectors by smooth gradients.

Lemma 6.1. *Let $\mu = \rho\mathscr{L}^D \in \mathscr{P}_2(\mathbf{R}^D)$ be satisfying $\rho \geq m_r > 0$ $\mathscr{L}^D$–a.e. on B_r for any $r > 0$. If $C > 0$, $\mathbf{v} \in T_\mu\mathscr{P}_2(\mathbf{R}^D)$ and*

(6.1) $$|\mathbf{v}(z)| \leq C(1+|z|) \qquad \text{for } \mu\text{–almost every } z \in \mathbf{R}^D$$

then there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbf{R}^D)$ such that

$$|\nabla\varphi_n(z)| \leq C(2+|z|) \qquad \forall z \in \mathbf{R}^D$$

and

$$\lim_{n\to+\infty} \|\mathbf{v} - \nabla\varphi_n\|_{L^2(\mu;\mathbf{R}^D)} = 0.$$

Proof. Let $\{\phi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbf{R}^D)$ be such that $\|\mathbf{v}-\nabla\phi_n\|_{L^2(\mu)} \to 0$ as $n \to +\infty$. For all $r > 0$ we have

$$\limsup_{n\to+\infty} \|\mathbf{v}-\nabla\phi_n\|^2_{L^2(B_r,\mathscr{L}^D,\mathbf{R}^D)} \leq \frac{1}{m_r}\limsup_{n\to+\infty} \|\mathbf{v}-\nabla\phi_n\|^2_{L^2(\mu)} = 0.$$

This proves that $\mathbf{v} \in L^2_{loc}(\mathbf{R}^{2d},\mathscr{L}^{2d})$ and that $\operatorname{curl}\mathbf{v} = 0$. Let $l_1 \in C_c^\infty$ be a nonnegative probability density whose support is contained in the unit ball of $\mathbf{R}^{2d}$ and set

$$\mathbf{v}_h = l_h * \mathbf{v}, \qquad \text{with} \qquad l_h(z) = \frac{1}{h^{2d}} l_1(\frac{z}{h}).$$

[1] Even though the test function $(x,y) \mapsto \langle \mathbf{v}_t(x); y\rangle$ is possibly discontinuous and unbounded, one can use the boundedness of 2-moments of η_h and the fact that their first marginal does not depend on h to pass to the limit, see for instance §5.1.1 in [4]

Clearly, $\mathbf{v}_h \in C^\infty(\mathbf{R}^{2d}, \mathbf{R}^{2d})$ and $\operatorname{curl} \mathbf{v}_h = 0$. Hence, there exist $A_h \in C^\infty(\mathbf{R}^{2d})$ such that $\mathbf{v}_h = \nabla A_h$ and $A_h(0) = 0$. Thanks to Jensen's inequality, (6.1) implies that

$$\begin{aligned} |\mathbf{v}_h(z)| = |\int_{\mathbf{R}^{2d}} l_h(w)\mathbf{v}(z-w)dw| &\leq C\int_{\mathbf{R}^{2d}} l_h(w)(1+|z-w|)dw \\ &\leq C(1+|z|) + C\int_{\mathbf{R}^{2d}} l_h(w)|w|dw \\ &= C(1+|z|) + hC\int_{\mathbf{R}^{2d}} l_1(w')|w'|dw' \\ &\leq C(1+|z|) + hC\int_{B_1(0)} l_1(w')dw' \\ &\leq C(2+|z|), \end{aligned} \tag{6.2}$$

for $h \leq 1$. Since $\{\mathbf{v}_h\}_{h>0}$ converges $\mathscr{L}^{2d}$–almost everywhere to $\mathbf{v}$, the uniform bound in (6.2) and the fact that $\mu \in \mathscr{P}_2(\mathbf{R}^D)$ imply, by the dominated convergence theorem,

$$\lim_{h\to 0} \|\mathbf{v} - \nabla A_h\|^2_{L^2(\mu;\mathbf{R}^D)} = 0. \tag{6.3}$$

Define

$$B_h^r(z) = \begin{cases} A_h(z) & \text{for } |z| \leq r \\ 0 & \text{for } |z| \geq 2r. \end{cases} \tag{6.4}$$

Note that B_h^r is a $C(2+r)$–Lipschitz function and so it admits an extension to $\mathbf{R}^D$, that we still denote by B_h^r, which is $C(2+r)$–Lipschitz. We use (6.1), (6.2) and the fact that

$$|\nabla B_h^r(z)| \leq C(2+r) \leq C(2+|z|) \qquad \text{on } B_r^c(0) \tag{6.5}$$

to conclude that for all $h \leq 1$

$$\begin{aligned} \int_{\mathbf{R}^{2d}} |\mathbf{v} - \nabla B_h^r|^2 d\mu &= \int_{B_r(0)} |\mathbf{v} - \nabla A_h|^2 d\mu + \int_{B_r^c(0)} |\mathbf{v} - \nabla B_h^r|^2 d\mu \\ &\leq \int_{\mathbf{R}^{2d}} |\mathbf{v} - \nabla A_h|^2 d\mu + 4C^2\int_{B_r^c(0)} (2+|z|)^2 d\mu. \end{aligned} \tag{6.6}$$

We combine (6.3) and (6.6) to conclude that

$$\lim_{h,1/r\to 0} \|\mathbf{v} - \nabla B_h^r\|^2_{L^2(\mu;\mathbf{R}^D)} = 0. \tag{6.7}$$

This, together with (6.2) and (6.5) yields the lemma. QED.

The following lemma provides a discrete solution of the Hamiltonian ODE in a small time interval, whose iteration will lead to a discrete solution. To make the iteration possible, one has to show that the flow preserves in some sense the bounds on the initial datum: this is possible thanks to the fact that the flow is incompressible.

Lemma 6.2. *Let $h > 0$, let $\mu = \rho\mathscr{L}^D \in \mathscr{P}_2^a(\mathbf{R}^D)$ be satisfying*

$$\rho \geq m_r > 0 \qquad \mathscr{L}^D\text{–a.e. on } B_r, \text{ for any } r > 0 \tag{6.8}$$

and let $\mathbf{v} \in T_\mu\mathscr{P}_2(\mathbf{R}^D)$ be satisfying (6.1), with $e^{Ch} \leq 2$. Then there exists a family of measures $\mu_t = \rho_t\mathscr{L}^D$, $t \in [0,h]$, satisfying

(a) *$\int_{\mathbf{R}^D} S(\rho_t)\,dz \leq \int_{\mathbf{R}^D} S(\rho)\,dz$ for any convex function $S : [0,+\infty) \to [0,+\infty)$;*
(b) *$t \mapsto \mu_t \in \mathscr{P}_2(\mathbf{R}^D)$ is absolutely continuous, $\mu_0 = \mu$ and the continuity equation*

$$\frac{d}{dt}\mu_t + \nabla\cdot(J\mathbf{v}\mu_t) = 0, \qquad (t,z) \in (0,h)\times\mathbf{R}^D \tag{6.9}$$

holds;
(c) *$\rho_t \geq m_{r'}$ $\mathscr{L}^D$–a.e. on B_r, with $r' = e^{Ch}r + 2(e^{Ch}-1)$.*

Finally, we have also that $t \mapsto \mu_t$ is Lipschitz continuous, with Lipschitz constant less than $L_o = C\sqrt{24(1+M_2(\mu))}$ and, in particular,

$$W_2(\mu_t,\mu) \leq hL_o \qquad \forall t \in [0,h]. \tag{6.10}$$

Remark 6.3. Assumption (6.8) is used twice. First, it is used to conclude that since $\mathbf{v}$ is defined μ–almost everywhere, then it is defined $\mathscr{L}^D$–almost everywhere, hence μ_t–almost everywhere, if $\mu_t \ll \mathscr{L}^D$. More importantly, it is used to apply Lemma 6.1, to treat $\mathbf{v}$ as a gradient and to obtain that $J\mathbf{v}$ is divergence free with respect to $\mathscr{L}^D$. This leads to the conclusion that the flow $\Phi(t,\cdot)$ associated to $J\mathbf{v}$ preserves $\mathscr{L}^D$ for each t fixed.

Proof of lemma 6.2 We assume first that $\mathbf{v} = \nabla\phi \in C_c^\infty(\mathbf{R}^D;\mathbf{R}^D)$ and that the weaker condition $|\mathbf{v}(z)| \leq C(2+|z|)$ is fulfilled. Under this assumption the autonomous vector field $J\mathbf{v}$ is smooth and divergence-free, so the flow $\Phi : [0,h]\times\mathbf{R}^D \to \mathbf{R}^D$ associated to $J\mathbf{v}$ is smooth and measure-preserving. In this case we simply define $\mu_t = \Phi(t,\cdot)_\#\mu$, so that the continuity equation (6.9) is satisfied. The measure preserving property gives that $\mu_t = \rho_t\mathscr{L}^D$, with

$$\rho_t \circ \Phi(t,\cdot) = \rho. \tag{6.11}$$

Notice that (a) (with an equality, and even for nonconvex S) follows immediately by (6.11), and (c) as well, provided we show that $\Phi(t,\cdot)^{-1}(B_r) \subset B_{r'}$. To show the latest inclusion, notice that $\Psi(t,y) = \Phi(t,\cdot)^{-1}(y)$ is the flow associated to $-J\mathbf{v}$, hence

$$\frac{d}{dt}|\Psi(t,y)| \leq |J\mathbf{v}|(\Psi(t,y)) \leq C(2+|\Psi(t,y)|).$$

By integrating this differential inequality we immediately obtain that

$$2+|\Psi(t,y)| \leq e^{Ct}(2+|y|).$$

Hence, $|y| < r$ implies $|\Psi(t,y)| < r'$ for $t \in [0,h]$. An analogous argument gives $2+|\Phi(t,z)| \leq e^{Ct}(2+|z|)$, hence when $e^{Ch} < 2$ we obtain

$$|\Phi(t,z)| \leq 2(|z|+1).$$

Using this inequality we can estimate

$$\int_{\mathbf{R}^D} |J\mathbf{v}|^2 d\mu_t \le 2C^2 \int_{\mathbf{R}^D} (4+|y|^2)\, d\mu_t = 8C^2 + 2C^2 \int_{\mathbf{R}^D} |\Phi(t,z)|^2\, d\mu$$
$$\le 8C^2 + 16C^2 \int_{\mathbf{R}^D} (1+|z|^2)\, d\mu = 24C^2 + 16C^2 M_2(\mu) \le L_o^2.$$

Using this estimate in conjunction with (3.2) and (6.9) yields that $t \mapsto \mu_t$ is L_o–Lipschitz .

In the general case we consider a sequence $\mathbf{v}_n = \nabla\phi_n$ with all properties stated in Lemma 6.1. As $\rho > 0$ $\mathscr{L}^D$–a.e., we can also assume with no loss of generality that $\mathbf{v}_n \to \mathbf{v}$ $\mathscr{L}^D$–a.e. in $\mathbf{R}^{2d}$. Let μ_t^n be the measures built according to the previous construction relative to $\mathbf{v}_n$ and notice that $t \mapsto \mu_t^n$ are equi-bounded in $\mathscr{P}_2(\mathbf{R}^D)$, and L_o–Lipschitz continuous. Furthermore, $\mu_t^n = \rho_t^n \mathscr{L}^D$ with ρ_t^n locally uniformly bounded from below. Hence, we may assume with no loss of generality that $\mu_t^n \to \mu_t$ narrowly for any $t \in [0,h]$.

By the lower semicontinuity of moments we get $\mu_t \in \mathscr{P}_2(\mathbf{R}^D)$ for any t, and the lower semicontinuity of Wasserstein distance (see for instance Proposition 7.1.3 in [4]) gives that the Lipschitz bound and the distance bound (6.10) are preserved in the limit. Also the inequality $\int S(\rho_t^n)\, dz \le \int S(\rho)\, dz$ with S convex and the local lower bound in (c) are easily seen to be stable under weak convergence, and imply (choosing $S = \underline{S}$ convex, growing faster than linearly at infinity, such that $\int \underline{S}(\rho)\, dz < +\infty$) that $\mu_t = \rho_t \mathscr{L}^D \in \mathscr{P}_2^a(\mathbf{R}^D)$ with $\rho_t \ge m_{r'}$ $\mathscr{L}^D$–a.e. on B_r for any $r > 0$.

It remains to show the validity of the continuity equation in (b). To this aim, it suffices to show that, for t fixed, $J\mathbf{v}_n\rho_t^n$ converge in the sense of distributions to $J\mathbf{v}\rho_t$. As $\underline{S}$ grows faster than linearly at infinity, we obtain from the inequality $\int \underline{S}(\rho_t^n)\, dz \le \int \underline{S}(\rho)\, dz$, that ρ_t^n is equi-integrable (see for instance Proposition 1.27 of [3]). Hence for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\mathscr{L}^D(B) < \delta \quad \Longrightarrow \quad \int_B \rho_t\, dz + \sup_n \int_B \rho_t^n\, dz < \varepsilon.$$

We fix $r > 0$ and choose as $B \subset B_r$ an open set given by Egorov theorem, so that $\mathbf{v}_n \to \mathbf{v}$ uniformly on $B_r \setminus B$; let also $\mathbf{v}' : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ be a continuous function coinciding with $\mathbf{v}$ on $B_r \setminus B$, with $|\mathbf{v}'| \le C(2+r)$. For any $\phi \in C_c(B_r)$ we have then

$$\int_{\mathbf{R}^D} \phi J\mathbf{v}_n\rho_t^n\, dz = \int_{\mathbf{R}^D} \phi(J\mathbf{v}_n - J\mathbf{v}')\rho_t^n\, dz + \int_{\mathbf{R}^D} \phi J\mathbf{v}\rho_t\, dz$$
$$+ \int_{\mathbf{R}^D} \phi(J\mathbf{v}' - J\mathbf{v})\rho_t\, dz + \int_{\mathbf{R}^D} \phi J\mathbf{v}'(\rho_t^n - \rho_t)\, dz,$$

so that

$$\limsup_{n\to+\infty} \left| \int_{\mathbf{R}^D} \phi J\mathbf{v}_n\rho_t^n\, dz - \int_{\mathbf{R}^D} \phi J\mathbf{v}\rho_t\, dz \right| \le 2C \sup|\phi|(2+r)\varepsilon.$$

As ε is arbitrary, this proves the weak convergence. QED.

Remark 6.4 (Stability of upper bounds)**.** By the same argument one can show that if $\rho \le M_r\ \mathscr{L}^D$–a.e. on B_r for any $r > 0$, then $\rho_t \le M_{r'}\ \mathscr{L}^D$–a.e. on B_r with $r' = e^{Ch}r + 2(e^{Ch} - 1)$.

The main result of this section is concerned with Hamiltonians H satisfying the following properties:

(H1) *There exist constants $C_o \in (0,+\infty)$, $R_o \in (0,+\infty]$ such that for all $\mu \in \mathscr{P}^2_a(\mathbf{R}^D)$ with $W_2(\mu,\bar\mu) < R_o$ we have $\mu \in D(H)$, $\partial H(\mu) \neq \emptyset$ and $\mathbf{w} = \nabla H(\mu)$ satisfies $|\mathbf{w}(z)| \le C_o(1+|z|)$ for μ–almost every $z \in \mathbf{R}^D$.*

(H2) *If $\mu = \rho\mathscr{L}^D$, $\mu_n = \rho_n\mathscr{L}^D \in \mathscr{P}^a_2(\mathbf{R}^D)$, $\sup_n W_2(\mu_n,\bar\mu) < R_o$ and $\mu_n \to \mu$ narrowly, then there exist a subsequence $n(k)$ and functions $\mathbf{w}_k, \mathbf{w} : \mathbf{R}^D \to \mathbf{R}^D$ such that $\mathbf{w}_k = \nabla H(\mu_{n(k)})\ \mu_{n(k)}$-a.e., $\mathbf{w} = \nabla H(\mu)\ \mu$-a.e. and $\mathbf{w}_k \to \mathbf{w}\ \mathscr{L}^D$–a.e. in $\mathbf{R}^D$ as $k \to +\infty$.*

To ensure the constancy of H along the solutions of the Hamiltonian system we consider also:

(H3) *$H : \mathscr{P}_2(\mathbf{R}^D) \to (-\infty,+\infty]$ is proper, lower semicontinuous and λ–convex for some $\lambda \in \mathbf{R}$.*

Recalling that $\mathscr{P}^a_2(\mathbf{R}^D)$ is dense in $\mathscr{P}_2(\mathbf{R}^D)$ it would be not difficult to show, by the same argument used at the beginning of the proof of Theorem 5.2, that (H3) and (H1) imply that H is Lipschitz continuous on the ball $\{\mu \in \mathscr{P}_2(\mathbf{R}^D) :\ W_2(\mu,\bar\mu) \le R_o\}$. Assumption (H2), instead, is a kind of "C^1-regularity" assumption on H. Thinking to the finite-dimensional theory (for instance to Peano's existence theorems for ODE's with a continuous velocity field) some assumption of this type seems to be necessary in order to get existence. In the following remark we discuss, instead, existence in the "flat" infinite-dimensional case and uniqueness in the finite-dimensional case.

Remark 6.5. Assume that we are given a convex (or λ-convex for some $\lambda \in \mathbf{R}$) Lipschitz function $H : \mathbf{R}^{2d} \to \mathbf{R}$. Then, $\partial H(x)$ is not empty for all $x \in \mathbf{R}^{2d}$ and we may define solutions of the Hamiltonian ODE those absolutely continuous maps $x : [0,+\infty) \to \mathbf{R}^{2d}$ satisfying $\mathbb{J}_d\dot{x}(t) \in \partial H(x(t))$ for a.e. $t \in [0,+\infty)$.

The same subdifferentiability argument used in the proof of Theorem 5.2 then shows that $t \mapsto H(x(t))$ is constant along Hamiltonian flows. Existence of Hamiltonian flows can be achieved by the following discrete scheme: fix a time parameter $h > 0$ and an initial datum $\bar{x} \in \mathbf{R}^{2d}$. Then, choose $p_0 \in \partial H(x_0)$ and set $x_h(t) = x_0 + \mathbb{J}_d p_0 t$ for $t \in [0,h]$, choose $p_1 \in \partial H(x_h(h))$ and set $x_h(t) = x_1 + \mathbb{J}_d p_1(t-h)$ for $t \in [h,2h]$ and so on. In this way $x_h(t)$ solves the "delayed" Hamiltonian equation

$$\mathbb{J}_d\dot{x}_h(t) \in \partial H\left(x_h(h[\frac{t}{h}])\right) \quad \text{for a.e. } t \ge 0. \tag{6.12}$$

Using a compactness and equi-continuity argument we can find a sequence $(h_i) \downarrow 0$ and a Lipschitz map $x : [0,\infty) \to \mathbf{R}^{2d}$ such that $x_{h_i}(t)$ converge to $x(t)$ as $i \to \infty$ for any $t \ge 0$ and $\dot{x}_{h_i}$ weakly converge in $L^2_{\text{loc}}([0,\infty);\mathbf{R}^{2d})$ to $\dot{x}$.

In order to show that $\mathbb{J}_d\dot{x} \in \partial H(x)$ a.e., we use an integral version of the discrete subdifferential inclusion, namely

$$H(y) \geq \int_0^\infty H(x_{h_i}(h_i[\frac{t}{h_i}]))\rho(t)\,dt + \int_0^\infty \langle y - x_{h_i}(h_i[\frac{t}{h_i}]), \mathbb{J}_d\dot{x}_{h_i}(t)\rangle\rho(t)\,dt,$$

with $\rho(t)$ nonnegative, with compact support and satisfying $\int \rho\,dt = 1$, and pass to the limit as $i \to \infty$ to find

$$H(y) \geq \int_0^\infty H(x(t))\rho(t)\,dt + \int_0^\infty \langle y - x(t), \mathbb{J}_d\dot{x}(t)\rangle\rho(t)\,dt.$$

Choosing properly a family ρ_i of approximations of δ_t, this yields

$$H(y) \geq H(x) + \langle y - x(t), \mathbb{J}_d\dot{x}(t)\rangle$$

at any Lebesgue point t of $\dot{x}$. This proves existence of Hamiltonian flows. We also refer the reader to a work in progress by Ghoussoub and Moameni [32] on related questions.

Notice that this scheme doesn't seem to work in the infinite-dimensional case, when $\mathbf{R}^{2d}$ is replaced by an infinite-dimensional phase space X, due to the difficulty of handling terms $\int\langle f_h(t), g_h(t)\rangle dt$ with f_h weakly converging in $L^2_{\mathrm{loc}}([0,+\infty);X)$ and $g_h(t)$ only pointwise weakly converging to $g(t)$. Indeed, we are not aware of any existence result in this direction.
Coming back to the finite-dimensional case $X = \mathbf{R}^{2d}$, the results in [5] (see also [6] for special classes of Hamiltonians) ensure a kind of "generic" uniqueness property, or uniqueness in the flow sense, in the same spirit of DiPerna–Lions' theory [25] (see §6 of [5] for a precise formulation). In brief, among all families of solutions $x(t,\overline{x})$ of the ODE, the condition

$$x(t,\cdot)_\#\mathscr{L}^{2d} \leq C\mathscr{L}^{2d} \quad \text{with } C \text{ independent of } t \tag{6.13}$$

determines x up to $\mathscr{L}^d$–negligible sets (i.e. if x and $\tilde{x}$ fulfil (6.13), then $x(\cdot,\overline{x}) = \tilde{x}(\cdot,\overline{x})$ for $\mathscr{L}^d$–a.e. $\overline{x}$) and the unique x satisfying (6.13) is stable within the class of approximations fulfilling (6.13) (in particular, one finds that $x(t,\cdot)$ is measure-preserving for all t). It turns out that the scheme described here produces a discrete flow $x_h(t,\overline{x})$ satisfying (6.13) with $C = 1$, and therefore is a good approximation of the unique Hamiltonian flow x. See also [45] for discrete schemes (called leap-frog schemes) that really preserve the symplectic forms and therefore the symplectic volume.

Theorem 6.6. *Assume that* (H1) *and* (H2) *hold and that* $T > 0$ *satisfies (6.18). Then there exists a Hamiltonian flow* $\mu_t = \rho_t\mathscr{L}^D : [0,T] \to D(H)$ *starting from* $\bar{\mu} = \bar{\rho}\mathscr{L}^D \in \mathscr{P}_2^a(\mathbf{R}^D)$*, satisfying* (5.1)*, such that the velocity field* $\mathbf{v}_t$ *coincides with* $\nabla H(\mu_t)$ *for a.e.* $t \in [0,T]$*. Furthermore,* $t \to \mu_t$ *is L–Lipschitz, with*

$$L^2 = 2C_o^2(1+M) \quad \textit{and} \quad M = e^{(25C_o^2+1)T}(1 + M(\bar{\mu})).$$

Finally, there exists a function $l(r)$ *depending only on* T *and* C_o *such that*

$$\bar{\rho} \geq m_r\ \mathscr{L}^D\text{-a.e. on } B_r\ \forall r > 0 \quad \Longrightarrow \quad \rho_t \geq m_{l(r)}\ \mathscr{L}^D\text{-a.e. on } B_r\ \forall r > 0 \tag{6.14}$$

and

(6.15) $\bar\rho \le M_r\ \mathscr{L}^D$*-a.e. on* $B_r\ \forall r>0 \quad \Longrightarrow \quad \rho_t \le M_{l(r)}\ \mathscr{L}^D$*-a.e. on* $B_r\ \forall r>0$.

If in addition (H3) *holds, then* $t\mapsto H(\mu_t)$ *is constant.*

Proof. In the first two steps of the proof, we shall assume existence of positive numbers m_r such that the initial datum satisfies $\bar\rho \ge m_r > 0\ \mathscr{L}^D$ a.e. on B_r for any $r>0$. That technical assumption will be removed only in the last step of the proof of the theorem.

Step 1. (a time discrete scheme). Since $\bar\rho$ is integrable, standard arguments give existence of a convex function $S:[0,+\infty)\to[0,+\infty)$, which grows faster than linearly at infinity and such that $\int S(\bar\rho)\,dz$ is finite. We fix an integer N sufficiently large, so that $C_oh<1/8$ and $1+C_oh/2<e^{C_oh}<1+2C_oh$ with $h=T/N$, and we divide $[0,T]$ into N equal intervals of length h. We shall next argue how, for any such N, Lemma 6.2 gives time discrete solutions $\mu_t^N=\rho_t^N\mathscr{L}^D$ satisfying:

(a) the Lipschitz constant of $t\mapsto\mu_t^N$ is less than $\bar L$, with $\bar L$ independent of N;
(b) $\sup_{N,t}W_2(\mu_t^N,\bar\mu)<R_o$, $\int S(\rho_t^N)\,dz\le\int S(\bar\rho)\,dz$ and $\rho_t^N\ge m_{l(r)}\ \mathscr{L}^D$-a.e. on B_r for any $r>0$;
(c) the "delayed" Hamiltonian equation

$$\frac{d}{dt}\mu_t^N+\nabla\cdot(J\mathbf{v}_t^N\mu_t^N)=0 \tag{6.16}$$

holds in the sense of distributions in $(0,T)\times\mathbf{R}^D$, with $\mathbf{v}_t^N=\nabla H(\mu_{ih}^N)$ for $0\le i\le N-1$ and $t\in[ih,(i+1)h)$.

In order to build μ_t^N, we apply Lemma 6.2 N times with $C=C_o$: we start with $\rho=\bar\rho$ and $\mathbf{v}=\nabla H(\bar\rho\mathscr{L}^D)$ to obtain a solution μ_t^N of (6.16) in $[0,h]$. Then, we apply the lemma again with $\rho=\rho_h^N$ and $\mathbf{v}=\nabla H(\rho_h^N\mathscr{L}^D)$ to extend it continuously to a solution of (6.16) in $[h,2h]$. In N steps we build the solution in $[0,T]$.

However, in order to be sure that the lemma can be applied each time, we have to check that the inequality $W_2(\mu_{ih}^N,\bar\mu)<R_o$ is valid for $i=0,\dots,N-1$, and this is where the restriction on T comes from: first notice that since

$$W_2(\mu_{(i+1)h}^N,\mu_{ih}^N)\le hC_o\sqrt{24(1+M_2(\mu_{ih}^N))}\ ,$$

by the triangle inequality we need only to prove by induction an upper bound of the form

$$M_2(\mu_{ih}^N)\le M, \tag{6.17}$$

for some M such that $C_oT\sqrt{24(1+M)}<R_o$. To estimate inductively the moments, we recall that $M_2(\mu)=W_2^2(\mu,\delta_0)$ and we use the triangle inequality to find

$$\begin{aligned} M_2(\mu^N_{(i+1)h}) &\leq \left(\sqrt{M_2(\mu^N_{ih})}+hC_o\sqrt{24(1+M_2(\mu^N_{ih}))}\right)^2 \\ &\leq (1+h)M_2(\mu^N_{ih})+24(1+\frac{1}{h})h^2C_o^2(1+M_2(\mu^N_{ih})) \\ &\leq (1+(25C_o^2+1)h)M_2(\mu^N_{ih})+25C_o^2h \end{aligned}$$

as soon as $24(h+1)<25$. Hence, setting for brevity $P=25C_o^2+1$, we have the inequality

$$M_2(\mu^N_{(i+1)h})\leq(1+Ph)M_2(\mu^N_{ih})+Ph.$$

By induction we get

$$M_2(\mu^N_{ih})\leq(1+Ph)^i(M_2(\bar{\mu})+1)-1$$

and setting $i=N$ we find that $M=e^{PT}(1+M_2(\bar{\mu}))$ is a good upper bound on all moments. We have proved that the lemma can be iterated N times, provided

$$C_oT\sqrt{24(1+e^{(25C_o^2+1)T}(1+M_2(\bar{\mu})))}<R_o. \tag{6.18}$$

Finally, let us find an explicit expression for the function $l(r)$ in (b) (the argument for (6.15) is similar, and based on Remark 6.4). As the constant r' in Lemma 6.2 is less than $re^{C_oh}+4C_oh$, by our choice of h, by induction on i we get

$$\rho_t^N\geq m_{r_i}\ \mathscr{L}^D\text{-a.e. on } B_r \text{ with } r_i=re^{iC_oh}+4C_oh(e^{(i-1)C_oh}+\cdots+1)\ \forall t\in[0,ih],\ 1\leq i\leq N.$$

Since

$$r_N=re^{NC_oh}+4C_oh\frac{e^{NC_oh}-1}{e^{C_oh}-1}\leq(r+8)e^{NC_oh}=(r+8)e^{C_oT},$$

it suffices to set $l(r)=(r+8)e^{C_oT}$.

Step 2. (passage to the limit). By (a), (b), $t\mapsto\mu_t^N$ are equi-bounded in $\mathscr{P}_2(\mathbf{R}^D)$, and equi-Lipschitz continuous. Hence, we may assume with no loss of generality that $\mu_t^N\to\mu_t$ narrowly for any $t\in[0,T]$.

By the lower semicontinuity of moments we get $\mu_t\in\mathscr{P}_2(\mathbf{R}^D)$ for any t, and the narrow lower semicontinuity of the Wasserstein distance (see for instance Proposition 7.1.3 of [4]) gives that the L-Lipschitz bound in (a) and the distance bound in (b) are preserved in the limit. Also the inequality $\int S(\rho_t^N)\,dz\leq\int S(\bar{\rho})\,dz$ and the local lower bounds in (b) are easily seen to be stable under weak convergence, hence $\mu_t=\rho_t\mathscr{L}^D$, and the conclusion of (6.14) holds with $l(r)=(r+8)e^{C_oT}$ (the argument for (6.15) is similar, and based on Remark 6.4).

It remains to show that μ_t is an Hamiltonian flow. To this aim, it is enough to show that, for any t fixed, $\mathbf{v}_t^N\mu_t^N$ converges, in the sense of distributions, to

$J\nabla H(\mu_t)\mu_t$. Assume by contradiction that this does not happen, i.e. there exist a subsequence N_i and a smooth test function φ such that

$$\inf_i \left| \int_{\mathbf{R}^D} \langle \mathbf{v}_t^{N_i}; \varphi \rangle d\mu_t^{N_i} - \int_{\mathbf{R}^D} \langle \mathbf{v}_t; \varphi \rangle d\mu_t \right| > 0. \tag{6.19}$$

Let us denote by $[\cdot]$ the greatest integer function. Notice that by assumption (H2) and the narrow convergence of $\mu^{N_i}_{[N_i t]/N_i}$ to μ_t we can assume with no loss of generality that

$$\mathbf{v}_t^{N_i} = J\nabla H(\mu^{N_i}_{[N_i t]/N_i}) \to J\nabla H(\mu_t) \qquad \mathscr{L}^D\text{–a.e. in } \mathbf{R}^{2d} \text{ as } i \to +\infty.$$

By the same argument used at the end of the proof of Lemma 6.2, based on Egorov theorem and the equi-integrability of $\rho_t^{N_i}$, we prove that $\mathbf{v}_t^{N_i}\mu_t^{N_i}$ converge in the sense of distributions to $J\nabla H(\mu_t)\mu_t$, thus reaching a contradiction with (6.19).

Therefore, it suffices to pass to the limit as $N \to \infty$ in (6.16) to obtain that μ_t is an Hamiltonian flow with velocity field $\mathbf{v}_t = \nabla H(\mu_t)$.

Step 3. Now we consider the general case. We strongly approximate $\bar\rho$ in $L^1(\mathbf{R}^D)$ by functions $\bar\rho^k$ such that $\bar\rho^k \mathscr{L}^D \in \mathscr{P}_2(\mathbf{R}^D)$ and, for any k, there exist constants $m_r^k > 0$ such that $\bar\rho^k \geq m_r^k\ \mathscr{L}^D$-a.e. on B_r for any $r > 0$ (for instance, convex combinations of $\bar\rho$ with a Gaussian). We also notice that the equi-integrability of $\{\bar\rho_k\}_{k=1}^\infty$ ensures the existence of a convex function S having a more than linear growth at infinity, and independent of k, such that $\int S(\bar\rho^k)\,dz \leq 1$ for any k.

The construction performed in Step 1 and Step 2 can then be applied for each k, yielding solutions of the Hamiltonian ODE $\mu_t^k = \rho_t^k \mathscr{L}^D$, $t \in [0,T]$, satisfying $\rho_0^k = \bar\rho^k$, $\int S(\rho_t^k)dx \leq 1$, and

$$\frac{d}{dt}\mu_t^k + \nabla \cdot (J\nabla H(\mu_t^k)\mu_t^k) = 0 \qquad \text{in } (0,T) \times \mathbf{R}^{2d}. \tag{6.20}$$

As, by construction, $t \mapsto \mu_t^k$ are L-Lipschitz, we can also assume, possibly extracting a subsequence, that $\mu_t^k \to \mu_t$ narrowly as $k \to +\infty$ for any $t \in [0,T]$. The upper bound on $\int S(\rho_t^k)dx$ then ensures that $\mu_t \in \mathscr{P}_2^a(\mathbf{R}^D)$ for all $t \in [0,T]$.

The same argument used in Step 2, based on (H2) and the equi-integrability of ρ_t^k, shows that for any $t \in [0,T]$, $J\nabla H(\mu_t^k)\mu_t^k$ converges to $J\nabla H(\mu_t)\mu_t$ as $k \to +\infty$ in the sense of distributions. Therefore, passing to the limit as $k \to +\infty$ in (6.20) we obtain that μ_t is a solution of the Hamiltonian ODE with velocity field $J\nabla H(\mu_t)$.

Let us next give a more explicit expression for the Lipschitz constant of $t \to \mu_t$. Recall that by (6.17), we have

$$M_2(\mu_{ih}^N) \leq M = e^{PT}(1 + M_2(\bar\mu)) \tag{6.21}$$

and $W_2(\mu_\tau, \bar\mu) < R_o$ for $\tau \in [0,T]$. Thus, (6.21) and (H1) imply that

$$\|\nabla H(\mu_\tau)\|^2_{L^2(\mu_\tau;\mathbf{R}^D)} \leq C_o^2 \int_{\mathbf{R}^D} (1+|z|)^2 d\mu_\tau(z) \leq 2C_o^2(1 + M(\mu_\tau)) \leq 2C_o^2(1+M). \tag{6.22}$$

This, together with (3.2), yields

$$W_2(\mu_s,\mu_t) \le \int_s^t \|\nabla H(\mu_\tau)\|_{L^2(\mu_\tau;\mathbf{R}^D)} d\tau \le L(t-s). \tag{6.23}$$

Finally, the constancy of $t \mapsto H(\mu_t)$ follows by the (essential) boundedness of $\|\mathbf{v}_t\|_{L^2(\mu_t;\mathbf{R}^D)}$ and Theorem 5.2. QED.

We conclude this section by showing a class of Hamiltonians satisfying the assumptions of Theorem 6.6.

Lemma 6.7. *Let $\nu \in \mathscr{P}_2(\mathbf{R}^D)$ with a bounded support and let $V : \mathbf{R}^D \to \mathbf{R}$ be λ_V convex, $W : \mathbf{R}^D \times \mathbf{R}^D \to \mathbf{R}$ convex and even, both differentiable and with at most quadratic growth at infinity. Then, for $a > 0$ the function*

$$H(\mu) = H_0(\mu) + \mathscr{V}(\mu) + \mathscr{W}(\mu) = -\frac{a}{2}W_2^2(\mu,\nu) + \int_{\mathbf{R}^{2d}} V\, d\mu + \frac{1}{2}\int_{\mathbf{R}^D\times\mathbf{R}^D} W\, d\mu\times\mu \tag{6.24}$$

is $(\lambda_V - a)$–convex, lower semicontinuous and satisfies (H1) *and* (H2).

Proof. Possibly rescaling V and W, we shall assume that $a = 1$. It is well known (see for instance [46] or Chaper 10 of [4]) that the potential energy $\mathscr{V}$ is λ_V–convex and lower semicontinuous, and that the interaction energy $\mathscr{W}$ is convex and lower semicontinuous. As a consequence, H is $(\lambda_V - 1)$–convex and lower semicontinuous.

In order to show (H1) it suffices to notice that both ∇W and ∇W have a growth at most linear at infinity, and prove that

$$\partial H(\mu) = \partial H_0(\mu) + \nabla V + (\nabla W * \mu) \qquad \forall \mu \in \mathscr{P}_2(\mathbf{R}^D), \tag{6.25}$$

taking also into account that Proposition 4.3 yields, in the case when $\mu \in \mathscr{P}_2^a(\mathbf{R}^D)$, $\partial H_0(\mu) = \{\mathbf{t}_\mu^\nu - \mathbf{id}\}$, and that $\mathbf{t}_\mu^\nu \in L^\infty(\mu;\mathbf{R}^D)$ (by the boundedness of the support of ν).

The inclusion $\supset$ in (6.25) is a direct consequence of the characterization (4.4) of the subdifferential and of the inequalities

$$\mathscr{V}(\nu) \ge \mathscr{V}(\mu) + \int_{\mathbf{R}^D} \langle \nabla V, \bar\gamma - \mathbf{id}\rangle\, d\mu + \frac{\lambda_V}{2} W_2^2(\mu,\nu)$$
$$\mathscr{W}(\nu) \ge \mathscr{W}(\mu) + \int_{\mathbf{R}^D} \langle (\nabla W) * \mu, \bar\gamma - \mathbf{id}\rangle\, d\mu$$

for $\gamma \in \Gamma_o(\mu,\nu)$ (see for instance [4]). In order to prove the inclusion $\subset$, we fix a vector $\xi \in \partial H(\mu)$ and define, for $\gamma \in \Gamma_o(\mu,\nu)$, the measures $\mu_t = ((1-t)\pi_1 + t\pi_2)_\# \gamma$ and $\gamma_t := (\pi_1, (1-t)\pi_1 + t\pi_2)_\# \gamma \in \Gamma_o(\mu,\mu_t)$. As $(\bar\gamma_t - \mathbf{id})\mu = t(\bar\gamma - \mathbf{id})\mu$, by applying the definition of subdifferential we obtain

$$\liminf_{t\downarrow 0} \frac{H(\mu_t) - H(\mu)}{t} \ge \int_{\mathbf{R}^D} \langle w, \bar\gamma - \mathbf{id}\rangle\, d\mu.$$

Now, the dominated convergence theorem gives

$$\lim_{t\downarrow 0}\frac{\mathscr{V}(\mu_t)-\mathscr{V}(\mu)}{t}=\int_{\mathbf{R}^{2d}}\langle\nabla V,\bar\gamma-\mathbf{id}\rangle\,d\mu,\qquad \lim_{t\downarrow 0}\frac{\mathscr{W}(\mu_t)-\mathscr{W}(\mu)}{t}=\int_{\mathbf{R}^{2d}}\langle(\nabla W)*\mu,\bar\gamma-\mathbf{id}\rangle\,d\mu,$$

so that

$$\liminf_{t\downarrow 0}\frac{H_0(\mu_t)-H(\mu)}{t}\geq\int_{\mathbf{R}^D}\langle\xi_0,\bar\gamma-\mathbf{id}\rangle\,d\mu$$

with $\xi_0=\xi-\nabla V-(\nabla W)*\mu$. Then, by (-1)–convexity of H_0 we get

$$H_0(\nu)\geq H_0(\mu)+\int_{\mathbf{R}^D}\langle\xi_0,\bar\gamma-\mathbf{id}\rangle\,d\mu-\frac{1}{2}W_2^2(\mu,\nu).$$

The previous inequality, together with Propositions 4.2 and 4.3, gives that $\xi_0\in\partial H_0(\mu)$.

Property (H2) follows directly from the identity

$$\partial H(\mu)=\{(\mathbf{t}_\mu^\nu-\mathbf{id})+\nabla V+(\nabla W)*\mu\}$$

and from Lemma 3.3. QED.

As shown in [38], another important class of convex functionals in $\mathscr{P}_2(\mathbf{R}^D)$ is provided by the so-called internal energy functional $\mu=\rho\mathscr{L}^D\mapsto\int S(\rho)\,dz$. However, as the subdifferential of this functional is not empty only when $L_S(\rho)$ is a $W^{1,1}$ function (here $L_S(y)=yS'(y)-S(y)$), these functionals fail to satisfy (H1).

The previous result can be extended to Hamiltonians generated from those of Lemma 6.7 through a sup-convolution. For simplicity we consider the case when neither potential nor interaction energies are present, but their inclusion does not present any substantial difficulty.

Lemma 6.8. *Assume that $\Omega\subset\mathbf{R}^D$ is a bounded open set, and that*

(a) *$K\subset\mathscr{P}(\Omega)$ is a convex set, with respect to the standard linear structure of $\mathscr{P}(\Omega)$, closed with respect to the narrow convergence;*
(b) *$\tilde J:K\to\mathbf{R}\cup\{+\infty\}$ is strictly convex with respect to the standard linear structure of $\mathscr{P}(\Omega)$, bounded from below and lower semicontinuous with respect to the narrow convergence.*

Define the Hamiltonian H on $\mathscr{P}_2(\mathbf{R}^D)$ by

$$-H(\mu)=\inf_{\nu\in K}\{\frac{1}{2}W_2^2(\mu,\nu)+\tilde J(\nu)\}.\tag{6.26}$$

Then H is (-1)–convex and lower semicontinuous, and satisfies (H1) *and* (H2).

Proof of Lemma 6.8. Since $\mu\mapsto -W_2^2(\mu,\nu)-\tilde J(\nu)$ is (-2)-convex for each $\nu\in K$, we obtain that H is (-1)-convex and so (H3) holds.

1. Notice first that $W_2^2(\cdot,\nu)$ is lower semicontinuous with respect to the narrow convergence (see for instance Proposition 7.1.3 of [4]). Since $\tilde J$ is bounded from below and lower semicontinuous, and since bounded sets in $\mathscr{P}_2(\mathbf{R}^D)$ are sequentially compact with respect to the narrow convergence, we obtain that the infimum in the definition of $-H$ is attained. Strict convexity of $\tilde{J}$ and convexity of $W_2^2(\cdot,\nu)$ give uniqueness of the minimizer, which we denote by $\nu(\mu)$. A compactness argument based on the uniqueness of $\nu(\mu)$ then shows that $\mu_n \to \mu$ in $\mathscr{P}_2(\mathbf{R}^D)$ implies $\nu(\mu_n) \to \nu(\mu)$ narrowly in $\mathscr{P}(\Omega)$. As Ω is bounded the map $\mu \mapsto \nu(\mu)$ is also continuous between $\mathscr{P}_2(\mathbf{R}^D)$ and $\mathscr{P}_2(\Omega)$.

2. Let $\mu_o \in \mathscr{P}_2^a(\mathbf{R}^D)$ and $\mu \in \mathscr{P}_2(\mathbf{R}^D)$. Clearly,

$$H(\mu) - H(\mu_o) \geq -\frac{1}{2}\Big(W_2^2(\mu,\nu(\mu_o)) - W_2^2(\mu_o,\nu(\mu_o))\Big).$$

This, together with the fact that the Wasserstein gradient of $\mu \to -\frac{1}{2}W_2^2(\mu,\nu(\mu_o))$ at μ_o is $\mathbf{t}_{\mu_o}^{\nu(\mu_o)} - \mathbf{id}$ (see (4.8)), yields that $\mathbf{t}_{\mu_o}^{\nu(\mu_o)} - \mathbf{id} \in \partial H(\mu_o)$ and so $\partial H(\mu_o)$ is nonempty.

To characterize the elements of $\partial H(\mu_o)$, let $\phi \in C_c^\infty(\mathbf{R}^D)$ and set

$$\mathbf{g}_s = \mathbf{id} + s\nabla\phi, \qquad \mu_s = \mathbf{g}_{s\#}\mu_o, \qquad \nu_s = \nu(\mu_s).$$

If $\xi \in \partial H(\mu_o)$, the fact that H is (-1)–convex implies that

$$H(\mu_s) - H(\mu_o) - \int_{\mathbf{R}^{2d}} \langle \xi; \mathbf{t}_{\mu_o}^{\mu_s} - \mathbf{id}\rangle d\mu_o + \frac{1}{2}W_2^2(\mu_o,\mu_s) \geq 0.$$

For $|s| << 1$, $\mathbf{g}_s$ is the gradient of a convex function and so, the previous inequality yields

$$\begin{aligned}
-s\int_{\mathbf{R}^D} \langle \xi;\nabla\phi\rangle d\mu_o \,+\, \frac{s^2}{2}\int_{\mathbf{R}^{2d}} |\nabla\phi|^2 d\mu_o &\geq H(\mu_o) - H(\mu_s) \\
&\geq \frac{1}{2}\Big(W_2^2(\mu_s,\nu_s) - W_2^2(\mu_o,\nu_s)\Big) \\
&\geq \frac{1}{2}\int_{\mathbf{R}^D} |\mathbf{id} - \mathbf{t}_{\mu_s}^{\nu_s}|^2 d\mu_s - \frac{1}{2}\int_{\mathbf{R}^D} |\mathbf{id} - \mathbf{k}_s \circ \mathbf{t}_{\nu_s}^{\mu_s}|^2 d\nu_s \\
&= \frac{1}{2}\int_{\mathbf{R}^D} |\mathbf{id} - \mathbf{t}_{\mu_s}^{\nu_s}|^2 d\mu_s - \frac{1}{2}\int_{\mathbf{R}^{2d}} |\mathbf{t}_{\mu_s}^{\nu_s} - \mathbf{k}_s|^2 d\mu_s. \qquad (6.27)
\end{aligned}$$

Here, we have set $\mathbf{k}_s = \mathbf{g}_s^{-1}$. One can easily check that

$$\mathbf{k}_s(y) = y - s\nabla\phi(y) + \frac{s^2}{2}\nabla^2\phi(y)\nabla\phi(y) + \varepsilon(s,y), \qquad (6.28)$$

where ε is a function such that $|\varepsilon(s,y)| \leq |s|^3\,||\phi||_{C^3(\mathbf{R}^{2d})}$. We combine (6.27) and (6.28) to conclude that

$$-s\int_{\mathbf{R}^D} \langle \xi;\nabla\phi\rangle d\mu_o + \frac{s^2}{2}\int_{\mathbf{R}^{2d}} |\nabla\phi|^2 d\mu_o \geq s\int_{\mathbf{R}^D} \langle \mathbf{id} - y;\nabla\phi\rangle d\gamma_s + o(s),$$

where γ_s is the unique optimal plan between μ_s and ν_s. Recall now that $\mu_s \to \mu_o$ in $\mathscr{P}_2(\mathbf{R}^D)$ and $\nu_s \to \nu$ in $\mathscr{P}_2(\Omega)$ as $s \to 0$, hence Lemma 3.3 gives

$$-s\int_{\mathbf{R}^D}\langle \xi;\nabla\phi\rangle d\mu_o + \frac{s^2}{2}\int_{\mathbf{R}^D}|\nabla\phi|^2 d\mu_o \geq s\int_{\mathbf{R}^D}\langle \mathbf{id}-\mathbf{t}_{\mu_o}^{\nu_o};\nabla\phi\rangle d\mu_o + o(s). \tag{6.29}$$

We divide both sides of (6.29) first by $s>0$ then $s<0$; letting $|s|\to 0$ we find

$$-\int_{\mathbf{R}^D}\langle \xi;\nabla\phi\rangle d\mu_o = \int_{\mathbf{R}^D}\langle \mathbf{id}-\mathbf{t}_{\mu_o}^{\nu_o};\nabla\phi\rangle d\mu_o.$$

This proves that $\pi_{\mu_0}\xi = \mathbf{t}_{\mu_o}^{\nu_o}-\mathbf{id}$. The minimality of the norm of the gradient then gives

$$\nabla H(\mu_o) = \mathbf{t}_{\mu_o}^{\nu_o}-\mathbf{id}. \tag{6.30}$$

¿From this representation of $\nabla H(\mu_o)$ and from (3.13) we obtain both (H1) and (H2). QED.

7 An alternative algorithm yielding existence of Hamiltonian flows for general initial data

In this section we provide a new discrete scheme providing existence of solutions to Hamiltonian flows for general initial data, i.e. not necessarily absolutely continuous with respect to Lebesgue measure. Being based on a linear interpolation at the level of transports, when particularized to Dirac masses this algorithm coincides with the one considered in Remark 6.5.

Lemma 7.1. *Let $f: X \to Y$ be a Borel map, $\mu \in \mathscr{P}(X)$, and let $\mathbf{v} \in L^2(\mu;\mathbf{R}^D)$. Then, setting $\nu = f_\#\mu$, we have $f_\#(\mathbf{v}\mu) = \mathbf{w}\nu$ for some $\mathbf{w} \in L^2(\nu;\mathbf{R}^D)$ with*

$$\|\mathbf{w}\|_{L^2(\nu;\mathbf{R}^D)} \leq \|\mathbf{v}\|_{L^2(\mu;\mathbf{R}^D)}. \tag{7.1}$$

Proof. Let $\sigma := f_\#(\mathbf{v}\mu)$ and $\varphi \in L^\infty(Y;\mathbf{R}^D)$; denoting by σ^α, $\alpha = 1,\cdots,N$, the components of σ we have

$$\left|\sum_{i=1}^D \int_Y \varphi^i\, d\sigma^i\right| \leq \|\varphi\circ f\|_{L^2(\mu;\mathbf{R}^D)}\|\mathbf{v}\|_{L^2(\mu;\mathbf{R}^D)} = \|\varphi\|_{L^2(\nu;\mathbf{R}^D)}\|\mathbf{v}\|_{L^2(\mu;\mathbf{R}^D)}.$$

Since φ is arbitrary this proves (7.1). QED.

Lemma 7.2. *Let $T>0$, $C\geq 0$, $\mu_t^n : [0,T] \to \mathscr{P}_2(\mathbf{R}^D)$ and $\mathbf{v}_t^n \in L^2(\mu_t;\mathbf{R}^k)$ be satisfying:*

(a) *$\mu_t^n \to \mu_t$ narrowly as $n \to +\infty$, for all $t \in [0,T]$;*
(b) *$\|\mathbf{v}_t^n\|_{L^2(\mu_t;\mathbf{R}^k)} \leq C$ for a.e. $t \in [0,T]$;*
(c) *the $\mathbf{R}^k$-valued space-time measures $\mathbf{v}_t^n\mu_t^n dt$ are weakly* converging in $(0,T)\times\mathbf{R}^D$ to σ.*

Then there exist $\mathbf{v}_t \in L^2(\mu_t;\mathbf{R}^k)$*, with* $\|\mathbf{v}_t\|_{L^2(\mu_t;\mathbf{R}^k)} \le C$ *for a.e.* t*, such that* $\sigma = \mathbf{v}_t\mu_t dt$*.*

Proof. Possibly extracting a subsequence we can also assume that the scalar space-time measures $|\mathbf{v}_t^n|\mu_t^n dt$ weak*-converge to ν, and it is well-known (see for instance Proposition 1.62(b) of [3]) that $|\sigma| \le \nu$. Since, by Hölder inequality, the projection of $|\mathbf{v}_t^n|\mu_t^n dt$ on $[0,T]$ is less than Cdt, the same is true for ν. Hence the disintegration theorem (see for instance Theorem 2.28 in [3]) provides us with the representation $\sigma = \sigma_t dt$ for suitable $\mathbf{R}^k$-valued measures in $\mathbf{R}^D$ having total variation less than C for a.e. t.

Now, for any $\varphi \in C_c^\infty(0,T)$, $\psi \in C_c^\infty(\mathbf{R}^D;\mathbf{R}^k)$ we have

$$\left|\int_0^T \varphi(t)\langle\psi;\sigma_t\rangle dt\right| = |\langle\varphi\psi;\sigma\rangle| = \lim_{n\to+\infty}\left|\int_0^T \varphi(t)\langle\psi;\mathbf{v}_t^n\mu_t^n\rangle dt\right| \le C\int_0^T |\varphi|(t)\sqrt{\langle|\psi|^2;\mu_t\rangle}dt.$$

As φ is arbitrary, this means that $|\langle\psi;\sigma_t\rangle| \le C\sqrt{\langle|\psi|^2;\mu_t\rangle}$ for a.e. t. By a density argument we can find a Lebesgue negligible set $N \subset (0,T)$ such that

$$|\langle\psi;\sigma_t\rangle| \le C\sqrt{\langle|\psi|^2;\mu_t\rangle} \qquad \forall\psi \in C_c^\infty(\mathbf{R}^D;\mathbf{R}^k), \quad \forall t \in (0,T)\setminus\mathscr{N}.$$

Hence, for any $t \in (0,T)\setminus\mathscr{N}$ we have $\sigma_t = \mathbf{v}_t\mu_t$ for some $\mathbf{v}_t \in L^2(\mu_t;\mathbf{R}^k)$ with $L^2(\mu_t;\mathbf{R}^k)$ norm less than C. QED.

We consider now two basic assumptions on the Hamiltonian, that are variants of those considered in the previous section.

(H1') *There exist constants* $C_o \in [0,+\infty)$*,* $R_o \in (0,+\infty]$ *such that for all* $\mu \in \mathscr{P}_2(\mathbf{R}^D)$ *with* $W_2(\mu,\bar\mu) < R_o$ *we have* $\mu \in D(H)$*,* $\partial H(\mu) \ne \emptyset$ *and* $\|\nabla H(\mu)\|_{L^2(\mu)} \le C_o$*.*

(H2') *If* $\sup_n W_2(\mu_n,\bar\mu) < R_o$ *and* $\mu_n \to \mu$ *narrowly, then*

$$\bigcap_{m=1}^{\infty} \overline{\mathrm{co}}\left(\{\nabla H(\mu_n)\mu_n :\ n \ge m\}\right) \subset \left\{\mathbf{w}\mu :\ \mathbf{w} \in \partial H(\mu)\cap T_\mu\mathscr{P}_2(\mathbf{R}^D)\right\}, \tag{7.2}$$

where $\overline{\mathrm{co}}$ *denotes the closed convex hull, with respect to weak***-topology.*

Remark 7.3. *(a) Assumption* (H1') *is weaker than* (H1)*, with the replacement of a pointwise bound with an integral one. Also* (H2') *is essentially weaker than* (H2)*, as it does not impose any "strong" convergence property on* $\nabla H(\mu_n)$*; however, this forces to consider a stability with respect to closed convex hulls.*

(b) A sufficient condition which ensures (H2') *is the following:*

(H2") If $\sup_n W_2(\mu_n,\bar\mu) < R_o$ and $\mu_n \to \mu$ narrowly, then

$$\nabla H(\mu_n)\mu_n \to \nabla H(\mu)\mu$$

in the sense of distribution.

(c) As in the previous section, the condition (H3) *ensures constancy of the Hamiltonian along the Hamiltonian flows. We can apply the same argument used*

at the beginning of the proof of Theorem 5.2, to obtain that (H3) *and* (H1') *imply that* H *is Lipschitz continuous on the ball* $\{\mu \in \mathscr{P}_2(\mathbf{R}^D):\ W_2(\mu,\bar\mu) \le R_o\}$.

Theorem 7.4. *Assume that* (H1') *and* (H2') *hold and that* $C_oT < R_o$*. Then there exists a Hamiltonian flow* $\mu_t : [0,T] \to D(H)$ *starting from* $\bar\mu \in \mathscr{P}_2(\mathbf{R}^D)$*, satisfying* (5.1)*, such that* $t \to \mu_t$ *is* C_o*–Lipschitz. Furthermore, if* (H3) *holds, then* $t \mapsto H(\mu_t)$ *is constant.*
In particular, if $\partial H(\mu) \cap T_\mu \mathscr{P}_2(\mathbf{R}^D) = \{\nabla H(\mu)\}$ *for all* μ *such that* $W_2(\mu,\bar\mu) < R_o$*, then the velocity field* $\mathbf{v}_t$ *in* (5.1) *coincides with* $\nabla H(\mu_t)$ *for a.e.* $t \in [0,T]$.

Proof. **Step 1.** (construction of a discrete solution). We fix an integer N sufficiently large and we divide $[0,T]$ in N equal intervals of length $h = T/N$. We build discrete solutions μ_t^N satisfying:

(a) the Lipschitz constant of $t \mapsto \mu_t^N$ is less than C_o;
(b) $W_2(\mu_t^N,\bar\mu) \le C_oT$;
(c) the "delayed" Hamiltonian equation

$$\frac{d}{dt}\mu_t^N + \nabla\cdot(\mathbf{w}_t^N \mu_t^N) = 0 \tag{7.3}$$

holds in the sense of distributions in $(0,T) \times \mathbf{R}^D$, with

$$\mathbf{w}_t^N \mu_t^N = \big(\mathbf{id} + (t-ih)J\nabla H(\mu_{ih}^N)\big)_\# \big(J\nabla H(\mu_{ih}^N)\mu_{ih}^N\big) \tag{7.4}$$

for $0 \le i \le N-1$ and $t \in [ih,(i+1)h)$.

We build first the solution in $[0,h]$, setting $\mathbf{w}_o^N = J\nabla H(\bar\mu)$. We then set

$$\mu_t^N = \big(\mathbf{id} + t\mathbf{w}_o^N\big)_\# \bar\mu, \qquad \mathbf{w}_t^N = \frac{\big(\mathbf{id} + t\mathbf{w}_o^N\big)_\# \big(\mathbf{w}_o^N\bar\mu\big)}{\mu_t^N}, \qquad t \in [0,h].$$

We claim that $\mathbf{w}_t^N$ is an admissible velocity field for μ_t^N. Indeed, for any $\varphi \in C_c^\infty(\mathbf{R}^D)$ we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^D}\varphi\, d\mu_t^N &= \frac{d}{dt}\int_{\mathbf{R}^D}\varphi(\mathbf{id}+t\mathbf{w}_o^N)d\bar\mu = \int_{\mathbf{R}^D}\langle\nabla\varphi(x+t\mathbf{w}_o^N);\mathbf{w}_o^N\rangle d\bar\mu\\ &= \sum_{i=1}^D\int_{\mathbf{R}^D}\frac{\partial\varphi}{\partial x_i}d\Big(\big(\mathbf{id}+t\mathbf{w}_o^N\big)_\#(\mathbf{w}_{oi}^N\bar\mu)\Big) = \int_{\mathbf{R}^D}\langle\nabla\varphi;\mathbf{w}_t^N\rangle d\mu_t^N.\end{aligned}$$

As φ is arbitrary, this proves that (7.3) is fulfilled in $[0,h]$. Notice also that Lemma 7.1 gives

$$\int_{\mathbf{R}^D}|\mathbf{w}_t^N|^2 d\mu_t^N \le \int_{\mathbf{R}^D}|\mathbf{w}_o^N|^2 d\bar\mu \le C_o^2 \qquad \forall t \in [0,h],$$

hence (3.2) gives that the Lipschitz constant of $t \mapsto \mu_t^N$ in $[0,h]$ is bounded by C_o. In particular $W_2(\bar\mu,\mu_t^N) \le C_oh$ for $t \in [0,h]$. We can repeat this process, setting $\mathbf{w}_h^N = J\nabla H(\mu_h^N)$ and introduce the following extensions on $(h,2h]$:

$$\mu_t^N = \big(\mathbf{id} + (t-h)\mathbf{w}_h^N\big)_\# \mu_h^N, \qquad \mathbf{w}_t^N := \frac{\big(\mathbf{id} + (t-h)\mathbf{w}_h^N\big)_\# \big(\mathbf{w}_h^N\mu_h^N\big)}{\mu_t^N}$$

for $t \in [h, 2h]$, with the Lipschitz constant of $t \mapsto \mu_t^N$ is bounded by C_o and the continuity equation (c) holding. By iterating this process N times we build a solution of (7.3), provided $NhC_o < R_o$. In summary, we have obtained that

$$(7.5)\qquad W_2(\mu_t^N, \bar{\mu}) \le C_o T, \quad \|\nabla H(\mu_t^N)\|_{L^2(\mu_t^N; \mathbf{R}^D)} \le C_o, \quad \|\mathbf{w}_t^N\|_{L^2(\mu_t^N; \mathbf{R}^D)} \le C_o$$

for $t \in [0, T]$. The first inequality in (7.5) is due to our choice of T and to the fact that $t \to \mu_t$ is C_o–Lipschitz. The second inequality is a consequence of (H1'). To obtain the last inequality in (7.5), we have used Lemma 7.1. By (7.5), we can readily conclude (a) and (b). The construction of μ_t^N and $\mathbf{w}_t^N$ is made such that (c) holds.

Step 2. (passage to the limit). By (a), (b), $t \mapsto \mu_t^N$ are equi-bounded in $\mathscr{P}_2(\mathbf{R}^D)$, and equi-Lipschitz continuous. Hence, we may assume with no loss of generality that $\mu_t^N \to \mu_t$ narrowly for any $t \in [0, T]$.

By the lower semicontinuity of moments we get $\mu_t \in \mathscr{P}_2(\mathbf{R}^D)$ for any t, moreover, the lower semicontinuity of $W_2(\cdot, \cdot)$ under narrow convergence gives that the C_o-Lipschitz bound in (a) and the distance bound in (b) are preserved in the limit.

It remains to show that μ_t solves the Hamiltonian ODE. To this aim, taking into account Lemma 7.2 and possibly extracting a subsequence (not relabelled for simplicity) we can assume that there exist $\mathbf{w}_t \in L^2(\mu_t; \mathbf{R}^D)$, with $\|\mathbf{w}_t\|_{L^2(\mu_t)} \le C_o$ for a.e. t, such that the space-time measures $\mathbf{w}_t^N \mu_t^N dt$ weak*-converge to $\mathbf{w}_t \mu_t dt$. We have to show that $\mathbf{w}_t = J\mathbf{v}_t$ for some $\mathbf{v}_t \in T_\mu \mathscr{P}_2(\mathbf{R}^D)$. To this aim, notice that

$$\lim_{N \to +\infty} \int_0^T \varphi(t) \langle \psi; \mathbf{w}_t^N \mu_t^N \rangle dt = \int_0^T \varphi(t) \langle \psi; \mathbf{w}_t \mu_t \rangle dt \qquad \forall \varphi \in C_c^\infty(0, T), \ \psi \in C_c^\infty(\mathbf{R}^D; \mathbf{R}^D).$$

For ψ fixed, this means that the maps $t \mapsto \langle \psi; \mathbf{w}_t^N \mu_t^N \rangle$ weakly converge in $L^2(0, T)$ to $\langle \psi; \mathbf{w}_t \mu_t \rangle$. Therefore, a sequence of convex combinations of them converges a.e. to $\langle \psi; \mathbf{w}_t \mu_t \rangle$ and we obtain

$$(7.6)\qquad \langle \psi; \mathbf{w}_t \mu_t \rangle \le \limsup_{N \to +\infty} \langle \psi; \mathbf{w}_t^N \mu_t^N \rangle$$

for a.e. $t \in [0, T]$. By a density argument we can find a Lebesgue negligible set $\mathscr{N} \subset (0, T)$ such that, for all $t \in (0, T) \setminus \mathscr{N}$, (7.6) holds for all $\psi \in C_o(\mathbf{R}^D; \mathbf{R}^D)$ (the closure, in the sup norm, of $C_c(\mathbf{R}^D; \mathbf{R}^D)$).

Now, fix $t \in (0, T) \setminus N$ where (7.6) holds for all $\psi \in C_o(\mathbf{R}^D; \mathbf{R}^D)$ and apply Hahn-Banach theorem to obtain that

$$\mathbf{w}_t \mu_t \in \bigcap_{M=1}^\infty K_{M,t}$$

where $K_{M,t}$ is the closed convex hull of $\{\mathbf{w}_t^N \mu_t^N\}_{N \ge M}$ with respect to the weak* topology. Indeed, fix M and assume by contradiction that $\mathbf{w}_t \mu_t$ does not belong to $K_{M,t}$. Then, we can strongly separate $\mathbf{w}_t \mu_t$ and $K_{M,t}$ by a continuous linear functional, induced by some function $\psi \in C_c(\mathbf{R}^D;\mathbf{R}^D)$, to obtain a contradiction with (7.6). As

$$\begin{aligned}\mathbf{w}_t^N \mu_t^N &= \Big(\mathbf{id} + (t-[Nt]/N)\mathbf{w}^N_{[Nt]/N}\Big)_\# (\mathbf{w}^N_{[Nt]/N}\mu^N_{[Nt]/N}) \\ &= \Big(\mathbf{id} + (t-[Nt]/N)J\nabla H(\mu^N_{[Nt]/N})\Big)_\# (J\nabla H(\mu^N_{[Nt]/N})\mu^N_{[Nt]/N})\end{aligned}$$

we obtain also that

$$\mathbf{w}_t\mu_t \in \bigcap_{M=1}^{\infty} \overline{\mathrm{co}}\left(\left\{J\nabla H(\mu^N_{[Nt]/N})\mu^N_{[Nt]/N} :\ N \geq M\right\}\right),$$

hence (H2') gives that $\mathbf{w}_t\mu_t = J\mathbf{v}_t\mu_t$ for some $\mathbf{v}_t \in \partial H(\mu_t) \cap T_{\mu_t}\mathscr{P}_2(\mathbf{R}^D)$.

Finally, the constancy of $t \mapsto H(\mu_t)$ follows by the (essential) boundedness of $\|\mathbf{v}_t\|_{L^2(\mu_t;\mathbf{R}^D)}$ and Theorem 5.2. QED.

Remark 7.5. *One can readily check that if we assume that* (H1') *and* (H2") *hold and that $C_oT < R_o$, then there exists a Hamiltonian flow $\mu_t : [0,T] \to D(H)$ starting from $\bar{\mu} \in \mathscr{P}_2(\mathbf{R}^D)$, satisfying* (1.2)*, such that $t \to \mu_t$ is C_o–Lipschitz. Furthermore, if* (H3) *holds, then $t \mapsto H(\mu_t)$ is constant.*

We can prove now the following extension of Lemma 6.7, where we drop the boundedness assumption on the support of ν.

Lemma 7.6. *Let $\nu \in \mathscr{P}_2(\mathbf{R}^D)$ and let V, W as in Lemma 6.7. Then the function H defined in* (6.24) *satisfies* (H1'), (H2') *and* (H3).

Proof. (H3) has already been proved in Lemma 6.7, while (H1') follows by the identity (6.25), taking into account that

$$\int_{\mathbf{R}^D} |\bar{\gamma} - \mathbf{id}|^2 d\mu \leq \int_{\mathbf{R}^D\times\mathbf{R}^D} |y-x|^2 d\gamma = W_2^2(\mu,\nu) \qquad \forall \gamma \in \Gamma_o(\mu,\nu).$$

Finally, let us check property (H2'). Let $\mathbf{w}\mu$ be the weak* limit of the convex combinations

$$\sum_{i=n}^{l(n)} \lambda_i^n \mathbf{w}_i\mu_i \quad \text{with } 0 \leq \lambda_i^n \leq 1,\ \sum_{i=n}^{l(n)} \lambda_i^n = 1,$$

and, representing as $\mathbf{w}_n = \bar{\gamma}_n - \mathbf{id}$ for suitable $\gamma_n \in \Gamma_o(\mu_n,\nu)$, define

$$\hat{\mu}_n = \sum_{i=n}^{l(n)} \lambda_i^n \mu_i, \qquad \hat{\gamma}_n = \sum_{i=n}^{l(n)} \lambda_i^n \gamma_i \in \Gamma(\hat{\mu}_n,\nu).$$

Let δ be a distance in $\mathscr{P}(\mathbf{R}^D \times \mathbf{R}^D)$ inducing the narrow convergence (see for instance Remark 5.1.1 of [4]). As any limit point with respect to the narrow topology of $\{\gamma_n\}_{n=1}^\infty$ belongs to $\Gamma_o(\mu,\nu)$ (see for instance Proposition 7.1.3 of [4]), a compactness argument gives an infinitesimal sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,+\infty)$ and $\eta_n \in$

$\Gamma_o(\mu,\nu)$ such that $\delta(\gamma_n,\eta_n)<\varepsilon_n$. In particular, setting $\hat\eta_n=\sum_{i=n}^{l(n)}\lambda_i^n\eta_i\in\Gamma_o(\mu,\nu)$ and noticing that δ is induced by a norm, we have

$$\delta(\hat\gamma_n,\hat\eta_n)\le\sup_{i\ge n}\varepsilon_i.$$

In particular, since $\Gamma_o(\mu,\nu)$ is narrowly closed, we infer that any limit point γ, in the narrow topology, of $\hat\gamma_n$, belongs to $\Gamma_o(\mu,\nu)$. Let γ be any of these limit points, along a subsequence $n(k)$, and notice that for any $\varphi\in C_c^\infty(\mathbf{R}^{2d};\mathbf{R}^{2d})$ we have

$$\begin{aligned}\langle\mathbf{w}\mu;\varphi\rangle &= \lim_{k\to+\infty}\langle\sum_{i=n(k)}^{l(n(k))}\lambda_i^{n(k)}(\overline{\gamma}_i-\mathbf{id})\mu_i;\varphi\rangle=\lim_{k\to+\infty}\int_{\mathbf{R}^D\times\mathbf{R}^D}\langle y-x;\varphi(x)\rangle d\hat\gamma_{n(k)}\\ &=\int_{\mathbf{R}^D\times\mathbf{R}^D}\langle y-x;\varphi(x)\rangle d\gamma=\langle(\overline{\gamma}-\mathbf{id})\mu;\varphi\rangle.\end{aligned}$$

As φ is arbitrary, this proves that $\mathbf{w}=\overline{\gamma}-\mathbf{id}$, hence (3.10) and Proposition 4.3 yield $\mathbf{w}\in T_\mu\mathscr{P}_2(\mathbf{R}^D)$ and $\mathbf{w}\in\partial H(\mu)$. QED.

8 Examples

In this section we briefly illustrate some PDE's fitting in our framework.

Semigeostrophic equations.

(a) If we set $d=1$ and $\nu=\chi_\Omega\mathscr{L}^2$ in Lemma 6.7, where $\Omega\subset\mathbf{R}^2$ is a bounded Borel set with $\mathscr{L}^2(\Omega)=1$, then

$$\frac{d}{dt}\mu_t+D_x\cdot\left(\mathbb{J}_1(T_{\mu_t}^\nu-\mathbf{id})\mu_t\right)=0$$

is the Hamiltonian ODE relative to $-W_2^2(\mu,\nu)/2$, thanks to (4.6). This PDE is a variant of the semigeostrophic equation. Notice that the (-1)–convexity of H is ensured by Proposition 4.3.

(b) When $d=1$ and $\tilde J(\rho)=\frac{1}{2}\int_\Omega\rho^2dx$, then the Hamiltonian ODE relative to

$$H(\mu):=\sup_{\rho\in K}-\frac{1}{2}W_2^2(\mu,\rho\mathscr{L}^3)-\tilde J(\rho)$$

corresponds to the semigeostrophic shallow water equation, studied in [17]. It suffices to apply Lemma 6.8.

(c) Finally, if $D=3$, $J(x,y,z)=(-y,x,0)$ and $H(\mu)=-W_2^2(\mu,\nu)/2$, with $\nu=\chi_\Omega\mathscr{L}^3$, then the Hamiltonian ODE is the 3-d semigeostrophic equation studied in [10] and [16].

Vlasov-Poisson and Vlasov-Monge-Ampére equations.

Suppose that $d\ge1$, $\nu=(\chi_\Omega\mathscr{L}^d)\times\delta_0$, where $\Omega\subset\mathbf{R}^d$ is a bounded Borel set with $\mathscr{L}^d(\Omega)=1$, and δ_0 is the Dirac mass in $\mathbf{R}^d$. Then, as shown in [18], the Hamiltonian in Lemma 6.7 decouples into

$$H(\mu)=-\frac{1}{2}M_2(\mu^2)-\frac{1}{2}W_2^2(\mu^1,\chi_\Omega\mathscr{L}^d),$$

where μ^1 (resp. μ^2) is the first (resp. second) marginal of μ. This is due to the fact the optimal transport map $\mathbf{t}_\mu^\nu$ between $\mu \in \mathscr{P}_2^a(\mathbf{R}^{2d})$ and ν has necessarily the form $(\mathbf{t},0)$, where $\mathbf{t}$ is the optimal transport map between μ_1 and $\chi_\Omega \mathscr{L}^d$, and an analogous property holds at the level of optimal plans, when μ is a general measure in $\mathscr{P}_2(\mathbf{R}^{2d})$.

Setting $\mu_t = f(t,\cdot)\mathscr{L}^{2d}$ and $\rho_t(x) = \int_{\mathbf{R}^d} f(t,x,v)\,dv$ (i.e. the first marginal of μ, we have then obtained the Hamiltonian for the Vlasov-Monge-Ampère (VMA) equation studied in [12] and more recently in [18], which is (up to a scaling argument)

$$\begin{cases} \frac{d}{dt} f(t,x,v) + D_x \cdot \Big(v f(t,x,v) \Big) & = D_v \cdot \Big(f(t,x,v) \nabla_x \Phi_{\rho_t}(x) \Big) \\ (\mathbf{id} - \nabla_x \Phi_{\rho_t})_\# \rho_t = \chi_\Omega \mathscr{L}^d, & \text{with } |x|^2/2 - \Phi_{\rho_t}(x) \text{ convex.} \end{cases} \tag{8.1}$$

Note that when $d=1$ the relation between ρ_t and Φ_{ρ_t} reduces to $\rho_t = 1 - \partial_{xx}\Phi_{\rho_t}$ and so (8.1) is nothing but the well-known Vlasov-Poisson equation. Our existence result Theorem 6.6 covers the case of absolutely continuous solutions, while Theorem 7.4 covers, thanks to Lemma 7.6, also the case of general initial data: in this case (VMA) has to be understood as follows:

$$\begin{cases} \frac{d}{dt} \mu_t + D_x \cdot (v \mu_t) = D_v \cdot ((\mathbf{id} - \overline{\gamma}) \mu_t) \\ \gamma \in \Gamma_0(\mu_t^1, \chi_\Omega \mathscr{L}^d). \end{cases} \tag{8.2}$$

Indeed, any $\gamma' \in \Gamma_o(\mu_t, \chi_\Omega \mathscr{L}^d \times \delta_0)$ can be written as a product $\gamma \times (\mathbf{id} \times 0)_\# \mu_t^2$, with $\gamma \in \Gamma_o(\mu_t^1, \chi_\Omega \mathscr{L}^d)$, so that $\overline{\gamma'} = (\overline{\gamma}, 0)$. Finally, it would be interesting to compare carefully, in one space dimension, our existence result for the Vlasov-Poisson equation with the existence result in [47]. Here we just mention that on the one hand our result allows more general initial data (no exponential decay of the velocities is required), on the other hand the solution built in [47] has additional space-time *BV* regularity properties related to velocity averaging, that are used to define the product $D_v \cdot (f \nabla_x \Phi_{\rho_t})$.

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