A Weak KAM theorem; from finite to infinite dimension

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Abstract

These notes contain a series of lectures given by the first author in the 2008 GNAMPA–INDAM School in Pisa. It is based on recent results by both authors who initiated a study of an infinite dimensional weak KAM theory. While some of the results presented here have already appeared in their joint work [16], the core of this manuscript, section 3, has never been submitted for publication anywhere. The current manuscript should be regarded as a companion to [16].

In [16] it is shown that asymptotic behaviour of a class of Vlasov systems can be studied via a cell problem (C): $H(M, c + \nabla U) = \bar{H}(c)$. (C) is to be satisfied in the sense of viscosity on what will be referred to as the $L^2(0,1)$–torus (cf. (10)), a quotient space of the Hilbert space $L^2(0,1)$. More importantly, existence of solutions U for (C) and existence of calibrated curves associated to U are obtained by studying the limit as ϵ tend to zero of Hamilton-Jacobi equations of the form $(HJE)_\epsilon$: $\epsilon V + H(M, c + \nabla V) = 0$. The purpose of these lectures is to show that if H satisfies appropriate invariance properties, a Galerkin type approximation method can be used to establish existence of solutions for $(HJE)_\epsilon$. This result is in contrast with [8], where it is shown that Galerkin type methods are not expected to provide solutions for $(HJE)_\epsilon$. We could have identified the largest class of Hamiltonians for which the results in these notes hold. We chose not to work in the greatest generality for two reasons: first of all, our study was motivated by the Vlasov systems appearing in kinetic theory and we restrict ourselves to Hamiltonians corresponding to these PDEs. Secondly, we tried to keep the computations as simple as possible to separate the main ideas from technical details.

1 Introduction

To understand the subtlety of the so-called cell problem appearing in the KAM and the Weak KAM theories, let us start with a very simple Hamiltonian. Consider the one-dimensional Hamiltonian $h(x,p) = |p|^2 - \sin^2(\pi x)$ for $x, p \in I\!R$. As done in the *KAM theory* let us proceed

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with the exercise of searching for real numbers λ and functions $s \in C^1(I\!R)$ such that s' is $\mathbb{Z}$-periodic and satisfies the equation $h(x, s'(x)) = \lambda$. Since every function $s \in C^1(I\!R)$ such that s' is $\mathbb{Z}$-periodic can be decomposed as the sum of a $\mathbb{Z}$-periodic function $u \in C^1(I\!R)$ and a linear function $x \to cx$, the problem at hand is to find two reals numbers λ, c and a $\mathbb{Z}$-periodic function $u \in C^1(I\!R)$ such that

$$h(x, c + u'(x)) = \lambda \tag{1}$$

has a solution. For such a simple Hamiltonian, at a first glance, (1) might look like a simple problem. It is wrong to think that existence of solutions depends mainly on whether or not λ is in the range of h. If $c = 0$ or $|c| = 2/\pi$ one readily checks that unless $\lambda = 0$ there is no $\mathbb{Z}$-periodic function $u \in C^1(I\!R)$ satisfying (1). If $0 < |c| < 2/\pi$ then (1) has no $\mathbb{Z}$–periodic solution $u \in C^1(I\!R)$. In the *Weak KAM theory* one seeks for real numbers λ and $\mathbb{Z}$-periodic Lipschitz functions $u \in C(I\!R)$ viscosity solutions of (1). For $c \in I\!R$ prescribed it is well-known that (1) admits a periodic viscosity solution for exactly one value of λ denoted $\bar{h}(c)$. This is one way of defining the effective Hamiltonian $\bar{h}$ which can be explicitly computed as a function of c [18]. The so-called cell problem (1) plays a central role in understanding the dynamics of the Hamiltonian flow of h. Especially as we are dealing with a Hamiltonian defined on the cotangent bundle to a one-dimensional manifold, (1) can be used to identify periodic solutions of the flow associated h (cfr. [22]).

The purpose of [16] has been to extend techniques of the Weak KAM theory– proven to be a powerful tool for understanding finite dimensional Hamiltonian systems– to some classes of PDEs viewed as infinite dimensional Hamiltonian systems. In these notes, we show that the infinite dimensional analogue of (1) can be obtained by studying finite dimensional Hamiltonian systems. In order to explain the link between the existing (finite dimensional) Weak KAM theory and our approach let us fix a $\mathbb{Z}$-periodic function $W \in C^2(I\!R)$. For each positive integer n consider the evolutive system

$$\begin{cases} \dot{x}_i &= np_i \\ \dot{p}_i &= -\frac{1}{n^2}\sum_{j=1}^n W'(x_i - x_j) \\ x_i(0) &= \bar{x}_i, \quad p_i(0) = \bar{p}_i. \end{cases} \tag{2}$$

Here, $\bar{x}, \bar{p} \in I\!R^n$ are prescribed. Clearly, (2) is a Hamiltonian system where the Hamiltonian and the Lagrangian are defined on $I\!R^n \times I\!R^n$ by

$$h(x,p) = \frac{n}{2}|p|^2 + \frac{1}{2n^2}\sum_{i,j=1}^n W(x_i - x_j), \quad l(x,v) = \frac{1}{2n}|v|^2 + \frac{1}{2n^2}\sum_{i,j=1}^n W(x_i - x_j). \tag{3}$$

Let $(\bar{x}, \bar{p})$ be the initial conditions in (2). The Hamiltonian flow is ϕ defined by $\phi_t(\bar{x}, \bar{p}) = (x(t), p(t))$. Let $\mathcal{G}_n$ be the group of permutations of n letters. For $\tau \in \mathcal{G}_n$ and $k \in \mathbb{Z}^n$ we define

$$P_\tau(x,p) = (x^\tau, p^\tau), \quad T_k(x,p) = (x + k, p)$$

where $x, p \in I\!R^n$ and $x^\tau := (x_{\tau(1)}, \cdots, x_{\tau(n)})$. Since $h \circ P_\tau = h = h \circ T_k$ we have

$$\phi_t \circ P_\tau = \phi_t = \phi_t \circ T_k. \tag{4}$$

Thus, h can be viewed as a function defined on $(\mathcal{M}_n/\mathcal{G}_n) \times I\!R^n$ and ϕ can be viewed as a flow on the cotangent bundle of the n–symmetric product of the circle $\mathcal{M}_n/\mathcal{G}_n$, known to be a manifold [23]. Here, $\mathcal{M}_n$ is the n–dimensional flat torus and we shall identify the cotangent bundle of $\mathcal{M}_n/\mathcal{G}_n$ with $(\mathcal{M}_n/\mathcal{G}_n) \times I\!R^n$.

If a probability measure μ on the cotangent bundle $T^*\mathcal{M}_n$ is invariant under the flow ϕ, then

$$\int_{T^*\mathcal{M}_n} \langle \nabla u(x), \nabla_p h(x,p)\rangle \mathrm{d}\mu(x,p) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{T^*M} f(\phi_t(x,p))\mathrm{d}\mu(x,p)\Big|_{t=0} = 0, \tag{5}$$

where $f(x,p) = u(x)$ and u is continuous and $\mathbb{Z}^n$-periodic. We write $u \in C^1(\mathcal{M}_n)$. Let $F(x,p) = (x, \nabla_p h(x,p))$ be the Legendre map associated to h. The push-forward of μ by F is the measure $\nu := F_\# \mu$ defined on the tangent bundle $T\mathcal{M}_n$ by

$$\nu(B) = \mu\big(F^{-1}(B)\big)$$

for all Borel sets $B \subset T\mathcal{M}_n$. Note that (5) reads off

$$\int_{T\mathcal{M}_n} \langle \nabla u(x), v\rangle \mathrm{d}\nu(x,v) = 0. \tag{6}$$

One says that ν is weakly invariant under the flow ϕ, although the definition in (6) does not involve h or ϕ. Suppose $S \in C^2(\mathbb{R}^n)$ and ∇S is $\mathbb{Z}^n$-periodic. In other words, we are selecting a closed one–form $(x,\xi) \in \mathcal{M}_n \times \mathbb{R}^n \to \Lambda_x(\xi)$ of class C^1 on $\mathcal{M}_n$. One readily checks existence of a $\vec{c} \in \mathbb{R}^n$ (characterizing the cohomology class of Λ) such that

$$\int_{\mathcal{M}_n \times \mathbb{R}^n} \langle \nabla S(x), v\rangle \mathrm{d}\nu(x,v) = \langle \vec{c}, R(\nu)\rangle.$$

Here,

$$R(\nu) := \int_{T\mathcal{M}_n} v \mathrm{d}\nu(x,v)$$

and is referred to as the rotation number of ν.

In the Weak KAM theory one seeks for Borel measures on the tangent bundle $T\mathcal{M}_n$ that minimize the action

$$\nu \to \mathcal{A}_n(\nu) := \int_{T\mathcal{M}_n} l \mathrm{d}\nu.$$

The minimization is performed over the set of weakly invariant measures ν of prescribed rotation vector $\vec{r} \in \mathbb{R}^n$. For a class of Lagrangians including those appearing in (3) such minimal measures are known to exist and are supported by the subdifferentials of functions $x \to \langle \vec{c}, x\rangle + u(x)$. Here $u \in C(\mathcal{M}_n)$ is a viscosity solutions of the cell problem

$$h(x, \vec{c} + \nabla u) = \bar{h}(\vec{c}) \tag{7}$$

and $\vec{c} \in \mathbb{R}^n$ is related to $\vec{r}$. We have denoted by $\bar{h}$ the effective Hamiltonian of h, defined by the fact that $\bar{h}(\vec{c})$ is the unique real number λ such that $h(x, \vec{c} + \nabla u) = \lambda$ admits a viscosity solution $u \in C(\mathcal{M}_n)$.

In these notes, not only are we interested in measures ν that minimize $\mathcal{A}_n$ over the set of weakly invariant measures of precribed rotation number $\vec{r}$, but we also require these measures to be invariant under the action of the group $\mathcal{G}_n : P_{\tau\#}\nu = \nu$ for all $\tau \in \mathcal{G}_n$. The latter condition yields that $\vec{r}$ must be parallel to $(1, \cdots, 1) \in \mathbb{R}^n$. It becomes natural to impose in (7) that $\vec{c}$ must be parallel to $(1, \cdots, 1) \in \mathbb{R}^n$. As a matter of fact, only for these special $\vec{c}$, were we able to show that as we let n tend to infinity, the finite dimensional solutions of (7) converge to their infinite dimensional analogue.

A formal explanation for restricting ourselves to $\vec{c}$ which are parallel to $(1, \cdots, 1) \in \mathbb{R}^n$ is based on the link between (2) and the Vlasov systems. The starting point is to view $T^*\mathcal{M}_n$ as a subset of $\mathcal{P}_2(I\!R^2)$, the set of Borel probability measures on $I\!R^2$ of bounded second moments. The embedding is given by $(\bar{x}, \bar{p}) \to 1/n \sum_{i=1}^n \delta_{(\bar{x}_i, \bar{p}_i)}$. Hence, to the path $t \to (x(t), p(t)) \in T^*\mathcal{M}_n$ satisfying (2) we associate the path $t \to f_t \in \mathcal{P}_2(I\!R^2)$ defined for each $t \geq 0$ by

$$f_t = \frac{1}{n} \sum_{i=1}^n \delta_{(x_i(t), \dot{x}_i(t))}.$$

Let ϱ_t be the first marginal of f_t: $\varrho_t = \frac{1}{n}\sum_{i=1}^n \delta_{x_i(t)}$ and set $P_t = \varrho_t * W$. The system of equations (2) translates into the so-called Vlasov system

$$\begin{cases} \partial_t f_t + v\partial_x f_t &= \partial_x P_t \, \partial_v f_t \\ \qquad\quad P_t(x) &= \int_{I\!R} W(x - \bar{x}) \mathrm{d}\rho_t(x) \\ \qquad\qquad f_0 &= \bar{f} := \frac{1}{n}\sum_{i=1}^n \delta_{\left(\bar{x}_i, n\bar{p}_i\right)}. \end{cases} \tag{8}$$

The first equation in (8) must be understood in the sense of distributions. While (8) is a richer system than (2) in the sense that it encompasses the case $n = \infty$, both systems coincide when $n < \infty$. In the latter case both systems represents the evolution of n undistinguishable particles of same mass. The fact that the particles are undistinguishable explains why the rotation vectors of interest, from the point of view of the Vlasov systems, must be vectors whose components are equal.

Gangbo [27] has noticed that (8) can be regarded as an infinite-dimensional Hamiltonian ODE on the space of Borel probability measures on $I\!R^2$ with finite second-order moments (cfr. also [1] and [14]). Indeed, if

$$\mathcal{H}(f) := \iint_{I\!R^2} \left[\frac{v^2}{2} + \frac{1}{2}\iint_{I\!R^2} W(x-y)\mathrm{d}f(y,w)\right] \mathrm{d}f(x,v),$$

then one may regard (8) as

$$\partial_t f + \mathrm{div}\big[J\nabla_{\mathrm{w}}\mathcal{H}(f) f\big] = 0,$$

where J is the clockwise rotation matrix of angle $\pi/2$, and ∇_{w} is the Wasserstein gradient [1]. In the current manuscript we look for some special solutions, which allow for a connection with a more conventional way of regarding (8) as Hamiltonian. Assume the initial data is in the set of probabilities on $I\!R^2$ such that $f_0 = (M_0, N_0)_{\#}\nu_0$, where ν_0 is the Lebesgue measure on $(0,1)$ and $M_0,\ N_0 \in L^2(0,1)$. This means

$$\int_{I\!R^2} \varphi(x,v)\mathrm{d}f_0(x,v) = \int_0^1 \varphi(M_0(y), N_0(y))\mathrm{d}y \text{ for all } \varphi \in C_c(I\!R^2).$$

Let us introduce the initial value problem

$$\ddot{\sigma}_t z = -\int_0^1 W'(\sigma_t z - \sigma_t w)\mathrm{d}w, \qquad \sigma_0 = M, \quad \dot{\sigma}_0 = N. \tag{9}$$

This is an evolutive system on the infinite dimensional manifold $L^2(0,1)$, which is a separable Hilbert space. We denote its inner product by $\langle\cdot,\cdot\rangle$ and its norm by $\|\cdot\|$. The space $L^2(0,1)$ has a natural differential structure and at each $M \in L^2(0,1)$ the tangent space at M is $\mathcal{T}_M L^2(0,1) = L^2(0,1)$. Hence, the tangent bundle is $\mathcal{T}L^2(0,1) := L^2(0,1) \times L^2(0,1)$ which we identify with the cotangent bundle.

Let $L^2_{\mathbb{Z}}(0,1)$ be the set of $M \in L^2(0,1)$ whose ranges are contained in $\mathbb{Z}$. We define the $L^2_{\mathbb{Z}}(0,1)$–torus by

$$\mathbb{T} := L^2(0,1)/L^2_{\mathbb{Z}}(0,1). \tag{10}$$

We say that $\mathcal{W} : L^2(0,1) \to I\!R$ is $L^2_{\mathbb{Z}}(0,1)$-periodic $\mathcal{W}(M+Z) = \mathcal{W}(M)$ for all $M \in L^2(0,1)$ and all $Z \in L^2_{\mathbb{Z}}(I)$. We view $\mathcal{W}$ as a function defined on the $\mathbb{T}$. If, in addition, $\mathcal{W}$ is continuous, we write $\mathcal{W} \in C(\mathbb{T})$.

Suppose Λ is a $L^2_{\mathbb{Z}}(0,1)$-periodic, differentiable, closed one-form on $L^2(0,1)$ in the sense of [16] section 5. Suppose that $M \to \Lambda_M(M)$ is Lipschitz and rearrangement invariant and $L^2_{\mathbb{Z}}(0,1)$-periodic. Suppose the second moment of γ, a Borel probability on $\mathcal{T}L^2(0,1)$, is finite:

$$\int_{\mathcal{T}L^2(0,1)} l_2(N) \mathrm{d}\gamma(M,N) < \infty, \qquad l_2(N) := \|N\|^2_{L^2(0,1)}.$$

By a *rearrangement invariant* map U defined on $L^2(0,1)$ we understand a map satisfying $U(M) = U(N)$ for all $M,\ N \in L^2(0,1)$ such that $M_{\#}\nu_0 = N\#\nu_0$. Then there exists a real number c and a Lipschitz function $U \in C^1(\mathbb{T})$ such that

$$\Lambda_M(N) = c + \mathrm{d}U_M(N)$$

for $M, N \in L^2(0,1)$. If γ is a Borel measure on $\mathcal{T}L^2(0,1)$ invariant under the flow Ψ in the sense that $\Psi_{t\#}\gamma = \gamma$ for all $t > 0$ we use arguments similar to those appearing in (5) to obtain that

$$\int_{\mathcal{T}L^2(0,1)} \Lambda_M(N) \mathrm{d}\gamma(M,N) = R(\gamma)\ c,$$

where

$$R(\gamma) := \int_{\mathcal{T}L^2(0,1)} l(N) \mathrm{d}\gamma(M,N) \qquad l(N) := \int_0^1 N \mathrm{d}\nu_0.$$

We refer to $R(\gamma)$ as the *rotation number* of γ.

If $W \in C^{1,1}(I\!R)$, we apply the Cauchy-Lipschitz-Picard Theorem [4] to obtain that for any initial data $(\bar{M}, \bar{N}) \in L^2(0,1) \times L^2(0,1)$ the problem (9) admits a unique solution $\sigma \in H^2(0,\infty; L^2(0,1))$. We define the Eulerian flow

$$\Psi(t,M,N) = (\Psi^1(t,M,N), \Psi^2(t,M,N)) = (\sigma_t, \dot{\sigma}_t). \tag{11}$$

We can then easily check that

$$f_t := (M(t,\cdot), \dot{M}(t,\cdot))_{\#}\chi_{(0,1)} \text{ with } f_0 = (M_0, N_0)_{\#}\chi_{(0,1)}$$

satisfies (8). Note that (9) is Hamiltonian and the energy $E(t) := H(\Psi(t,M,N))$ is conserved: $E(0) = E(t)$. Here, the Hamiltonian and the Lagrangian $H, L : L^2(0,1) \times L^2(0,1) \to I\!R$ are given by

$$H(M,N) = \frac{1}{2}\|N\|^2 + \frac{1}{2}\mathcal{W}(M), \qquad L(M,N) = \frac{1}{2}\|N\|^2_{\nu_0} - \frac{1}{2}\mathcal{W}(M) \tag{12}$$

where

$$\mathcal{W}(M) := \int_{(0,1)^2} W(Mz - Mw) \mathrm{d}z \mathrm{d}w.$$

It is not a loss of generality to assume that $W(0) = 0$ and W is even. Indeed, we may substitute W by $W - W(0)$ without altering (9). Also, substituting W by $z \to [W(z) + W(-z)]/2$ will not alter $\mathcal{W}$. In order to make some computations simpler, we further assume that

$$W(z) = W(-z) \leq W(0) = 0 \text{ for all } z \in I\!R. \tag{13}$$

The only restrictive assumption here is that the maximum of W is attained at 0.

Let $\mathcal{G}$ be the set of bijections $G : [0,1] \to [0,1]$ such that G, G^{-1} are Borel and push ν_0 forward to itself. The group $\mathcal{G}$ acts on $L^2(0,1) : (G,M) \in \mathcal{G} \times L^2(I) \to M \circ G$. It also acts on the topological subspace $L^2_{\mathbb{Z}}(0,1)$ and so, induces a natural action on $\mathbb{T}$ and on the tangent bundle $L^2(0,1) \times L^2(0,1)$. Note that L and H are invariant under the action of $\mathcal{G}$.

Our goal is to prove the following result: for each fixed positive integer n, let $\mathcal{C}_n$ be the set of real valued functions M on $[0,1]$, constant on each subinterval $I^n_i := ((i-1)/n, i/n)$, $i = 1, \cdots, n$. Let $L^n \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $H^n \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ be the Lagrangian and Hamiltonian defined in (25) and (26). We fix $c \in \mathbb{R}$. The standard Hamilton-Jacobi theory provides us with an explicit way for constructing $u^n(\cdot; c) \in C(\mathbb{R}^n)$, $\mathbb{Z}^n$-periodic viscosity solutions of $H^n(x, \nabla u^n(x;c) + \mathbf{c}_n) = c^2/2$ in $\mathbb{R}^n$, where $\mathbf{c}_n := (c, c, ..., c) \in \mathbb{R}^n$. Let us introduce the notation

$$L_c(M,N) := L(M,N) - c \int_0^1 M \mathrm{d}x. \tag{14}$$

Theorem 1.1. *There exists a rearrangement invariant $\mathcal{U}(\cdot;c)$, Lipschitz continuous in the strong $L^2(0,1)$-topology satisfying the following properties: for all $n \geq 1$ integer and $x \in \mathbb{R}^n$, $u^n(x;c) = \mathcal{U}\big(\sum_{i=1}^n x_i \chi_{I_i^n}; c\big)$. Furthermore, $\mathcal{U}(\cdot;c)$ is a viscosity solution for*

$$H(M, \nabla_{L^2} \mathcal{U}(M;c) + c) = \frac{c^2}{2} \tag{15}$$

and $\mathcal{U}(M;c) \in C(\mathbb{T})$. Similarly, there exists a rearrangement invariant $\mathcal{U}_*(\cdot;c)$ Lipschitz continuous in the strong $L^2(0,1)$–topology which satisfies the following conditions: for each non-decreasing $M \in L^2(0,1)$ there exists a so-called calibrated curve σ^c associated to $\mathcal{U}_*(\cdot;c)$ in the sense that $\sigma^c \in H^2(0,\infty; L^2(0,1))$, $\sigma^c_0 = M$ and whenever $T > 0$,

$$\mathcal{U}_*(\sigma^c_T; c) = \int_0^T L_c(\sigma^c_t, \dot\sigma^c_t)\mathrm{d}t + \mathcal{U}_*(M;c) + \frac{1}{2}c^2 T.$$

Furthermore, for all $\sigma \in H^2(0,\infty; L^2(0,1))$ we have

$$\mathcal{U}_*(\sigma_T; c) \leq \int_0^T L_c(\sigma_t, \dot\sigma_t)\mathrm{d}t + \mathcal{U}_*(\sigma_0;c) + \frac{1}{2}c^2 T.$$

It is proven in [16] that the following corollaries are direct consequences of theorem 1.1.

Corollary 1.2. *For each $c \in \mathbb{R}$ and each $M \in L^2(0,1)$ which is monotone nondecreasing, there exists $N \in L^2(I)$ such that*

$$\sup_{t>0} \sqrt{t} \Big\| \frac{\Psi^1(t,M,N)}{t} - c \Big\|_{\nu_0} < \infty, \qquad \lim_{t\to\infty} \Psi^2(t,M,N) = c.$$

Corollary 1.3. *Given $c \in \mathbb{R}$ and a Borel probability measure μ on $\mathbb{R}$ of bounded second moment, there exists a path $t \to \rho_t \in AC^2_{loc}(0,\infty; \mathcal{P}_2(\mathbb{R}))$ and $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ Borel satisfying the following properties: $u_t \in L^2(\rho_t)$ for $\mathcal{L}^1$–almost every $t > 0$, and $\rho_0 = \mu$. The pair (ρ, u) satisfies the Euler system*

$$\begin{cases} \partial_t(\rho_t u_t) + \partial_x(\rho_t u_t^2) = -\rho_t \partial_x P_t \\ \partial_t \rho_t + \partial_x(\rho_t u_t) = 0 \\ P_t(x) = \int_{\mathbb{R}} W(x-y)\mathrm{d}\rho_t(y). \end{cases} \tag{16}$$

Furthermore,

$$\sup_{t>0} \sqrt{t} \|\mathbf{id}/t - c\|_{\rho_t} < \infty, \qquad \lim_{t\to\infty} \|u_t - c\|_{\rho_t} = 0. \tag{17}$$

We have chosen the Vlasov system as a simple model to illustrate the use of the weak KAM theory for understanding qualitative behavior of PDEs appearing in kinetic theory, for several reasons. Firstly, they provide a simple link between finite and infinite dimensional systems. Secondly, they are one of the most frequently used kinetic models in statistical mechanics. Existence and uniqueness of global solutions for the initial value problem are well understood [3], [9], [19]. In this paper we have searched for special solutions which allow for a connection with a more conventional way of regarding (8) as Hamiltonian. We assume the initial data to be of the form $f_0 = (M,N)_{\#}\nu_0$ where $M,\ N \in L^2(X)$ so that the unique solution of (8) retains the same structure.

2 Effective Hamiltonian

In this section we define the effective Hamiltonian $\bar{H}$ in our infinite-dimensional setting and compute $\bar{H}(c)$. The choice of constant functions as "rotation numbers" in this context is fully justified in [16].

We begin by recalling some results from [17], adapted to our setting. In [17] we proved the existence of an infinite-dimensional *effective Lagrangian* under the following assumption: suppose L is a Lagrangian on $L^2(0,1)\times L^2(0,1)$ satisfying the growth conditions

$$c\|N\|^2 \leq L(M,N) \leq C\big(1+\|N\|^2\big), \text{ for all } (M,N)\in [L^2(X)]^2, \tag{18}$$

where c, C are given positive constants. Assume L is $L^2_{\mathbb{Z}}(0,1)$–periodic in M, i.e.

$$L(M+Z,N) = L(M,N) \text{ for all } Z\in L^2_{\mathbb{Z}}(0,1),\ M,\ N\in L^2(0,1). \tag{19}$$

Assume further that there exists $\Lambda > 0$ such that

$$L(M,N_1) - L(M,N_2) \leq \Lambda \int_{\{N_1\neq N_2\}} |N_1|^2 \mathrm{d}x \tag{20}$$

for all $M,\ N_1, N_2\ \in L^2(0,1)$. Also, it satisfies, for some continuous, nondecreasing $\omega : I\!R \to I\!R$ such that $\omega(0)=0$,

$$|L(M_1,N) - L(M_2,N)| \leq \omega\big(\mathcal{L}^1\big(\{M_1\neq M_2\}\big)\big) \tag{21}$$

for all $M_1,\ M_2, N\ \in L^2(0,1)$. Fix $T>0$ and consider $\mathcal{H} := H^1(0,T;L^2(0,1))$ endowed with the topology τ given by

$$M_n \stackrel{\tau}{\to} M \Longleftrightarrow \|M_n - M\|_{L^2((0,T)\times\Omega)} \to 0 \text{ and } \{\dot{M}_n\} \text{ is bounded in } L^2((0,T)\times X) \tag{22}$$

for every $\Omega \subset\subset X$. By following mostly the techniques in [10], we have proved in [17] (in even more generality) that

$$\int_0^T \bar{L}(\dot\sigma)\mathrm{d}t = \Gamma(\tau)\lim_{\varepsilon\to 0}\int_0^T L\left(\frac{\sigma}{\varepsilon},\dot\sigma\right)\mathrm{d}t$$

for

$$\bar{L}(N) := \liminf_{T\to\infty}\ \inf_{\phi\in\mathcal{H}_0} ⨍_0^T L(tN+\phi(t), N+\dot\phi(t))\mathrm{d}t, \tag{23}$$

where $\Gamma(\tau)$ denotes the Γ-convergence with respect to the topology τ. The set $\mathcal{H}_0$ represents all functions in $\mathcal{H}$ with null trace. The continuity of the map $\bar{L}$ with respect to the strong $L^2(0,1)$ topology was obtained as a consequence of its convexity and local boundedness.

Definition 2.1. *The map $\bar{L}$ is called the* effective Lagrangian *corresponding to L. Its Legendre transform defined for $\zeta \in L^2(0,1)$ by*

$$\bar{H}(\zeta) = \sup_{\xi \in L^2(0,1)} \left\{ \langle \xi, \zeta \rangle_{L^2(X)} - \bar{L}(\xi) \right\}$$

is called the effective Hamiltonian *associated to H (the Legendre transform of L).*

We proved in [17] that the viscosity solutions (given by the Lax-Oleinik variational formula) for the evolutionary Hamilton-Jacobi equations with oscillating Hamiltonians $H(\cdot/\epsilon, \cdot)$ converge to the Hopf-Lax solution of the Hamilton-Jacobi equation with the effective $\bar{H}$. In the classical but unpublished paper [18] the authors arrived to the effective Hamiltonian by performing this homogenization. They showed that for every $P \in \mathbb{R}^n$ (rotation vector) there exists a unique $\lambda \in \mathbb{R}$ such that the *cell problem*

$$H(x, \nabla u(x) + P) = \lambda$$

admits a periodic viscosity solution; then $\bar{H}$ was defined by $\lambda = \bar{H}(P)$. E [10] showed that if $H(x,p)$ is convex in p, then one can use Lax-Oleinik's representation for the viscosity solutions to obtain the homogenization result in [18]. Indeed, in his approach, $\bar{L}$ is obtained first as an object giving the Γ-limit of oscillating integral functionals. Note that in [17] we followed E's approach and arrived to the effective Hamiltonian by means of the Legendre transform of the effective Lagrangian.

We now return to $L : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ given by (12). For $c \in \mathbb{R}$, not only are we able to compute $\bar{H}(c)$ explicitly, but we will see in the next section that its discrete counterparts have precisely the same value. This feature turns out to be crucial for our analysis, as our approach is of the finite-to-infinite-dimensions kind.

Proposition 2.2. *If we identify $c \in \mathbb{R}$ with the constant function $f \equiv c$ over $(0,1)$, then*

$$\bar{L}(c) = \bar{H}(c) = \frac{1}{2}c^2 \text{ for all } c \in \mathbb{R}. \tag{24}$$

Proof: Since $W \leq 0$, (12) implies $L(M,N) \geq \|N\|^2/2$ for all $M,\ N \in L^2(0,1)$. From (23) we deduce $\bar{L}(N) \geq \|N\|^2/2$, so

$$c\int_0^1 N\mathrm{d}x - \bar{L}(N) \leq c\int_0^1 N\mathrm{d}x - \frac{1}{2}\int_0^1 N^2\mathrm{d}x \leq \frac{1}{2}c^2 \text{ for all } N \in L^2(0,1).$$

Thus, $\bar{H}(c) \leq c^2/2$ and we now need to prove the opposite inequality. For this we observe that

$$\inf_{\phi \in \mathcal{H}} \int_0^T L(tc + \phi(t), c + \dot{\phi}(t))\mathrm{d}t = \int_0^T L(tc, c)\mathrm{d}t = \frac{1}{2}c^2$$

for all $T > 0$ because $W \leq 0 = W(0)$ and the infimum is taken over $H^1(0,T; L^2(0,1))$ functions such that $\phi(0) = \phi(T) = 0$. According to (23), we obtain $\bar{L}(c) = c^2/2$. Therefore, $\bar{L}(c) + \bar{H}(c) \geq c^2$ yields $\bar{H}(c) \geq c^2/2$ which concludes the proof. QED.

3 From finite to infinite-dimensions

In this section we introduce the discrete versions of the particle interaction Lagrangian and Hamiltonian. We study the corresponding cell problems, then we show that the viscosity solutions obtained by a linear perturbation approximation argument are finite-dimensional restrictions of a rearrangement invariant, periodic, Lipschitzian functional on $L^2(0,1)$.

3.1 Discrete Hamiltonians

We endow $I\!R^n$ with the inner product $\langle x, \mathbf{y}\rangle_n = x \cdot \mathbf{y}/n$, denote by $|\cdot|_n$ the induced norm and by ∇_n the induced gradient. Let us define $L^n : I\!R^n \times I\!R^n \to I\!R$ by

$$L^n(x, v) = \frac{1}{2}|v|_n^2 - \frac{1}{2n^2}\sum_{i,j=1}^{n} W(x_i - x_j). \tag{25}$$

Its Legendre transform is, clearly, $H^n : I\!R^n \times I\!R^n \to I\!R$ defined by

$$H^n(x, \mathbf{p}) = \frac{1}{2}|\mathbf{p}|_n^2 + \frac{1}{2n^2}\sum_{i,j=1}^{n} W(x_i - x_j). \tag{26}$$

Note that $L^n(x, v) = L(M_n, N_n)$, where M_n and N_n are piecewise constant $M_n \equiv x_i$, $N_n \equiv v_i$ on the n-regular partitions of X. One can easily adapt the proof of Proposition 2.2 above to prove:

Lemma 3.1. *If we denote by $c^n := (c, ..., c) \in I\!R^n$*

$$\overline{H^n}(\mathbf{c}^n) = \frac{1}{2}c^2 \text{ for all integers } n \geq 1 \text{ and all } c \in I\!R. \tag{27}$$

We know from the classical, finite-dimensional theory, that (27) implies that $c^2/2$ is the unique real number λ for which the equation

$$H^n(x, \nabla_n v(x) + c^n) = \lambda \tag{28}$$

admits a $\mathbb{Z}^n$-periodic viscosity solution [11], [12] denoted by $u^n(\cdot; c)$. Since these solutions are, in general, not unique, we choose specific ones, obtained by a standard approximation argument.

Remark 3.2. *For a generic Lagrangian l and its associated Hamiltonian h, one can obtain [13], [5] a periodic viscosity solution for the cell problem (7) the following way: for $\mathbf{c} \in I\!R^n$ define $\tilde{h}(x, \mathbf{p}) = h(x, \mathbf{c} - \mathbf{p})$, so that its Legendre transform is $\tilde{l}(x, \mathbf{v}) = \mathbf{c} \cdot \mathbf{v} + l(x, -\mathbf{v})$. Then, for $\epsilon > 0$ there exists a unique periodic viscosity solution for $\epsilon w_\epsilon + \tilde{h}(x, -\nabla w_\epsilon) = 0$ such that the pair $(w_\epsilon - \min w_\epsilon, -\epsilon w_\epsilon)$ converges as $\epsilon \downarrow 0$ (possibly, up to subsequence) to $(w, \bar{h}(\mathbf{c}))$ uniformly on $I\!R^n$, where w is a periodic solution for the cell problem. It is known [13] that w_ϵ admits the representation*

$$w_\epsilon(x) = \inf_{\sigma(0)=x} \int_0^\infty e^{-\epsilon s}\tilde{l}(\sigma(s), \dot{\sigma}(s))\mathrm{d}s.$$

This fact will be used below.

We now return to the L^n case and introduce the Lagrangian L_c^n along with its corresponding Hamiltonian H_c^n by

$$L_c^n(x, \mathbf{v}) = L^n(x, \mathbf{v}) - \langle c^n, \mathbf{v}\rangle_n, \quad H_c^n(x, \mathbf{p}) = H^n(x, c^n + \mathbf{p}). \tag{29}$$

Note that they are, indeed, Legendre conjugates. Then (28) becomes

$$H_c^n(x, \nabla_n v(x)) = \overline{H^n}(c^n) = \frac{1}{2}c^2. \tag{30}$$

For $\epsilon > 0$ one looks at the unique viscosity solution $u_\epsilon^n(\cdot; c)$ for

$$\epsilon v + H_c^n(x, \nabla_n v(x)) = 0 \tag{31}$$

and shows [18] that $u^n_\epsilon(\cdot;c)-\min u^n_\epsilon(\cdot;c)$ converges uniformly to some $u^n(\cdot;c)$ which is a viscosity solution for (30) and, thus, for (28). By Remark 3.2, these unique $u^n_\epsilon(\cdot;c)$ have the following representation formula

$$u^n_\epsilon(x;c)=\inf_{\sigma(0)=x}\int_0^\infty e^{-\epsilon s}\tilde{L}^n_c(\sigma(s),\dot{\sigma}(s))\mathrm{d}s \tag{32}$$

where $\sigma\in H^1(0,\infty;I\!R^n)$ and $\tilde{L}^n_c(x,\mathbf{v})=L^n(x,\mathbf{v})+\langle c^n,\mathbf{v}\rangle_n$. Due to the permutation invariance of $\tilde{L}^n_c$, we infer that

$$u^n_\epsilon(x;c)=u^n_\epsilon(x_\tau;c)\text{ for any permutation of } n \text{ letters } \tau, \tag{33}$$

where we recall that $x^\tau=(x_{\tau(1)},...,x_{\tau(n)})$. Let us now prove a useful lemma.

Lemma 3.3. *For every $c\in I\!R$, $\epsilon>0$ and every positive integer n,*

$$\min_{x\in I\!R^n}u^n_\epsilon(x;c)=u^n_\epsilon(0;c)=-\frac{c^2}{2\epsilon}.$$

Proof: According to (32),

$$\begin{aligned}u^n_\epsilon(x;c)&=\inf_{\sigma(0)=x}\int_0^\infty e^{-\epsilon s}\Big[\frac{1}{2}|\dot{\sigma}|^2_n+\langle\dot{\sigma},c^n\rangle_n-\frac{1}{2n^2}\sum_{i,j=1}^n W(\sigma_i-\sigma_j)\Big]\mathrm{d}s\\&=\inf_{\sigma(0)=x}\int_0^\infty e^{-\epsilon s}\Big[-\frac{c^2}{2}+\frac{1}{2n}\sum_{i=1}^n|\dot{\sigma}_i+c|^2-\frac{1}{2n^2}\sum_{i,j=1}^n W(\sigma_i-\sigma_j)\Big]\mathrm{d}s.\end{aligned}$$

Since $W\le 0$, clearly the minimum with respect to x is attained at $x=0$ when $\sigma_i(s)=-cs$ for all $s\ge 0$. Thus, the conclusion follows. QED.

Next, we prove a "consistency" result.

Lemma 3.4. *Let m, n be positive integers. Then*

$$u^{mn}_\epsilon(x^m_1,...,x^m_n;c)=u^n_\epsilon(x;c), \tag{34}$$

where $x^m_j:=(x_j,...,x_j)\in I\!R^m$ for $1\le j\le n$.

Proof: Without loss of generality, we may assume $c=0$ and drop the dependence on c in the notation. We have

$$u^{mn}_\epsilon(x)=\inf_{\sigma_{mn}}\int_0^\infty e^{-\epsilon s}\Big[\frac{1}{2mn}\sum_{i=1}^{mn}|\dot{\sigma}^i_{mn}|^2-\frac{1}{2m^2n^2}\sum_{i,j=1}^{mn}W(\sigma^i_{mn}-\sigma^j_{mn})\Big]\mathrm{d}s, \tag{35}$$

where $\sigma^i_{mn}(0)=x_j$ for all $1\le j\le n$ and all $(j-1)m+1\le i\le jm$. Now let us consider the expression on the right hand side under the restriction $\sigma^{i_1}_{mn}=\sigma^{i_2}_{mn}=:\sigma^j_n$ for any $1\le j\le n$ and every $(j-1)m+1\le i_1,\ i_2\le jm$. Then,

$$\frac{1}{2mn}\sum_{i=1}^{mn}|\dot{\sigma}^i_{mn}|^2-\frac{1}{2m^2n^2}\sum_{i,j=1}^{mn}W(\sigma^i_{mn}-\sigma^j_{mn})=\frac{1}{2n}\sum_{j=1}^{n}|\dot{\sigma}^j_n|^2-\frac{1}{2n^2}\sum_{i,j=1}^{n}W(\sigma^i_n-\sigma^j_n),$$

which means that if we take the infimum in (35) with respect to x under this restriction we get

$$u^{mn}_\epsilon(x^m_1,...,x^m_n)\le u^n_\epsilon(x).$$

To prove the opposite inequality, denote by $J_k = \{(k-1)m+1, ..., km\}$ for $1 \le k \le n$. Since $W \le 0$, after throwing away some nonnegative terms, one has

$$\int_0^\infty e^{-\epsilon s}\Big[\frac{1}{2mn}\sum_{i=1}^{mn}|\dot\sigma^i_{mn}|^2 - \frac{1}{2m^2n^2}\sum_{i,j=1}^{mn} W(\sigma^i_{mn}-\sigma^j_{mn})\Big]\mathrm{d}s$$
$$\ge m^{-n}\int_0^\infty e^{-\epsilon s}\sum_{\mathbf{k}\in J_1\times\ldots\times J_n}\Big[\frac{1}{2n}\sum_{i=1}^{n}|\dot\sigma^{k_i}_{mn}|^2 - \frac{1}{2n^2}\sum_{i,j=1}^{n} W(\sigma^{k_i}_{mn}-\sigma^{k_j}_{mn})\Big]\mathrm{d}s$$
$$\ge m^{-n}\sum_{\mathbf{k}\in J_1\times\ldots\times J_n}\inf_{\sigma^{k_i}_{mn}(0)=x_i, 1\le i\le n}\int_0^\infty e^{-\epsilon s}\Big[\frac{1}{2n}\sum_{i=1}^{n}|\dot\sigma^{k_i}_{mn}|^2 - \frac{1}{2n^2}\sum_{i,j=1}^{n} W(\sigma^{k_i}_{mn}-\sigma^{k_j}_{mn})\Big]\mathrm{d}s$$
$$= m^{-n}m^n u^n_\epsilon(x) = u^n_\epsilon(x).$$

We conclude by taking the infimum in the left hand side. QED.

By Lemma 3.3 we infer

$$u^n_\epsilon(\cdot;c) + \frac{c^2}{2\epsilon} \to u^n(\cdot;c) \text{ uniformly in } I\!R^n \text{ as } \epsilon \downarrow 0.$$

Due to the periodicity of L^n_c and the uniqueness of $u^n_\epsilon(\cdot;c)$, it follows that $u^n_\epsilon(\cdot;c)$ is periodic. Lemma 3.3 and (33) imply that $u^n(\cdot;c)$ is also $\mathbb{Z}^n$-periodic and permutation invariant. Furthermore, Lemma 3.4 implies

$$u^{mn}(x^m_1, ..., x^m_n; c) = u^n(x;c) \text{ for all positive integers } m,\ n. \tag{36}$$

It is also known that $u^n(\cdot;c)$ is Lipschitz on $I\!R^n$, so it is differentiable a.e. and the equation (28) is satisfied pointwise at the points of differentiability. One can easily see then that the Lipschitz constant κ_n satisfies $0 < \kappa_n \le \sqrt{c^2 - 2\inf V} =: \kappa$.

3.2 An infinite-dimensional extension

Let us now consider partitioning the interval $X = (0,1)$ into n equal subintervals and denote by $\mathcal{C}_n$ the set of all real-valued functions defined on $(0,1)$ and constant on each such subinterval. Any function $f \in \mathcal{C}_n$ can be identified with a vector $x^n_f \in I\!R^n$ by its values. We now define

$$\tilde U : \cup_{n\ge1}\mathcal{C}_n =: \mathcal{C} \to I\!R \text{ by } \tilde U(f;c) = u^{n_0}(x^{n_0}_f;c) \text{ whenever } f \in \mathcal{C}_n. \tag{37}$$

According to Lemma 3.4, not only is this functional well-defined, but it is also Lipschitz on $\mathcal{C}$ with respect to the strong $L^2(0,1)$-norm. Indeed, let $f,\ g \in \mathcal{C}$. Then $f \in \mathcal{C}_n$ and $g \in \mathcal{C}_m$ for some positive integers $m,\ n$, so $f,\ g \in \mathcal{C}_{mn}$. Thus,

$$|\tilde U(f;c) - \tilde U(g;c)| = |u^{mn}(x^{mn};c) - u^{mn}(\mathbf{y}^{mn};c)| \le \kappa |x^{mn} - \mathbf{y}^{mn}|_{mn}.$$

But $|x^{mn} - \mathbf{y}^{mn}|_{mn} = \|f-g\|_{L^2(0,1)}$, so the claim is proved. Due to the density of $\mathcal{C}$ in $L^2(0,1)$, we conclude that $\tilde U$ can be uniquely extended by continuity to $L^2(0,1)$. More precisely:

Proposition 3.5. *For any $c \in I\!R$ there exists a unique $U(\cdot;c) : L^2(0,1) \to I\!R$ which is Lipschitz continuous with* $\mathrm{Lip}(U(\cdot;c)) \le \kappa$ *and such that* $U(\cdot;c)\big|_{\mathcal{C}_n} = u^n(\cdot;c)$.

In the next section we will prove that U is the viscosity solution we are looking for. Before that, let us show that it has some "nice" properties inherited from u^n.

Proposition 3.6. *For any $c \in I\!R$ the functional $U(\cdot;c)$ is RI and $L^2_{\mathbb{Z}}(0,1)$-periodic.*

Proof: We give up the c-dependence to unburden notation. Let $M \in L^2(0,1)$ and $\tilde{M}$ be its monotone rearrangement. Take a sequence of maps $M_n \in \mathcal{C}_n$ converging to M in $L^2(0,1)$, denote by $\tilde{M}_n$ their monotone rearrangements and set $\mu_n = M_{n\#}\nu_0$, $\mu = M_{\#}\nu_0$. By one-dimensional optimal transport [16], we have

$$\|\tilde{M}_n - \tilde{M}\| = W_2(\mu_n, \mu) \leq \|M_n - M\|$$

which gives $\tilde{M}_n \to \tilde{M}$ in $L^2(0,1)$. Here, W_2 denotes the 2–Wasserstein distance on the real line (cfr. [2], [15] [27]). Since u^n is permutation invariant, we conclude $U(M_n) = U(\tilde{M}_n)$ which, due to the continuity of U, implies $U(M) = U(\tilde{M})$. Thus, U is rearrangement invariant. To prove periodicity take $Z \in L^2_{\mathbb{Z}}(0,1)$ and a sequence $Z_n \in L^2_{\mathbb{Z}}(0,1)$ piecewise constant on the n-equipartition of $(0,1)$ such that $Z_n \to Z$ in $L^2(0,1)$. Then $M_n + Z_n \to M + Z$ in $L^2(0,1)$, so $U(M_n + Z_n) \to U(M + Z)$. But the $\mathbb{Z}^n$-periodicity of u^n yields $U(M_n + Z_n) = U(M_n)$ and the continuity of U concludes the proof. QED.

Remark 3.7. *In the proof we have used the fact that any $Z \in L^2_{\mathbb{Z}}(0,1)$ is the L^2-limit of functions that are integer-valued, piecewise constant on the n-equipartition of $(0,1)$. Indeed, to see that, note that we may first approximate Z by functions taking on only finitely many values. So it suffices to prove the statement for indicator functions of Borel sets $A \subset 0,1$. Since the Lebesgue measure is Borel regular, it is enough to consider open sets $O \subset (0,1)$. Furthermore, one can reduce these open sets to finite unions of disjoint open subintervals of $(0,1)$. For such sets, the property is easy to prove.*

4 The Weak KAM Theorem

Here we shall prove Theorem 1.1, i.e. we shall show that $U(\cdot;c)$ constructed in the previous section provides a viscosity solution for (15).

Definition 4.1. *Let V be a real valued proper functional defined on $L^2(0,1)$ with values in $I\!R \cup \{\pm\infty\}$. Let $M_0 \in L^2(0,1)$ and $\xi \in L^2(0,1)$. We say that ξ belongs to the (Fréchet) subdifferential of V at M_0 and we write $\xi \in \partial.V(M_0)$ if*

$$V(M) - V(M_0) \geq \langle \xi, M - M_0 \rangle + o\big(\|M - M_0\|\big)$$

for all $M \in L^(0,1)$.
We say that ξ belongs to the superdifferential of V at M_0 and we write $\xi \in \partial^{\cdot} V(M_0)$ if $-\xi \in \partial.(-V)(M_0)$.

Remark 4.2. *As expected, when the sets $\partial.V(M_0)$ and $\partial^{\cdot} V(M_0)$ are both nonempty, then they coincide and consist of a single element. That element is the L^2-gradient of V at M_0, denoted by $\nabla_{L^2} V(M_0)$.*

4.1 Viscosity solutions; solution semigroup

We can now define [6] the notion of viscosity solution for a general Hamilton-Jacobi equation of the type

$$F(M, \nabla_{L^2} U(M)) = 0. \tag{HJ}$$

Definition 4.3. *Let $V : L^2(0,1) \to I\!R$ be continuous.*
(i) We say that V is a viscosity subsolution *for (HJ) if*

$$F(M,\zeta) \leq 0 \text{ for all } M \in L^2(0,1) \text{ and all } \zeta \in \partial^{\cdot} V(M). \tag{38}$$

(ii) We say that V is a viscosity supersolution *for (HJ) if*

$$F(M,\zeta) \geq 0 \text{ for all } M \in L^2(0,1) \text{ and all } \zeta \in \partial_{\cdot} V(M). \tag{39}$$

(iii) We say that V is a viscosity solution *for (HJ) if V is both a subsolution and a supersolution for (HJ).*

Remark 4.4. *If U is a viscosity solution, then, in view of remark 4.2, we deduce that (HJ) is satisfied at all $M \in L^2(0,1)$ where $\partial_{\cdot} U(M) \cap \partial^{\cdot} U(M) \neq \emptyset$, which are precisely the points where U is differentiable.*

Let $M \in L^2(0,1)$ and $V : L^2(0,1) \to I\!R$ be continuous and bounded. For $t \geq 0$ define the operator $T_{L,t}$ on the space of uniformly continuous and bounded functionals $BUC(L^2(0,1))$ by

$$T_{L,t}V(M) := \inf_{S(t)=M} \left\{ V(S(0)) + \int_0^t L(S(\tau), \dot{S}(\tau)) \mathrm{d}\tau : \ S \in H^1(0,t;L^2(0,1)) \right\}. \tag{40}$$

Observe that $t \to T_{L,t}$ defines a (backward) semigroup on $[0,\infty)$. Furthermore, $\mathcal{U}(t,M) =: T_{L,t}V(M)$ yields the unique viscosity solution [6], [7] for the Cauchy problem associated with the evolutionary Hamilton-Jacobi equation

$$\partial_t \mathcal{U}(t,M) + H(M, \nabla_{L^2}\mathcal{U}(t,M)) = 0, \quad \mathcal{U}(0,M) = V(M).$$

As a consequence, we have the following:

Proposition 4.5. *The map $\mathcal{V} \in BUC$ is a fixed point for $\{T_{L,t}\}_{t\geq 0}$, i.e.*

$$T_{L,t}\mathcal{V} = \mathcal{V} \text{ for all } t \geq 0 \tag{41}$$

if and only if $\mathcal{V}$ is a viscosity solution for $H(M, \nabla_{L^2}\mathcal{V}(M)) = 0$.

Indeed, if we put $V(t,M) := \mathcal{V}(M)$, then according to the above discussion V solves (in the viscosity sense) the Cauchy problem with initial data $\mathcal{V}$. Since this V is, in fact, time-independent, we deduce that it is a viscosity solution for the stationary HJ equation. Similarly, we obtain that U constructed at the end of the previous section satisfies the requirements of Theorem 1.1 if it has the following property.

Proposition 4.6. *For any $c \in I\!R$ let $U(\cdot;c)$ be the functional from Proposition 3.5. Then*

$$T_{L_c,t}U(\cdot;c) = U(\cdot;c) - \frac{1}{2}c^2 t \text{ for all } t \geq 0, \tag{42}$$

where L_c is defined in (14).

The goal of the remainder of this section is proving Proposition 4.6. To achieve this, we consider the discrete L^n and use it to define (we use $T^n_{c,t}$ instead of $T^n_{L^n_c,t}$ to unburden notation)

$$T^n_{c,t}v(x) := \inf_{\sigma(t)=x} \left\{ v(\sigma(0)) + \int_0^t L^n_c(\sigma(\tau), \dot{\sigma}(\tau)) \mathrm{d}\tau : \ \sigma \in H^1(0,t;I\!R^n) \right\}, \tag{43}$$

where L_c^n is defined in (29). Likewise, any viscosity solution of $H^n(x, \nabla_n v(x) + c^n) = c^2/2$ satisfies the n-dimensional version of (42). We deduce that, in particular, the n-dimensional approximations (restrictions, rather) u^n of U satisfy

$$T_{c,t}^n u^n(\cdot; c) = u^n(\cdot; c) - \frac{1}{2}c^2 t \text{ for all } t \geq 0. \tag{44}$$

We would like to use this to prove (42) by passing to the limit as $n \to \infty$ in some sense.

Remark 4.7. *Note that if we further simplify notation by setting $T_t := T_{L_0,t}$ and $T_t^n := T_{0,t}^n$, easy calculations show that (44) becomes*

$$T_t^n \bar{u}^n(\cdot; c) = \bar{u}^n(\cdot; c) \text{ for all } t \geq 0, \text{ where } \bar{u}^n(\cdot; c) := u^n(\cdot; c) + \langle \cdot, c^n \rangle_n.$$

Likewise, (42) becomes

$$T_t \bar{U}(\cdot; c) = \bar{U}(\cdot; c) \text{ for all } t \geq 0, \text{ where } \bar{U}(M; c) := U(M; c) + c \int_0^1 M \mathrm{d}x.$$

4.2 The main result

Again, in this subsection, we consider the case $c = 0$ without loss of generality. Thus, we can use the notation from Remark 4.7.

Lemma 4.8. *Let $U : L^2(0,1) \to I\!R$ be Lipschitz continuous. Then, for any $t > 0$, $T_t U$ is uniformly continuous on $L^2(0,1)$.*

Proof: Let $\varepsilon > 0$ be fixed. Take $\delta > 0$ (to be fixed later), $M_1,\ M_2 \in L^2(0,1)$ such that $\|M_1 - M_2\| \leq \delta$. By definition, there exists $S_1 \in H^1(0,t; L^2(0,1))$ with $S_1(t) = M_1$ such that

$$U(S_1(0)) + \int_0^t L(S_1(s), \dot{S}_1(s))\mathrm{d}s \leq T_t U(M_1) + \delta. \tag{45}$$

Define

$$S^\delta(s) = \begin{cases} S_1(s) & \text{if } 0 \leq s \leq t - \delta \\ \frac{s-t+\delta}{\delta}(M_2 - M_1) + S_1(s), & \text{if } t - \delta \leq s \leq t, \end{cases}$$

a path connecting $S_1(0)$ and M_2. Thus,

$$\begin{aligned} T_t U(M_2) &\leq U(S_1(0)) + \int_0^t L(S_1, \dot{S}_1)\mathrm{d}s - \int_{t-\delta}^t L(S_1, \dot{S}_1)\mathrm{d}s + \int_{t-\delta}^t L(S^\delta, \dot{S}^\delta)\mathrm{d}s \\ &\leq T_t U(M_1) + \delta + C\delta + \frac{1}{2}\int_{t-\delta}^t \left(\|\dot{S}^\delta\|^2 - \|\dot{S}_1\|^2\right)\mathrm{d}s \\ &\leq T_t U(M_1) + C\delta + \frac{1}{2\delta}\|M_2 - M_1\|^2 + \frac{\|M_2 - M_1\|}{\delta}\int_{t-\delta}^t \|\dot{S}_1(s)\|\mathrm{d}s \\ &\leq T_t U(M_1) + C\delta + \frac{1}{2\delta}\|M_2 - M_1\|^2 + \frac{\|M_2 - M_1\|}{\sqrt{\delta}}\left(\int_0^t \|\dot{S}_1(s)\|^2\mathrm{d}s\right)^{1/2}. \end{aligned}$$

But if we consider the constant path M_1 in the variational principle we get

$$T_t U(M_1) \leq U(M_1) - \frac{t}{2}\iint_{X^2} V(M_1(x) - M_1(y))\mathrm{d}y\mathrm{d}x,$$

so, in view of (45), we obtain

$$\frac{1}{2}\int_0^t \|\dot{S}_1(s)\|^2 \mathrm{d}s \leq \delta + Ct + U(M_1) - U(S_1(0)).$$

Now we use

$$\|M_1 - S_1(0)\| \leq \int_0^t \|\dot{S}_1(s)\|\mathrm{d}s \leq \sqrt{t}\Big(\int_0^t \|\dot{S}_1(s)\|^2 \mathrm{d}s\Big)^{1/2}$$

and the fact that U is Lipschitz to infer that $\int_0^t \|\dot{S}_1(s)\|^2 \mathrm{d}s$ is bounded by some $C(\delta, t)$ (increasing in each variable). So, for $\delta < 1$, $\int_0^t \|\dot{S}_1(s)\|^2 \mathrm{d}s$ is bounded (since t is fixed). Thus, for $\delta < 1$, one has

$$T_t U(M_2) - T_t U(M_1) \leq C\delta + \frac{1}{2\delta}\|M_2 - M_1\|^2 + \frac{C}{\sqrt{\delta}}\|M_2 - M_1\| \leq C(\delta + \sqrt{\delta}) =: \varepsilon$$

whenever $\|M_2 - M_1\| \leq \delta$. One can now interchange the roles of M_1 and M_2 to conclude. QED.

Since we took $c = 0$, we denote $U(\cdot; 0)$ and $u^n(\cdot; 0)$ by U and u^n, respectively. We know that $U(M_n) = u^n(x)$ whenever $M_n \equiv x_i$ is piecewise constant on the n-regular partition of $(0, 1)$. Therefore, it makes sense to write $T_t^n U(M_n)$ which is nothing but $T_t^n u^n(x)$.

Lemma 4.9. *If U is the one defined in Proposition 3.5, then for any $t > 0$ and any $M \in L^2(0, 1)$ there exists a sequence $M_n \in \mathcal{C}_n$ such that*

$$\|M_n - M\| \to 0 \text{ and } \limsup_{n\to\infty} T_t^n U(M_n) \leq T_t U(M). \tag{46}$$

Proof: Let $\varepsilon_n \downarrow 0$. Since U is continuous and $\mathcal{C}$ is dense in $L^2(0, 1)$, for n sufficiently large we can find $M_n \in \mathcal{C}_n$ such that $\|M_n - M\| \leq \varepsilon_n/\kappa$. Then $|U(M_n) - U(M)| \leq \varepsilon_n$. Also, take $\gamma_n \in H^1(0, t; L^2(0, 1))$ such that $\gamma_n(t) = M$ and

$$\mathcal{A}(t; \gamma_n) - \varepsilon_n \leq T_t U(M) \leq U(\gamma_n(0)) + \int_0^t L(\gamma_n(s), \dot{\gamma}_n(s))\mathrm{d}s =: \mathcal{A}(t; \gamma_n). \tag{47}$$

Let $\sigma_n \in L^2(0, t; \mathcal{C}_n)$ such that $\|\dot{\gamma}_n - \sigma_n\|_{L^2((0,t)\times(0,1))} \leq \varepsilon_n/(\kappa\sqrt{t})$ (see, for example, [17] for the existence of such σ_n). Then place

$$S_n(s, x) = M_n(x) - \int_s^t \sigma_n(\tau, x)\mathrm{d}\tau.$$

Obviously, $S_n \in H^1(0, t; \mathcal{C}_n)$. We have

$$S_n(s) - \gamma_n(s) = M_n - M + \int_s^t (\dot{\gamma}_n - \sigma_n)\mathrm{d}\tau$$

which implies

$$\|S_n(s) - \gamma_n(s)\| \leq \frac{2\varepsilon_n}{\kappa} \text{ for } 0 \leq s \leq t. \tag{48}$$

Since $S_n(t) = M_n$, we can write

$$\begin{aligned} T_t^n U(M_n) &\leq U(S_n(0)) + \int_0^t L(S_n(s), \sigma_n(s))\mathrm{d}s \\ &\leq U(S_n(0)) - U(\gamma_n(0)) + \mathcal{A}(t; \gamma_n) + \int_0^t \big[L(S_n(s), \sigma_n(s)) - L(\gamma_n(s), \dot{\gamma}_n(s))\big]\mathrm{d}s \\ &\leq 3\varepsilon_n + T_t U(M) + \int_0^t \big[L(S_n(s), \sigma_n(s)) - L(\gamma_n(s), \dot{\gamma}_n(s))\big]\mathrm{d}s, \end{aligned} \tag{49}$$

where we have taken into account the Lipschitz property of U (with Lipschitz constant at most κ) and (47). As for the last term in the right hand side, that is nothing but

$$\frac{1}{2}\left[\|\sigma_n\|^2_{L^2(X_t)} - \|\dot\gamma_n\|^2_{L^2(X_t)}\right] - \frac{1}{2}\int_0^t \int_{(0,1)^2} \left[W(S_n(s,x) - S_n(s,y)) - W(\gamma_n(s,x) - \gamma_n(s,y))\right],$$

where $X_t := (0,t) \times (0,1)$. But (47) implies

$$\frac{1}{2}\|\dot\gamma_n\|^2_{L^2(X_t)} \leq T_t U(M) + \varepsilon_n + t \sup |W|,$$

so $\|\dot\gamma_n\|_{L^2(X_t)}$ is bounded and, since $\|\dot\gamma_n - \sigma_n\|_{L^2(X_t)} \leq \varepsilon_n/(\kappa\sqrt{t})$, $\|\sigma_n\|_{L^2(X_t)}$ is also bounded. These considerations, along with (48) and the Lipschitz continuity of V, imply

$$\lim_{n\to\infty} \int_0^t \left[L(S_n(s), \sigma_n(s)) - L(\gamma_n(s), \dot\gamma_n(s))\right] \mathrm{d}s = 0.$$

Therefore, (49) yields the second statement in (46). QED.

We have now all the tools to prove Proposition 4.6. Let us remind the reader that we do not lose generality by considering the case $c = 0$ only.

Proof of Proposition 4.6: Take $M \in L^2(0,1)$ and the corresponding sequence M_n from Lemma 4.9. Note that, since $M_n \in \mathcal{C}_n$ for all n, (44) enables us to write

$$T_t^n U(M_n) = U(M_n) \to U(M) \text{ as } n \to \infty,$$

where the convergence is due to the continuity of U. By using Lemma 4.9 again we pass to lim sup in the left hand side to deduce

$$T_t U(M) \geq U(M).$$

To prove the opposite inequality, note that

$$T_t U(M_n) \leq T_t^n U(M_n) \text{ for all } n \geq 1,\ t > 0$$

because the infimum in the definition of the left hand side is taken under fewer restrictions. Now Lemma 4.8 applies to yield the convergence of $T_t U(M_n)$ to $T_t U(M)$. But we have already seen that the right hand side converges to $U(M)$, so we conclude the proof. QED.

4.3 Forward semigroup

Define the (forward) semigroup $\tilde T_{L,t}$ on $C(\mathbb{T})$ by

$$\tilde T_{L,t} V(M) = \sup_{S(0)=M} \left\{ V(S(t)) - \int_0^t L(S(s), \dot S(s)) \mathrm{d}s \right\}.$$

One can modify the proof of Proposition 4.6 to prove:

Proposition 4.10. *For any $c \in I\!R$ there exists a Lipschitz continuous, periodic, rearrangement invariant map $\tilde U(\cdot; c) : L^2(0,1) \to I\!R$ such that*

$$\tilde T_{L_c,t} \tilde U(\cdot; c) = \tilde U(\cdot; c) + \frac{1}{2}c^2 t \text{ for all } t \geq 0. \tag{50}$$

Furthermore, $\tilde U(M_n; c) = \tilde u^n(x; c)$ whenever $M_n \in \mathcal{C}_n$ and x is the corresponding vector in $I\!R^n$, where $\tilde u^n(\cdot; c)$ is a forward semigroup Weak KAM solution on $\mathbb{T}^n$, i.e.

$$\tilde T^n_{L_c^n,t} \tilde u^n(\cdot; c) = \tilde u^n(\cdot; c) + \frac{1}{2}c^2 t \text{ for all } t \geq 0. \tag{51}$$

Indeed, this is, in some sense, the dual of Proposition 4.6 as it uses the forward semigroup $\tilde{T}_t$ instead of the more usual Lax-Oleinik backward semigroup T_t to construct viscosity solutions for (15). The following result yields the second statement of Theorem 1.1.

Proposition 4.11. *Let $M \in \mathcal{M}$ be fixed. Then for every $c \in I\!R$ there exists a global extremal curve $S \in H^2(0,\infty; L^2(0,1))$ such that $S(0) = M$, $S(t) \in \mathcal{M}$ for all $t \geq 0$, and*

$$\tilde{U}(S(t);c) - \tilde{U}(M;c) = \int_0^t L_c(S(s), \dot{S}(s))\mathrm{d}s + \frac{1}{2}c^2 t \text{ for all } t \geq 0. \tag{52}$$

Proof: Let $M_n \in \mathcal{C}_n$ be nondecreasing and such that $M_n \to M$ in $L^2(0,1)$. According to Proposition 4.10, the restriction $\tilde{u}^n$ of $\tilde{U}$ to $\mathcal{C}_n$ (or, equivalently, $I\!R^n$) is a Weak KAM solution for (51) on $\mathbb{T}^n$. Theorem 4.5.3 in [12] provides a global extremal $\{(\sigma_n(s), \dot{\sigma}_n(s))\}_{s\geq 0} \subset I\!R^n \times I\!R^n$ such that

$$\tilde{u}^n(\sigma_n(t);c) - \tilde{u}^n(x_n;c) = \int_0^t L_c^n(\sigma_n(s), \dot{\sigma}_n(s))\mathrm{d}s + \frac{1}{2}c^2 t \text{ for all } t \geq 0, \tag{53}$$

where x_n is the n-dimensional vector corresponding to M_n. Note that the path $s \to \sigma_n(s)$ we consider here is, in fact, a lift of the one from [12] to the universal cover $I\!R^n$. So, if we denote by $S_n(s)$ the function in $\mathcal{C}_n$ corresponding to $\sigma_n(s) \in I\!R^n$, then we can write

$$\tilde{U}(S_n(t);c) - \tilde{U}(M_n;c) = \int_0^t L_c(S_n(s), \dot{S}_n(s))\mathrm{d}s + \frac{1}{2}c^2 t \text{ for all } t \geq 0. \tag{54}$$

We first deduce

$$\tilde{U}(S_n(t);c) - \tilde{U}(M_n;c) \geq \frac{1}{2}\int_0^t \|\dot{S}_n(s) - c\|^2 \mathrm{d}s \text{ for all } t \geq 0 \tag{55}$$

which means $S_n^c(s) := S_n(s) - cs$ has functional time derivative bounded in $L^2(0,\infty; L^2(0,1))$ uniformly with respect to n. It follows that $t \to S_n^c(t)$ is uniformly Hölder. In particular, for each $t \geq 0$, $S_n^c(t)$ is bounded in $L^2(0,1)$, uniformly with respect to n. Note that, since M_n is nondecreasing we may assume [15], [17] that $S_n^c(t)$ is nondecreasing for all n and t. Therefore, for any $t \geq 0$, there exists a subsequence $n_k \to \infty$ such that $S_{n_k}^c(t) \to S^c(t)$ in $L^2_{loc}(0,1)$ for some $S^c(t) \in \mathcal{M}$ (see, e.g., [17]). By a standard diagonalization argument, we can use the same subsequence for all $t \in [0,\infty) \cap \mathbb{Q}$. Again, by a standard argument, one notices that $t \to S^c(t)$ is Hölder continuous on $[0,\infty) \cap \mathbb{Q}$, so it can be extended to the whole $[0,\infty)$ in a unique way. Furthermore, after relabeling $S_{n_k}^c(t)$ by $S_n^c(t)$, we use the uniform Hölder continuity of S_n^c and S^c (as well as the density of $\mathbb{Q}$ in $I\!R$) to deduce that $S_n(t) \to S^c(t) + ct =: S(t)$ in $L^2_{loc}(0,1)$ for all $t \geq 0$. But the uniform bound on $\dot{S}_n^c$ in $L^2(0,\infty; L^2(0,1))$ and some of the considerations above, also imply that $S \in H^1(0,\infty; L^2(0,1))$ and, up to an unrelabeled subsequence, $\dot{S}_n \rightharpoonup \dot{S}$ weakly in $L^2(0,\infty; L^2(0,1))$. Now let us assume $\tilde{U}(\cdot;c)$ is continuous with respect to the $L^2_{loc}(0,1)$-topology as well (stronger than the already known $L^2(0,1)$-continuity). If we pass to liminf in (54) as $n \to \infty$, we obtain (due to L_c being l.s.c.)

$$\tilde{U}(S(t);c) - \tilde{U}(M;c) \geq \int_0^t L_c(S(s), \dot{S}(s))\mathrm{d}s + \frac{1}{2}c^2 t \text{ for all } t \geq 0$$

which, in light of (50) and the definition of $\tilde{T}_{L_c,t}$, turns into the equality (52). To prove the L^2_{loc}-continuity of $\tilde{U}(\cdot;c)$ let us take $\Omega \subset\subset (0,1)$, $f \in L^2(0,1)$ and simply remark that

$$\begin{aligned} \left|\tilde{U}(f\chi_\Omega) - \tilde{U}(f)\right| &= \left|\tilde{U}(\widehat{f\chi_\Omega}) - \tilde{U}(\hat{f})\right| = \left|\tilde{U}(\hat{f}\chi_\Omega) - \tilde{U}(\hat{f})\right| \\ &\leq \mathrm{Lip}(\tilde{U}(\cdot;c))\left(\int_{(0,1)\setminus\Omega} |\hat{f}(x)|^2 \mathrm{d}x\right)^{1/2} \leq \mathrm{Lip}(\tilde{U}(\cdot;c))\mathcal{L}^1((0,1)\setminus\Omega), \end{aligned}$$

where we remind the reader that $\hat{f} = f - [f]$ (here $[\cdot]$ stands for the integer part function). Thus, the Lipschitz continuity of $\tilde{U}(\cdot; c)$ with respect to the L^2-topology implies its uniform continuity with respect to the L^2_{loc}-topology. QED.

5 A spatially periodic Vlasov-Poisson system

We claim that Proposition 4.6 holds even if W is not necessarily C^2. The need for this regularity assumption came from the use of Theorem 4.5.3 in [12] to provide us with a global extremal $\{(\sigma_n(s), \dot{\sigma}_n(s))\}_{s\geq 0} \subset I\!R^n \times I\!R^n$ satisfying (53), and it has no bearing on Proposition 4.6. Indeed, we proved Proposition 4.6 by approximation with finite-dimensional Weak KAM solutions $u^n(\cdot; c)$ which were extracted from [5] (therefore, independently of [12]). To further explain, we refer back to Remark 3.2 (where we indicate how our $u^n(\cdot; c)$ is constructed) and point to the conditions on the Hamiltonian (1), (2) and (3) in [5]. These conditions will still be satisfied by H^n defined in (26) if W is only Lipschitz continuous (instead of C^2) and $\mathbb{Z}$-periodic. If (13) is further assumed on W, then the entire construction from Section 3 carries through. Lemmas 4.8 and 4.9 will also hold, so Proposition 4.6 will remain true in this less regular case.

In order to be able to reproduce the proof of Proposition 4.11 in this case, it would suffice to know that the global extremals $\{(\sigma(s), \dot{\sigma}(s))\}_{s\geq 0} \subset I\!R^n \times I\!R^n$ employed in (53) still exist at the n-dimensional level. For that, we apply a standard compactness argument in $H^1(0, t; I\!R^n)$ for a maximizing sequence in

$$u^n(x; c) = \sup_{\sigma(0)=x} \left\{ u^n(\sigma(t); c) - \int_0^t L^n_c(\sigma(s), \dot{\sigma}(s))\mathrm{d}s \right\} - \frac{1}{2}c^2 t,$$

say it satisfies

$$u^n(x; c) - \frac{1}{m} \leq u^n(\sigma_m(t); c) - \int_0^t L^n_c(\sigma_m(s), \dot{\sigma}_m(s))\mathrm{d}s - \frac{1}{2}c^2 t \tag{56}$$

The maximizing sequence $\{\sigma_m\}_m$ is bounded in $H^1(0, t; I\!R^n)$ (because $u^n(\cdot; c)$ is bounded in L^∞) and, since $\sigma_m(0) = x$ for all m we infer that, at least up to a subsequence, σ_m converges to some σ uniformly on $[0, t]$ (in particular, $\sigma(0) = x$) while $\dot{\sigma}_m$ converges to $\dot{\sigma}$ weakly in $L^2(0, t; I\!R^n)$. We obtain the desired result by passing to liminf as $m \to \infty$ in (56). Thus, Theorem 1.1 and Corollary 1.3 remain true for W only Lipschitz continuous. Not the same can be said about Corollary 1.2, as W is not regular enough to define the flow Ψ.

A periodic version of the Vlasov-Poisson system is replacing (8) if the C^2 potential W is replaced by the less regular

$$W(z) = \frac{1}{2}\big(|z|^2_{\mathbb{T}^1} - |z|_{\mathbb{T}^1}\big). \tag{57}$$

Indeed, one checks by direct computation that the convolution $P := W * \rho$ satisfies

$$1 - \partial^2_{xx} P = \sum_{k\in\mathbb{Z}} \rho(\cdot + k). \tag{58}$$

Consequently, with the potential (57) the system (8) turns into

$$\begin{cases} \partial_t f_t + v\partial_x f_t &= \partial_x P_t \, \partial_v f_t \\ 1 - \partial^2_{xx} P_t &= \sum_{k\in\mathbb{Z}} \rho_t(\cdot + k) \\ \rho_t &= \int_{I\!R} f_t \mathrm{d}v. \end{cases} \tag{59}$$

Any Borel probability μ^* on $\mathbb{T}^1$ can be represented as the $\mathbb{Z}$-indexed sum of integer translations of a Borel probability μ supported in $[0,1)$. Thus, in light of (58) we deduce that

$$P_{\mu^*}(x) = \int_{I\!R} W(x-z)\mathrm{d}\mu(z) = \int_{\mathbb{T}^1} W(x-z)\mathrm{d}\mu^*(z)$$

satisfies

$$1 - \partial^2_{xx} P_{\mu^*} = \mu^* \text{ for all } \mu^* \in \mathcal{P}(\mathbb{T}^1).$$

Furthermore, note that if $[0,\infty) \ni t \to f_t \in \mathcal{P}_2(I\!R \times I\!R)$ satisfies (59) in the sense of distributions, then so does $t \to f_t(\cdot + k, \cdot)$ for any $k \in \mathbb{Z}$. By the linearity in f_t of the first equation in (59), we deduce that

$$f_t^* := \sum_{k\in\mathbb{Z}} f_t(\cdot + k, \cdot) \in \mathcal{P}(\mathbb{T}^1 \times I\!R) \tag{60}$$

satisfies (in the sense of distributions)

$$\begin{cases} \partial_t f_t^* + v\partial_x f_t^* &= \partial_x P_t\, \partial_v f_t^* \\ 1 - \partial^2_{xx} P_t &= \rho_t^* \\ \rho_t^* &= \int_{I\!R} f_t^* \mathrm{d}v \end{cases} \quad \text{in } \mathbb{T}^1 \times I\!R. \tag{61}$$

Let $c \in \mathbb{R}$ and $\mu^* \in \mathcal{P}(\mathbb{T}^1)$. We now have all the ingredients for a proof of the following:

Theorem 5.1. *There exists a path* $[0,\infty) \ni t \to f_t^* \in \mathcal{P}_2(\mathbb{T}^1 \times \mathbb{R})$ *satisfying in the distributional sense the spatially periodic Vlasov-Poisson system (61) with* $\rho_0^* = \mu^*$ *and such that*

$$\sup_{t>0} \sqrt{t}\|\mathbf{id}/t - c\|_{\rho_t} < \infty, \qquad \lim_{t\to\infty} \int_{\mathbb{T}^1}\int_{\mathbb{R}} |v-c|^2 \mathrm{d}f_t^*(x,v) = 0,$$

where $\rho_t \in \mathcal{P}_2(I\!R)$ *is such that*

$$\rho_t^* = \sum_{k\in\mathbb{Z}} \rho_t(\cdot + k).$$

Proof: Let $\mu \in \mathcal{P}([0,1)) \subset \mathcal{P}_2(I\!R)$ such that

$$\mu^* = \sum_{k\in\mathbb{Z}} \mu(\cdot + k).$$

According to Corollary 1.3 and our observations above for less regular potentials, there is a path $t \to \rho_t \in AC^2_{loc}(0,\infty;\mathcal{P}_2(I\!R))$ and $u : (0,\infty)\times I\!R \to I\!R$ Borel satisfying $u_t \in L^2(\rho_t)$ for $\mathcal{L}^1$–almost every $t > 0$, and $\rho_0 = \mu$. Also, (16) is satisfied with the third equation replaced by the second equation in (59). Set

$$f_t(x,v) := \rho_t(x)\delta_{u_t(x)}(v) \in \mathcal{P}_2(I\!R \times I\!R).$$

This path $t \to f_t$ satisfies (59) in the sense of distributions, and we have already shown above that f_t^* given by (60) solves (61) in the distributional sense with $\rho_0^* = \mu^*$. The asymptotic statement on the energy of f_t^* follows from the second equation in (17). QED.

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References

[1] L. Ambrosio, W. Gangbo, *Hamiltonian ODEs in the Wasserstein Space of Probability Measures*, Comm. Pure Appl. Math. **LXI**, 18–53 (2008).

[2] L. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and the Wasserstein spaces of probability measures*, Lectures in mathematics, E.T.H. Zurich, Birkhäuser, 2005.

[3] W. Brawn, K. Hepp, *The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles*, Comm. Math. Phys. **56** (2), 101–113 (1977).

[4] H. Brezis, *Analyse fonctionnelle; théorie et applications*, Masson, Paris (1983).

[5] M.C. Concordel. *Periodic homogenization of Hamilton-Jacobi equations: additive eigenvalues and variational formula*, Indiana Univ. Math. J. **45** (4), 1095–1117 (1996).

[6] M. G. Crandall, P. L. Lions. *Hamilton-Jacobi equations in infinite dimensions I. Uniqueness of viscosity solutions*, J. Funct. Anal. **62**, 379–396 (1985).

[7] M. Crandall, P.L. Lions. *Hamilton-Jacobi equations in infinite dimensions II. Existence of viscosity solutions*, J. Funct. Anal. **65**, 368–405 (1986).

[8] M. Crandall, P.L. Lions. *Hamilton-Jacobi equations in infinite dimensions III*, J. Funct. Anal. **68**, 214–247 (1986).

[9] R.L. Dobrushin. *Vlasov equations*, Funct. Anal. Appl. **13**, 115–123 (1979).

[10] W. E. *A class of homogenization problems in the Calculus of Variations*, Comm. Pure Appl. Math. **XLIV**, 733–759 (1991).

[11] L.C. Evans. *A survey of partial differential equations methods in weak KAM theory*, Comm. Pure Appl. Math. **57** (2004).

[12] A. Fathi. *Weak KAM theory in Lagrangian dynamics*, preliminary version, Lecture notes (2003).

[13] W.H. Fleming, H.M. Soner. *Controlled Markov processes and viscosity solutions*, Springer, New York (1993).

[14] W. Gangbo, H.K. Kim and T. Pacini. *Differential forms on Wasserstein space and infinite-dimensional Hamiltonian systems.* To appear in Memoirs of AMS.

[15] W. Gangbo, T. Nguyen, A. Tudorascu. *Euler-Poisson systems as action minimizing paths in the Wasserstein space*, Arch. Rat. Mech. Anal. **192**, 419–452 (2008).

[16] W. Gangbo, A. Tudorascu. *Lagrangian Dynamics on an infinite-dimensional torus; a Weak KAM theorem.* Preprint.

[17] W. Gangbo, A. Tudorascu. *Homogenization for Hamilton-Jacobi equations in probability spaces*, (in progress).

[18] P.L. Lions, G. Papanicolaou, S.R.S. Varadhan. *Homogenization of Hamilton-Jacobi equations*, unpublished, cca 1988.

[19] V.P. Maslov. *Self-consistent field equations.* Contemporary Problems in Mathematics **11**, VINITI, 153–234 Moscow (1978).

[20] J. Mather. *Minimal measures.* Comment. Math. Helv. **64**, no. 3, 375–394, (1989).

[21] J. Mather. *Action minimizing invariant measures for positive definite Lagrangian systems.* Math. Z. **207**, no. 2, 169–207, (1991).

[22] J. Mather. *Variational construction of connecting orbits.* Ann. Inst. Fourier (Grenoble) **43**, no. 5, 1349–1386, (1993).

[23] H.R. Morton, *Symmetric products of the circle*, Proc. Cambridge Philos. Soc. **63**, 349-352 (1967).

[24] T. Nguyen, A. Tudorascu. *Pressureless Euler/Euler-Poisson systems via adhesion dynamics and scalar conservation laws*, SIAM J. Math. Anal. (to appear).

[25] E. Spanier. *Infinite symmetric products, function spaces, and duality*, Annals of Mathematics **1**, 142–198 (1959).

[26] P. Vanhaecke, *Integrable systems and symmetric products*, Mathematische Zeitschriftt **227**, 93–127 (1998).

[27] C. Villani. *Topics in optimal transportation.* Graduate Studies in Mathematics **58**, American Mathematical Society (2003).