# STICKY PARTICLE DYNAMICS WITH INTERACTIONS

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ABSTRACT. We consider compressible pressureless fluid flows in Lagrangian coordinates in one space dimension. We assume that the fluid self-interacts through a force field generated by the fluid itself. We explain how this flow can be described by a differential inclusion on the space of transport maps, in particular when a sticky particle dynamics is assumed. We study a discrete particle approximation and we prove global existence and stability results for solutions of this system. In the particular case of the Euler-Poisson system in the attractive regime our approach yields an explicit representation formula for the solutions.

## 1. INTRODUCTION

In this paper, we consider a simple model for one-dimensional compressible fluid flows under the influence of a force field that is generated by the fluid itself. It takes the form of a hyperbolic conservation law for the density $\varrho$, which is a nonnegative measure in time and space and describes the distribution of mass or electric charge, and the real-valued Eulerian velocity field $v$. For suitable initial data $(\varrho, v)(t = 0, \cdot) =: (\bar{\varrho}, \bar{v})$, the unknowns $(\varrho, v)$ satisfy

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho v) &= 0 \\ \partial_t(\varrho v) + \partial_x(\varrho v^2) &= f[\varrho] \end{aligned}\right\} \quad \text{in } [0, \infty) \times \mathbb{R}. \tag{1.1}$$

The first equation in (1.1), called the continuity equation, describes the local conservation of mass or electric charge. Without loss of generality, we will assume in the following that the total mass/charge is equal to one initially and that the quadratic moment is finite so that $\varrho(t, \cdot) \in \mathscr{P}_2(\mathbb{R})$ for all $t \geqslant 0$, with $\mathscr{P}_2(\mathbb{R})$ the space of probability measures with finite quadratic moment. The second equation in (1.1) describes the conservation of momentum. We will assume in the following that $v(t, \cdot) \in \mathscr{L}^2(\mathbb{R}, \varrho(t, \cdot))$ for all $t \geqslant 0$ so the kinetic energy is finite.

The map $f : \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ (see §6 for its continuity properties) in (1.1) describes the force field, with $\mathscr{M}(\mathbb{R})$ the space of signed Borel measures with finite total variation. The force depends on the distribution of mass or electric charge and we will assume that $f[\varrho]$ is absolutely continuous with respect to $\varrho$. For further assumptions see Section 6. The typical (simplest) form of $f$ is

$$f[\varrho] = -\varrho\, \partial_x q_\varrho \quad \text{with} \quad q_\varrho(x) = V(x) + \int_{\mathbb{R}} W(x - y)\, d\varrho(y) \tag{1.2}$$

for suitable $C^1$ potential functions $V, W$ with (at most) linearly growing derivatives.

Another relevant example we have in mind is the Euler-Poisson system, for which

$$f[\varrho] = -\varrho\, \partial_x q_\varrho \quad \text{with } q_\varrho \text{ solution of} \quad -\partial^2_{xx} q_\varrho = \lambda(\varrho - \sigma). \tag{1.3}$$

Here $\lambda \in \mathbb{R}$ and $\sigma \in \mathscr{M}(\mathbb{R})$. When $\varrho$ is absolutely continuous with respect to the one-dimensional Lebesgue measure $\mathcal{L}^1$, then the function $q_\varrho$ admits a representation similar to (1.2), with

$$V(x) := -\frac{\lambda}{2} \int_{\mathbb{R}} |x - y|\, d\sigma(y), \quad W(x) := \frac{\lambda}{2}|x|. \tag{1.4}$$

If $\rho$ is not absolutely continuous with respect to $\mathcal{L}^1$, then we have a similar representation with a nondifferentiable $W$, so that $f[\rho]$ must be defined by a suitable approximation process.

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The Euler-Poisson equations in the repulsive regime (with $\lambda < 0$ and negative concave potential $W$) is a simple model for semiconductors. In this case, $\varrho$ describes the electron or hole distribution and the scalar function $q_\varrho$ represents the electric potential generated by the distribution of charges in the material. Here $\sigma$ is the concentration of ionized impurities. The Euler-Poisson equations in the attractive regime (with $\lambda > 0$ and positive convex potential $W$) is the one-dimensional version of a cosmological model for the universe at an early stage, describing the formation of galaxies. Now $q_\varrho$ represents the gravitational potential and $\sigma = 0$.

1.1. **Singular solutions and particle models.** Since there is no pressure in (1.1), there is no mechanism that forces the density $\varrho$ to be absolutely continuous with respect to the Lebesgue measure. In fact, the system (1.1) admits solutions that are singular measures. Assume that we are given initial data in the form of a finite linear combination of Dirac measures:

$$\bar\varrho = \sum_{i=1}^N \bar m_i \delta_{\bar x_i} \quad \text{and} \quad \bar\varrho\bar v = \sum_{i=1}^N \bar m_i \bar v_i \delta_{\bar x_i}, \tag{1.5}$$

where $\bar{\boldsymbol{x}} = (\bar x_1, \cdots, \bar x_N) \in \mathbb{R}^N$ are the initial locations of $N$ particles denoted $P_1, \cdots, P_N$, with corresponding masses $\bar{\boldsymbol{m}} = (\bar m_1, \cdots, \bar m_N)$ and initial velocities $\bar{\boldsymbol{v}} = (\bar v_1, \cdots, \bar v_N)$. We require that $\bar m_i > 0$ and $\sum_i \bar m_i = 1$ so that $\bar\varrho \in \mathscr{P}(\mathbb{R})$. For all times $t \geqslant 0$, we can assume that the positions $\boldsymbol{x}(t) = (x_1(t), \cdots, x_N(t))$ are monotonically ordered, so that they are unambiguously determined and attached to the particles. Then (at least formally) there is a solution of (1.1) in the form of a linear combination of Dirac measures:

$$\varrho(t,\cdot) = \sum_{i=1}^N m_i \delta_{x_i(t)} \quad \text{and} \quad (\varrho v)(t,\cdot) = \sum_{i=1}^N m_i v_i(t) \delta_{x_i(t)}, \tag{1.6}$$

where the functions $(x_i, v_i)$ solve the system of ordinary differential equations

$$\dot x_i(t) = v_i(t), \quad \dot v_i(t) = a_{\bar{\boldsymbol{m}},i}(\boldsymbol{x}(t)) \quad \text{and} \quad (x_i, v_i)(t=0) = (\bar x_i, \bar v_i) \tag{1.7}$$

between particle collisions. Here $a_{\boldsymbol{m},i}(\boldsymbol{x})$ is the value in the point $x_i(t)$ of the Radon-Nikodym derivative of the force $f[\varrho]$ with respect to the measure $\varrho$, so that

$$f[\varrho] = \sum_{i=1}^N a_{\boldsymbol{m},i}(\boldsymbol{x})\, m_i \delta_{x_i} \quad \text{if} \quad \varrho = \sum_{i=1}^N m_i\, \delta_{x_i}, \tag{1.8}$$

which is well defined when all $N$ particles are distinct.

Upon collision of, say, two particles with masses $m_k$ and $m_{k+1}$ at some time $t > 0$, the velocities of each one of them are changed to

$$v_k(t+) = v_{k+1}(t+) = \frac{m_k v_k(t-) + m_{k+1} v_{k+1}(t-)}{m_k + m_{k+1}}, \tag{1.9}$$

so that the momentum is preserved during the collision. Since both particles continue their journey with the same velocity, they may be considered as one bigger particle with mass $m_k + m_{k+1}$. Collisions of more than two particles can be handled in a similar fashion. We will refer to any solution of (1.1) in the form (1.6) as a discrete particle solution and we will say that it satisfies a *global sticky condition* if particles after collision are not allowed to split. In this case, after each collision, one could relabel the particles so that the system (1.7) still makes sense (with $N$ reduced in each particle collision) and induces a global in time evolution.

Let us denote by $\mathbb{K}^N$ the closed cone

$$\mathbb{K}^N := \big\{\boldsymbol{x} \in \mathbb{R}^N : x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N\big\}, \tag{1.10}$$

whose interior is $\operatorname{int}\mathbb{K}^N = \big\{\boldsymbol{x} \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\big\}$. The construction of discrete particle solutions as outlined above can be done rigorously whenever the functions $a_{\boldsymbol{m},i} : \operatorname{int}\mathbb{K}^N \to \mathbb{R}^N$ are uniformly continuous in each bounded set (so that they admit a continuous extension to $\mathbb{K}^N$ still denoted by $a_{\boldsymbol{m},i}$) and satisfies the compatibility condition

$$a_{\boldsymbol{m},k}(\boldsymbol{x}) = a_{\boldsymbol{m},k+1}(\boldsymbol{x}) \quad \text{if } x_k = x_{k+1} \text{ for some } 1 \leqslant k < N. \tag{1.11}$$

This is certainly the case when the potentials $V, W$ considered in (1.2) are of class $\mathrm{C}^1$. On the other hand, the case of the Euler-Poisson system is much more subtle and presents different features in the attractive or the repulsive case.

*The Euler-Poisson case in the repulsive regime: splitting and collapsing of masses.* Let us consider the simplest situation of $N$ distinct particles with equal initial velocities $\bar{v}$, in the repulsive regime $\lambda = -1$ with $\sigma = 0$. Let $M_0 := 0$, $M_i := \sum_{j=1}^{i} m_j$ for $i = 1, \ldots, N$ and set $A_i := \frac{1}{2}(M_{i-1} + M_i - 1)$. Then it is not difficult to check (see Example 6.9) that in the repulsive regime

$$a_{\boldsymbol{m},i}(\boldsymbol{x}) = A_i \quad \text{for all } i \text{ if } \boldsymbol{x} \in \operatorname{int} \mathbb{K}^N, \tag{1.12}$$

and so there is no continuous extension satisfying (1.11). Starting from distinct initial positions, particles follow (at least for a small time interval) the free motion paths

$$x_i(t) = \bar{x}_i + t\bar{v} + \frac{1}{2}A_i t^2. \tag{1.13}$$

Since $A_i \leqslant A_{i+1}$ for all $i$, there are no collisions. Taking the limit as the initial positions of two or more particles coincide we obtain the same representation for every $\boldsymbol{x} \in \mathbb{K}^N$. On the other hand, if two particles $P_k, P_{k+1}$ coincide at the time $t = 0$, i.e. $\bar{x}_k = \bar{x}_{k+1} = \bar{x}$ with the same initial velocity $\bar{v}$, then the "sticky" solution $x_k(t) = x_{k+1}(t) = \bar{x} + t\bar{v} + \frac{1}{4}(A_k + A_{k+1})t^2$ gives raise to an admissible solution to (1.1) which is different from the previous one. One could also consider a solution where $P_k, P_{k+1}$ stick in a small initial time interval $[0, s]$ and then evolve according to (1.13). Solutions are therefore not unique.

Considering a situation where the number $N$ of admissible particles grows to infinity with a uniform initial mass distribution concentrating at the origin, one can guess that a "repulsive" solution arising from a unit mass concentrated at $\bar{x}$ should instantaneously diffuse, becoming absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$: the explicit formula is

$$\varrho(t,\cdot) = u(t,\cdot)\mathcal{L}^1 \quad \text{with} \quad u(t,x) = \frac{1}{2t^2}\chi_{(\bar{x}+\bar{v}t-\frac{1}{4}t^2,\, \bar{x}+\bar{v}t+\frac{1}{4}t^2)}(x) \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0. \tag{1.14}$$

An even more complicated situation occurs e.g. if $\bar{v}_i = 0$ for $i \neq k, k+1$, but $\bar{v}_k > 0 > \bar{v}_{k+1}$ in such a way that a collision occurs between $P_k$ and $P_{k+1}$ at some time $t = r$, after which the particles could stick or wait for some time and then evolve as in the previous example.

It is therefore be important to find a selection mechanism that gives raise to a stable notion of solution and to obtain a continuous model by passing to the limit in the number of particles. In this paper we study a criterium of the following type: Assume that two particle $P_k, P_{k+1}$ collide at some time $r > 0$ with incoming velocities $v_k(r_-) \geqslant v_{k+1}(r_-)$. Then the particles will stick together for all times $r < t < s$ provided that $s$ is small enough so that

$$v_k(r_-) + \int_r^t a_{\boldsymbol{m},k}(\boldsymbol{x}(\tau))\,\mathrm{d}\tau \geqslant v_k(t) = v_{k+1}(t) \geqslant v_{k+1}(r_-) + \int_r^t a_{\boldsymbol{m},k+1}(\boldsymbol{x}(\tau))\,\mathrm{d}\tau \tag{1.15}$$

for all $r < t < s$. Conversely, if (1.15) becomes false for some time $s > r$, then the particles separate again. A rigorous formulation of condition (1.15) in the case of a simultaneous collision or separation of more than two particles, or of a continuous distribution of masses, can be better understood in the framework of differential inclusions in a Lagrangian setting, which we will describe in the next Section 1.2 and in Section 5.1. Before giving an idea of this approach, let us briefly consider how (1.15) simplifies in the attractive regime.

*The attractive Euler-Poisson system and the sticky condition.* In the attractive case, we can simply invert the signs in (1.12). It turns out, however, that the behaviour of the two-particles example considered in the previous paragraph changes completely, since the limit when two particles collapse exhibit a strong stability: after a collision, two or more particles stick together and do not split ever again, giving raise to a *global sticky solution.*

This reflects the fact that the sticky condition in the attractive regime implies (1.15) for all $s > r$: the functions $a_{\boldsymbol{m},i}$ defined by the negative of (1.12) always satisfy $a_{\boldsymbol{m},k}(\boldsymbol{x}) \geqslant a_{\boldsymbol{m},k+1}(\boldsymbol{x})$ and the incoming velocities of two particles $P_k, P_{k+1}$ colliding at some time $r$ satisfies $v_k(r-) \geqslant v_{k+1}(r_-)$, so that any sticky evolution corresponding to $x_k(t) = x_{k+1}(t)$ for $t \geqslant r$ will satisfy (1.15).

As we shall see, the differential description in the Lagrangian setting we will adopt encodes (1.15) and corresponds to a sticky condition whenever the acceleration field is continuous (as in (1.11)) or it is of attractive type. In the repulsive case it will model a suitable relaxation mechanism allowing for separation of particles after collision, still preserving the stability of the evolution.

1.2. **Lagrangian description and differential inclusions.** In this paper, we will give an interpretation of system (1.1) in the framework of differential inclusions. As before, let us first consider the simpler case of the dynamic of a finite number of particles. We can identify the positions of a collection of particles $P_1, \cdots, P_N$ with a vector $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$: since we labeled the particles in a monotone way, it is not admissible for particles to pass by one another, so the order of the locations must be preserved and the vector $\boldsymbol{x}$ is confined in the closed convex cone $\mathbb{K}^N$ defined by (1.10). Denoting by $\boldsymbol{v} = (v_1, \cdots, v_N) \in \mathbb{R}^N$ the vector of the velocities of the particles, their trajectories between collisions are determined by a system of differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \quad \dot{\boldsymbol{v}}(t) = \boldsymbol{a}_{\boldsymbol{m}}(\boldsymbol{x}(t)), \tag{1.16}$$

where $\boldsymbol{a}_{\boldsymbol{m}}(\boldsymbol{x}) := (a_{\boldsymbol{m},1}(\boldsymbol{x}), \cdots, a_{\boldsymbol{m},N}(\boldsymbol{x}))$ is a vector field defined for $\boldsymbol{x} \in \mathbb{K}^N$ as above, which in the simplest case is continuous. Whenever the vector $\boldsymbol{x}(t)$ hits the boundary

$$\partial\mathbb{K}^N = \big\{\boldsymbol{x} \in \mathbb{K}^N : \Omega_{\boldsymbol{x}} \neq \varnothing\big\}, \quad \Omega_{\boldsymbol{x}} := \big\{j\colon x_j = x_{j+1}, j = 1 \dots N-1\big\}, \tag{1.17}$$

however, an instantaneous force changes its velocity in such a way that it stays inside of $\mathbb{K}^N$.

In order to find a mathematical model that describes this situation, we must first identify the set of admissible velocities at each point $\boldsymbol{x} \in \mathbb{K}^N$, which is called the tangent cone of $\mathbb{K}^N$ at $\boldsymbol{x}$. It is defined by

$$T_{\boldsymbol{x}}\mathbb{K}^N := \operatorname{cl}\Big\{\theta(\boldsymbol{y} - \boldsymbol{x}) : \boldsymbol{y} \in \mathbb{K}^N,\ \theta \geqslant 0\Big\}. \tag{1.18}$$

In our situation, it is not difficult to check that

$$T_{\boldsymbol{x}}\mathbb{K}^N = \Big\{\boldsymbol{v} \in \mathbb{R}^N\colon v_j \leqslant v_{j+1} \text{ for all } j \in \Omega_{\boldsymbol{x}}\Big\}. \tag{1.19}$$

Assume now that $\boldsymbol{x}(t) \in \partial\mathbb{K}^N$ at some time $t$ and let $\boldsymbol{v}(t-)$ be the left-derivative of the curve $t \mapsto \boldsymbol{x}(t)$ at time $t$. The instantaneous force that is active on impact must change the velocity to a new value in the tangent cone $T_{\boldsymbol{x}(t)}\mathbb{K}^N$ of admissible velocities. Typically, there are many possibilities. Assuming inelastic collisions, we impose the impact law:

$$\boldsymbol{v}(t+) := \mathsf{P}_{T_{\boldsymbol{x}(t)}\mathbb{K}^N}\boldsymbol{v}(t-), \tag{1.20}$$

where $\boldsymbol{v}(t+)$ is the velocity immediately after impact. We denote by $\mathsf{P}_{T_{\boldsymbol{x}(t)}\mathbb{K}^N}$ the metric projection onto $T_{\boldsymbol{x}(t)}\mathbb{K}^N$ with respect to the weighted Euclidean distance induced by the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{m}} := \sqrt{\sum_{i=1}^N m_i v_i^2} \quad \text{for all } \boldsymbol{v} \in T_{\boldsymbol{x}(t)}\mathbb{K}^N \subset \mathbb{R}^N. \tag{1.21}$$

Hence $\boldsymbol{v}(t+)$ is the unique element in $T_{\boldsymbol{x}(t)}\mathbb{K}^N$ closest to $\boldsymbol{v}(t-)$.

It is well-known that the metric projection onto closed convex cones admits a variational characterization of its minimizers; see [25]. In particular, we have

$$\Big(\boldsymbol{v}(t-) - \boldsymbol{v}(t+)\Big) \cdot \boldsymbol{u} \leqslant 0 \quad \text{for all } \boldsymbol{u} \in T_{\boldsymbol{x}(t)}\mathbb{K}^N.$$

We deduce that the instantaneous force that changes the velocity upon impact onto the boundary $\partial\mathbb{K}^N$, must be an element of the normal cone $N_{\boldsymbol{x}(t)}\mathbb{K}^N$, which is defined as

$$N_{\boldsymbol{x}}\mathbb{K}^N := \Big\{\boldsymbol{n} \in \mathbb{R}^N\colon \boldsymbol{n} \cdot (\boldsymbol{y} - \boldsymbol{x}) \leqslant 0 \text{ for all } \boldsymbol{y} \in \mathbb{K}^N\Big\}. \tag{1.22}$$

Note that the normal cone $N_{\boldsymbol{x}}\mathbb{K}^N$ equals the subdifferential $\partial I_{\mathbb{K}^N}(\boldsymbol{x})$ of the indicator function $I_{\mathbb{K}^N}$ of $\mathbb{K}^N$ at the point $\boldsymbol{x}$. This follows immediately from the definition of the subdifferential.

This suggests to consider the second-order differential inclusion

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} + N_{\boldsymbol{x}}\mathbb{K}^N \ni \boldsymbol{a}_{\boldsymbol{m}}(\boldsymbol{x}) \quad \text{in } [0, \infty). \tag{1.23}$$

Notice that since $\boldsymbol{v}$ can exhibit jumps, solutions to (1.23) should be properly defined in a weak sense in the framework of functions of bounded variation. Second-order differential inclusion have been studied in the literature and existence of solutions has been shown in a *genuinely finite dimensional setting*. We refer the reader to [6,16,22] and the references therein for further information. Due to the possible *nonuniqueness* of solutions to second-order differential inclusions [22] and to the lack of estimates to pass to the limit when $N \to \infty$, we need a better understanding of the particular features of our setting, in particular of the convex cones $\mathbb{K}^N$.

*The sticky condition and an equivalent formulation of* (1.23). It has been shown in [18] that the one-parameter family of normal cones $N_{\boldsymbol{x}(t)}\mathbb{K}^N$ along an evolution curve $\boldsymbol{x} : [0,\infty) \to \mathbb{K}^N$ for which a global stickyness condition holds, satisfies the remarkable monotonicity property

$$N_{\boldsymbol{x}(s)}\mathbb{K}^N \subset N_{\boldsymbol{x}(t)}\mathbb{K}^N \quad \text{for all } s < t. \tag{1.24}$$

Consequently, for any selection $\boldsymbol{\xi} : [0,\infty) \to \mathbb{R}^N$ satisfying $\boldsymbol{\xi}(t) \in N_{\boldsymbol{x}(t)}\mathbb{K}^N$ (such as $\boldsymbol{a_m}(\boldsymbol{x}(t)) - \dot{\boldsymbol{v}}(t)$ in (1.23)) we have

$$\int_s^t \boldsymbol{\xi}(r)\,\mathrm{d}r \in N_{\boldsymbol{x}(t)}\mathbb{K}^N \quad \text{for all } s < t.$$

An integration of (1.23) yields, at least formally, that

$$\boldsymbol{v}(t) + N_{\boldsymbol{x}(t)}\mathbb{K}^N \ni \boldsymbol{v}(s) + \int_s^t \boldsymbol{a}(\boldsymbol{x}(r))\,\mathrm{d}r \tag{1.25}$$

and therefore the system (1.23) can be rewritten in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \quad \boldsymbol{v} + N_{\boldsymbol{x}}\mathbb{K}^N \ni \boldsymbol{y}, \quad \dot{\boldsymbol{y}} = \boldsymbol{a_m}(\boldsymbol{x}). \tag{1.26}$$

Introducing new unknowns $(\boldsymbol{x},\boldsymbol{y})$, we can rewrite (1.26) as a *first order* differential inclusion

$$\begin{aligned} \dot{\boldsymbol{x}} + N_{\boldsymbol{x}(t)}\mathbb{K}^N &\ni \boldsymbol{y} \\ \dot{\boldsymbol{y}} &= \boldsymbol{a_m}(\boldsymbol{x}) \end{aligned} \tag{1.27}$$

for which an existence and stability theory is available, at least when $\boldsymbol{a_m}$ is a Lipschitz map.

We will show that formulation (1.27) enjoys interesting features and always induces a measure-valued solution to (1.1). When the field $\boldsymbol{a_m}$ satisfies the compatibility condition (1.11), solutions to (1.27) satisfy the sticky condition, and the same property holds also for the Euler-Poisson equation in the attractive regime. In the repulsive case, we will see that (1.27) is a robust formulation of condition (1.15). Let us now consider the infinite-dimensional case.

### 1.3. Diffuse measures and differential inclusions for Lagrangian parametrizations.

In order to deal with general measure-valued solutions of (1.1), we had to recourse to Lagrangian coordinates, using ideas of optimal transport as considered in [18].

*Monotone Lagrangian rearrangements.* In this approach, the discrete parameter set $\{1,2,\cdots,N\}$ involved in the representation of discrete particle measures (1.5) will be substituted by $\Omega = (0,1)$. For every particle labeled by $m \in \Omega$, we will denote by $X(t,m) \in \mathbb{R}$ its position at time $t$. The map $X$ can be uniquely characterized in terms of the measure $\varrho$: it is the uniquely determined nondecreasing and right-continuous map $X : \Omega \to \mathbb{R}$ such that

$$X(m) \leqslant x \quad \Longleftrightarrow \quad m \leqslant \varrho\big((-\infty,x]\big) \quad \text{for all } x \in \mathbb{R}. \tag{1.28}$$

Equivalently, the push-forward $X_\#\mathfrak{m}$ of the one-dimensional Lebesgue measure $\mathfrak{m} := \mathcal{L}^1|_\Omega$ under the map $X$ equals $\varrho$. Recall that the push-forward measure is defined by

$$X_\#\mathfrak{m}(A) := \mathfrak{m}(X^{-1}(A)) \quad \text{for all Borel sets } A \subset \mathbb{R}. \tag{1.29}$$

Therefore the map $X$ is the optimal transport map pushing $\mathfrak{m}$ forward to $\varrho$. We refer the reader to Section 2 for further explanation.

In this way, to any solution $(\varrho, v)$ of (1.1), we can associate a map $X\colon [0,\infty) \times \Omega \longrightarrow \mathbb{R}$ with $X(t,\cdot)$ nondecreasing and a velocity $V : [0,+\infty) \times \Omega \to \mathbb{R}$ such that

$$X(t,\cdot)_\#\mathfrak{m} = \varrho(t,\cdot) \quad \text{for all } t \geqslant 0, \quad V(t,\cdot) = v(t,X(t,\cdot)) = \partial_t X(t,\cdot). \tag{1.30}$$

Our goal is to show that (1.1) can be associated to a differential inclusion in terms of $(X, V)$. This observation allows us to derive existence and stability results (see Sections 3 and 4) for (suitably defined) solutions of (1.1), which together with the existence of discrete particle solutions (see Section 5) imply a global existence result for (1.1) for general initial data; see Section 6.

*Differential inclusions.* The framework of first-order differential inclusions, analogous to the setting we already discussed for the discrete case (1.23), serves as a guiding principle for our discussion. The role of the cone $\mathbb{K}^N$ is now played by the cone of optimal transport maps

$$\mathscr{K} := \Big\{X \in \mathscr{L}^2(\Omega)\colon X \text{ is nondecreasing}\Big\} \tag{1.31}$$

in the Hilbert space $\mathscr{H} := \mathscr{L}^2(\Omega)$. Its normal cone $N_X\mathscr{K}$ at $X \in \mathscr{K}$ is given by

$$N_X\mathscr{K} := \Big\{W \in \mathscr{L}^2(\Omega)\colon \int_\Omega W\Big(\tilde{X} - X\Big)\,\mathrm{d}m \leqslant 0 \text{ for all } \tilde{X} \in \mathscr{K}\Big\}. \tag{1.32}$$

Again we have that $N_X\mathscr{K} = \partial I_{\mathscr{K}}(X)$. It can be shown that $N_X\mathscr{K} = \{0\}$ if and only if the map $X$ is not strictly increasing in $\Omega$. That is, whenever $\Omega_X \neq \varnothing$ where

$$\Omega_X := \Big\{m \in \Omega\colon X \text{ is constant in a neighborhood of } m\Big\}. \tag{1.33}$$

Consider now a family of densities $t \mapsto \varrho(t, \cdot)$ that satisfies (1.1). Let $t \mapsto X(t, \cdot) \in \mathscr{K}$ be the associated family of optimal transport maps; see (1.30). We want to interpret $X$ as a solution of differential inclusions, similar to (1.23) and (1.27).

Even at the continuous level, the monotonicity property (1.24) for sticky particle evolutions plays a crucial role. Note that the optimal transport map $X \in \mathscr{K}$ takes a constant value $x \in \mathbb{R}$ on some interval $(\alpha, \beta) \subset \Omega$ if the mass $\beta - \alpha$ (the Lebesgue measure of the interval) is moving to the same location, thereby forming a Dirac measure at $x$. Therefore sticky evolutions will be characterized as curves $t \mapsto X(t, \cdot)$ with the property that

$$\text{for any } t_1 \leqslant t_2 \text{ we have } \Omega_{X(t_1)} \subset \Omega_{X(t_2)}. \tag{1.34}$$

Notice that (1.34) implies that once a Dirac measure is formed, it may accrete more mass over time, but it can never lose mass.

A formulation via differential inclusions needs a Lagrangian expression of the force term in (1.1). That is, we must find a map $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ with the property that

$$\int_{\mathbb{R}} \psi(x)\, f[\varrho](dx) = \int_\Omega \psi(X(m))F[X](m)\,\mathrm{d}m \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}), \tag{1.35}$$

whenever $X \in \mathscr{K}$ and $X_\#\mathfrak{m} = \varrho$. We refer the reader to Section 6 for further discussion about the existence and properties of maps $F$ satisfying (1.35). In the following, we will assume that $F$ is continuous as a map of $\mathscr{K}$ into $\mathscr{L}^2(\Omega)$.

We then could expect $X$ to be a solution of a second-order differential inclusion, but arguing as for the discrete case (1.23) at least in the case of sticky evolutions (1.34) we end up with

$$\dot{X}(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar{V} + \int_0^t F[X(s)]\,\mathrm{d}s \tag{1.36}$$

for a.e. $t > 0$. This formulation and its consequences is at the heart of our argument.

It is a remarkable fact (see Theorem 3.5) that solutions to (1.36) always parametrize measure-valued solutions to the partial differential equation (1.1). Provided $F$ satisfies suitable continuity properties, it will be possible to prove existence (and uniqueness, when $F$ is Lipschitz) of solutions to (1.36) for any initial data $(\bar{X}, \bar{V}) \in \mathscr{K} \times \mathscr{L}^2(\Omega)$ by combining the theory of gradient flows of convex functionals in Hilbert spaces [9] with suitable compactness arguments.

When $F$ satisfies a suitable *sticking* condition, which is satisfied e.g. in the case of $\mathrm{C}^1$ potentials in (1.2) and of the Euler-Poisson system in the attractive regime, then solutions to (1.36) form a semigroup and have the sticky evolution property (1.34). Even for general $F$ (and in particular for the Euler-Poisson system in the repulsive regime) the differential inclusion (1.36) still selects a stable parametrization of solutions to (1.1). This is somewhat surprising since the reduction from second-order to first-order differential inclusion was motivated by the monotonicity (1.34), which typically is false without additional assumptions on $F$. In this Introduction we refer to such solutions as "robust."

*Representation formulae for the Euler-Poisson system.* In the case of the Euler-Poisson system (1.3) with $\sigma = 0$ one can show that the Lagrangian representation of the force $f$ is given by

$$F[X](m) = -\lambda A(m) \quad \text{where} \quad A(m) := m - \frac{1}{2}. \tag{1.37}$$

Note that the map $F[X]$ is *independent of* $X$ and (1.36) becomes

$$\dot{X}(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar{V} - \lambda t A$$

In the attractive regime when $\lambda \geq 0$, an explicit representation formula for the Lagrangian solution can be obtained (see Theorems 6.11). In fact a careful analysis shows that the solution $X$ to (1.36) can be computed by solving the trivial ODE in $X$ obtained by eliminating the $\mathscr{K}$-constraint:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}(t) = \bar{V} - \lambda t A$$

(whose solution is $\tilde{X}(t) = \bar{X} + t\bar{V} - \frac{1}{2}\lambda t^2 A$), and then projecting $\tilde{X}$ onto $\mathscr{K}$:

$$X(t) = \mathsf{P}_{\mathscr{K}}(\tilde{X}(t)) = \mathsf{P}_{\mathscr{K}}\Big(\bar{X} + t\bar{V} - \frac{1}{2}\lambda t^2 A\Big).$$

Applying the characterization given in [18], the metric projection of $\mathsf{P}_{\mathscr{K}}$ onto $\mathscr{K}$ can be found by introducing the primitive functions

$$\bar{\mathcal{X}}(m) := \int_0^m \bar{X}(\ell)\,\mathrm{d}\ell, \quad \bar{\mathcal{V}}(m) := \int_0^m \bar{V}(\ell)\,\mathrm{d}\ell, \quad \mathcal{A}(m) := \frac{1}{2}(m^2 - m),$$

and the time evolution

$$\mathcal{X}(t,m) := \bar{\mathcal{X}}(m) + t\,\bar{\mathcal{V}}(m) - \frac{1}{2}\lambda t^2 \mathcal{A}(m),$$

and then taking the derivative with respect to $m$ of the convex envelope $\mathcal{X}^{**}(t,\cdot)$ of $\mathcal{X}(t,\cdot)$:

$$X(t,m) = \frac{\partial}{\partial m}\mathcal{X}^{**}(t,m),$$

which defines a density $\varrho_t = X(t)_{\#}\mathfrak{m}$. It is then a simple exercise to recover formula (1.14) in the case $\bar{X}(m) \equiv \bar{x}$, $\bar{V}(m) \equiv \bar{v}$, since $\mathcal{X}(t,m) = \mathcal{X}^{**}(t,m)$ and $X(t,m) = \bar{x} + t\bar{v} + \frac{1}{2}t^2(m - \frac{1}{2})$.

1.4. **Time discrete schemes.** In this section, we show that the first-order differential inclusion (1.36) can be used to design a stable explicit numerical scheme to compute robust solutions to (1.1). This scheme is basically the same as the one introduced in [8] for "order-preserving vibrating strings" and "sticky particles", with just mild modifications. For simplicity, we concentrate on the pressureless repulsive Euler-Poisson system with a neutralizing background

$$\partial_t\varrho + \partial_x(\varrho v) = 0, \tag{1.38}$$

$$\partial_t(\varrho v) + \partial_x(\varrho v^2) = -\varrho\,\partial_x q_\varrho \qquad -\partial_{xx} q_\varrho = -(\varrho - 1), \tag{1.39}$$

(cf. (1.3) with $\lambda = -1$ and $\sigma = 1$). We assume the initial conditions to be 1-periodic in $x$ and the density $\varrho$ to have unit mean so that the system is globally neutral and the electric potential $q_\varrho$ is 1-periodic in $x$. Note that we choose the periodic setting only for convenience. In fact, for any non-periodic solution $\varrho$ of (1.1), one could consider the push-forward of $\varrho$ under the map $x \mapsto x - [x]$ for all $x \in \mathbb{R}$, with $[x]$ the largest integer not greater than $x$. One obtains a new density $\varrho^*$ that is concentrated on $[0,1)$ and therefore can be extended 1-periodically to the whole real line. One can then show that $\varrho^*$ satisfies the same equation. We refer the reader to [13] for details.

For smooth solutions without mass concentration, written in mass coordinates

$$V(t,m) = \dot{X}(t,m) = v(t, X(t,m)), \quad \partial_m X(t,m)\varrho(t, X(t,m)) = 1$$

(which requires that $\partial_m X(t,m) \geqslant 0$), one can show that the whole system reduces to a collection of independent linear pendulums labeled by their equilibrium position $m$ and subject to

$$\dot{X}(t,m) = V(t,m), \quad \dot{V}(t,m) + X(t,m) - m = 0. \tag{1.40}$$

(Notice that, due to the spatial periodicity of the initial conditions, the new unknown $X(t,m)-m$ and $V(t,m)$ are 1-periodic in $m$.) This reduction is valid as long as the pendulums stay "well-ordered" and do not cross each other, i.e., as long as $X(t,m)$ stays monotonically nondecreasing in $m$. This "non-crossing" condition is not sustainable for large initial conditions and collision generally occur in finite time. To handle sticky collisions, the concept of robust solutions introduced in Section 1.3 is a good way to obtain a well-posed mathematical model beyond collisions.

We are now ready to describe the semi-discrete scheme. Given a time step $\tau > 0$ and suitable initial data $(\bar{X}, \bar{V}) =: (X_{\tau,0}, V_{\tau,0})$, we denote by $(X_{\tau,n}(m), V_{\tau,n}(m))$ the approximate solution at time $t_n := n\tau$, for $n = 0, 1, 2, \ldots$, defined in two steps as follows:

(1) *Predictor Step:* We first integrate the ODE (1.40) and obtain the predictors

$$\hat{X}_{\tau,n+1}(m) = m + (X_{\tau,n}(m) - m)\cos(\tau) + V_{\tau,n}(m)\sin(\tau), \tag{1.41}$$

$$V_{\tau,n+1}(m) = -(X_{\tau,n}(m) - m)\sin(\tau) + V_{\tau,n}(m)\cos(\tau). \tag{1.42}$$

(2) *Corrector Step:* We rearrange $\hat{X}_{\tau,n+1}(m)$ in nondecreasing order with respect to $m$ and obtain $X_{\tau,n+1}(m)$. Because of the periodic boundary conditions, we have to perform this step with care. We rely on the existence, for each map $m \mapsto Y(m)$ such that $Y(m) - m$ is 1-periodic and locally Lebesgue integrable, of a unique map $m \mapsto Y^*(m)$ such that $Y^*(m)$ is nondecreasing in $m$ and

$$\int_0^1 \eta(Y^*(m))\,\mathrm{d}m = \int_0^1 \eta(Y(m))\,\mathrm{d}m$$

for all continuous 1-periodic function $\eta$.

This time discrete scheme becomes a fully discrete scheme, if the initial data $X_{\tau,0}(m) - m$ and $V_{\tau,0}(m)$ are piecewise constant on a uniform cartesian grid with step $h$. (We just have to be careful with the corrector step, by using a suitable sorting algorithm for periodic data.)

To illustrate the scheme, we show the numerical solutions corresponding to initial conditions

$$X_0(m) = m, \quad V_0(m) = 4\sin(2\pi m). \tag{1.43}$$

We use 400 equally spaced grid points $m$ (which corresponds to 400 "well-ordered" pendulums with $m$ as equilibrium position) and 5000 time steps (see Figures 1–3):

$$\tau = 0.001, \quad 0.005, \quad 0.01,$$

so that the final time $T$ of observation is respectively given by

$$T = 5, \quad 25, \quad 50.$$

On each picture, we show the space-time trajectories of 50 of the 400 pendulums, with space coordinate on the horizontal axis and time coordinate on the vertical one. On these pictures, we observe a strong concentration, with sticky collisions, of the pendulums at a very early stage (up to time $t = \pi/2$) around $x = 0.5$. Later on, some pendulums start to unstick and detach from each other (which allows new concentrations at later times $t \geqslant \pi$ around $x = 0$ and $x = 1$). Much later, after $t = 10\pi$, there is no further dissipation of energy, and, as pendulums touch each other, they always do so with zero relative speed. Then the corrector step is no longer active, and the scheme becomes exact (due to the exact integration of the predictor step). At this late stage, the solution becomes $2\pi$-periodic in time. We study the convergence of the scheme in Section 7.

1.5. **Plan of the paper.** We collect in Section 2 a few basic results on optimal transport in one dimension, on convex analysis (concerning in particular the properties of the convex cone $\mathscr{K}$), and on convex functionals in $\mathscr{L}^2(\Omega)$.

In Section 3, after a brief discussion of the basic properties of the Lagrangian force functional $F$, we introduce the notion of Lagrangian solutions to the differential inclusion (1.36). Theorem 3.5 collects their main properties, in particular in connection with measure-valued solutions to (1.1).

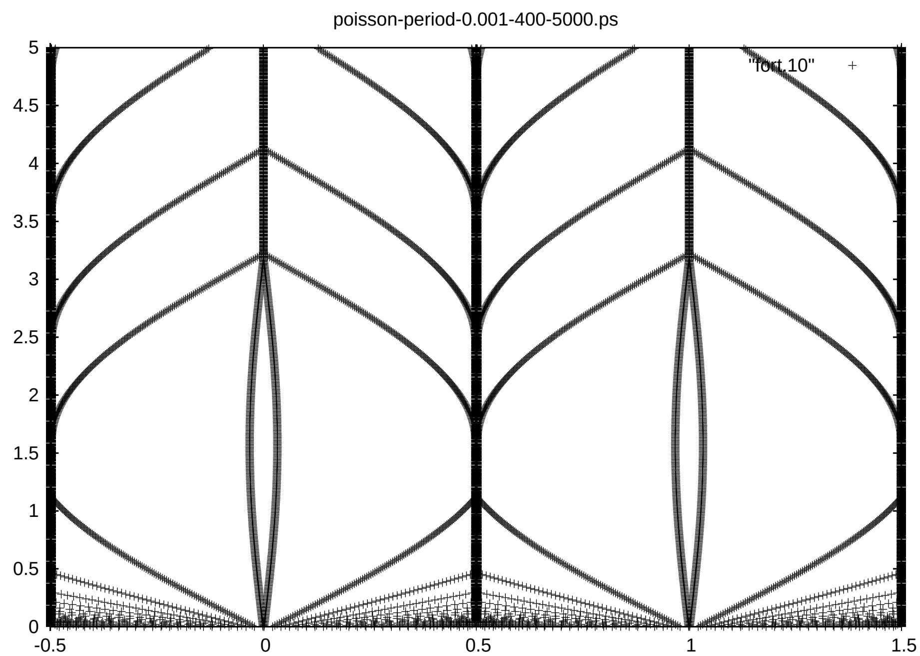


FIGURE 1. Space-time trajectories of pendulums, with timestep $\tau = 0.001$.

Sections 3.3 and 3.5 provide the main existence, uniqueness, and stability results for Lagrangian solutions, whereas Section 3.4 is devoted to the particular case of sticky evolutions.

We study in Section 4 a different class of solutions to (1.36), still linked to (1.1), that naturally arise as limit of sticky particle systems when $F$ does not obey the sticking condition. These solutions exhibit better semigroup properties than the Lagrangian solutions introduced in Section 3, but lack uniqueness.

In Section 5 we carefully study the dynamics of discrete particle systems, which we already briefly discussed in the Introduction. Discrete Lagrangian solutions associated to systems like (1.27) are treated in Section 5.1, where we also show that they can be used to approximate any continuous Lagrangian solution, such as the one considered in Section 3. The sticky dynamic at the particle level is considered in Section 5.2: the main Theorem 5.2 provides the basic results, which allow us to replace second-order with first-order differential inclusions at the discrete level and to get sticky evolutions for sticking forces. The particle approach is a crucial step of our analysis, since it avoids many technical difficulties arising at the continuous level. The general idea is to prove fine properties of the solutions (such as the monotonicity (1.34) in the sticking case or a representation formula) at the discrete level and then to extend them to the general case by applying suitable stability results with respect to the initial conditions. Those are typically obtained by applying contraction estimates (in the case when $F$ is Lipschitz) or compactness via Helly's Theorem, by exploiting higher integrability and monotonicity of transport maps.

Section 6 applies the Lagrangian formulation to (1.1), presenting some existence and stability results for solutions in the Eulerian formalism.

In Section 7 we prove the convergence of the time discrete scheme introduced in Section 1.4.

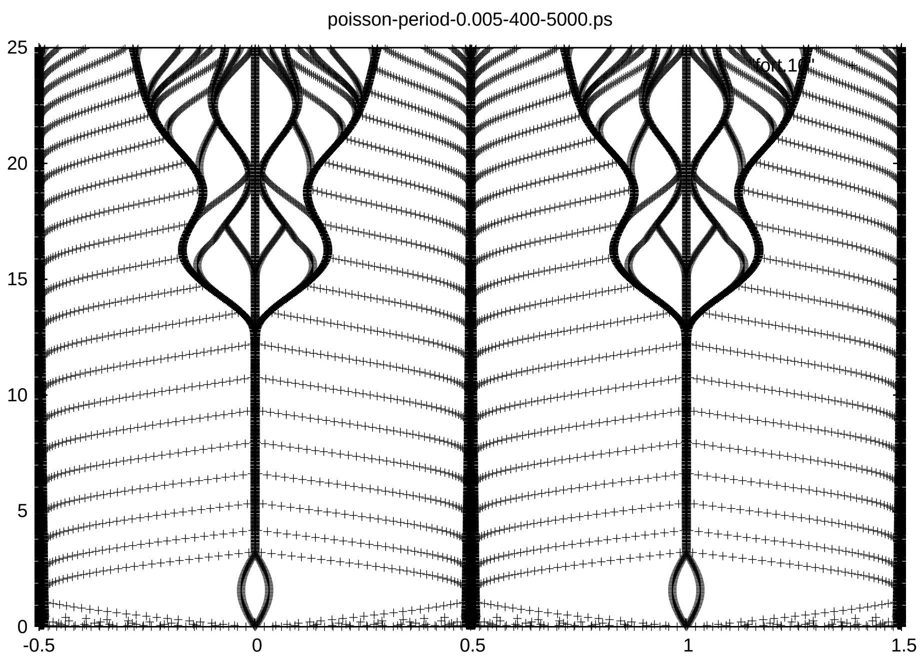


FIGURE 2. Space-time trajectories of pendulums, with timestep $\tau = 0.005$.

## 2. Preliminaries

Let us first gather some definitions and results that will be needed later.

2.1. **Optimal Transport.** We denote by $\mathscr{P}(\mathbb{R}^m)$ the space of all Borel probability measures on $\mathbb{R}^m$. The push-forward $\nu := Y_\# \mu$ of a given measure $\mu \in \mathscr{P}(\mathbb{R}^m)$ under a Borel map $Y : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is the measure defined by $\nu(A) := \mu(Y^{-1}(A))$ for all Borel sets $A \subset \mathbb{R}^n$. We will repeatedly use the change-of-variable formula

$$\int_{\mathbb{R}^n} \zeta(y)\,(Y_\#\mu)(\mathrm{d}y) = \int_{\mathbb{R}^m} \zeta(Y(x))\,\mu(\mathrm{d}x), \tag{2.1}$$

which holds for all Borel maps $\zeta : \mathbb{R}^n \longrightarrow [0, \infty]$.

We denote by $\mathscr{P}_2(\mathbb{R}^n)$ the space of all $\varrho \in \mathscr{P}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |x|^2\,\varrho(\mathrm{d}x) < \infty$. The Kantorovich-Rubinstein-Wasserstein distance $W_2(\varrho_1, \varrho_2)$ between $\varrho_1, \varrho_2 \in \mathscr{P}_2(\mathbb{R}^n)$ can be defined in terms of measures $\boldsymbol{\varrho} \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying $\pi^i_\# \boldsymbol{\varrho} = \varrho_i$ for $i = 1 \dots 2$, by the formula

$$W_2^2(\varrho_1, \varrho_2) := \min \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2\,\boldsymbol{\varrho}(\mathrm{d}x, \mathrm{d}y) : \boldsymbol{\varrho} \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n), \pi^i_\# \boldsymbol{\varrho} = \varrho_i \right\}. \tag{2.2}$$

Here $\pi^i(x_1, x_2) := x_i$ is the projection on the $i$th coordinate. It can be shown that there always exists an optimal transport plan $\boldsymbol{\varrho}$ for which the inf in (2.2) is attained. We denote by $\Gamma_{\mathrm{opt}}(\varrho_1, \varrho_2)$ the set of optimal transport plans.

In the one-dimensional case $n = 1$, there exists a unique $\boldsymbol{\varrho} \in \Gamma_{\mathrm{opt}}(\varrho_1, \varrho_2)$ realizing the minimum of (2.2) (when the cost is finite). It can be explicitly characterized as follows: for any $\varrho \in \mathscr{P}(\mathbb{R})$ we consider its cumulative distribution function, which is defined as

$$M_\varrho(x) := \varrho\big((-\infty, x]\big) \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

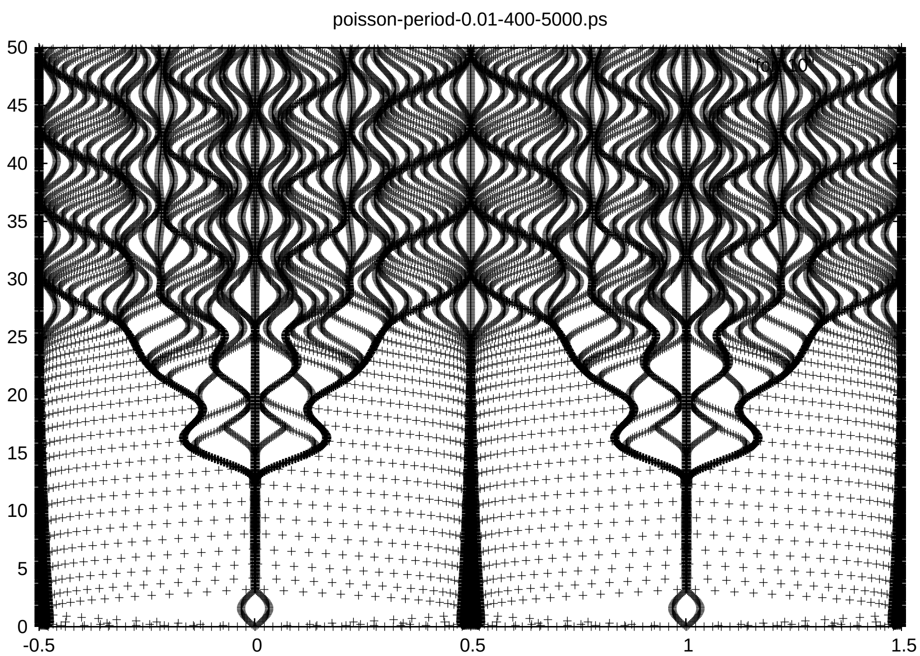


FIGURE 3. Space-time trajectories of pendulums, with timestep $\tau = 0.01$.

Note that then $\varrho = \partial_x M_\varrho$ in $\mathscr{D}'(\mathbb{R})$. Its monotone rearrangement is given by

$$X_\varrho(m) := \inf\{x\colon M_\varrho(x) > m\} \quad \text{for all } m \in \Omega, \tag{2.4}$$

where $\Omega := (0,1)$. The map $X_\varrho$ is right-continuous and nondecreasing. We have

$$\mathfrak{m} := \mathcal{L}^1|_\Omega, \quad (X_\varrho)_\# \mathfrak{m} = \varrho \quad \text{and} \quad \int_{\mathbb{R}} \zeta(x)\, \varrho(\mathrm{d}x) = \int_\Omega \zeta(X_\varrho(m))\, \mathrm{d}m \tag{2.5}$$

for all $\zeta\colon \mathbb{R} \longrightarrow [0,\infty]$ Borel. In particular, we have that $\varrho \in \mathscr{P}_2(\mathbb{R})$ if and only if $X_\varrho \in \mathscr{L}^2(\Omega)$. The Hoeffding-Fréchet theorem [19, Section 3.1] shows that the joint map $X_{\varrho_1,\varrho_2}\colon \Omega \longrightarrow \mathbb{R}\times\mathbb{R}$ defined by

$$X_{\varrho_1,\varrho_2}(m) := \Big(X_{\varrho_1}(m), X_{\varrho_2}(m)\Big) \quad \text{for all } m \in \Omega,$$

characterizes the transport plan $\boldsymbol{\varrho} \in \Gamma_{\mathrm{opt}}(\varrho_1,\varrho_2)$ by the formula

$$\boldsymbol{\varrho} = (X_{\varrho_1,\varrho_2})_\# \mathfrak{m}; \tag{2.6}$$

see [10, 19, 24] for further information. As a consequence, we obtain that

$$W_2^2(\varrho_1,\varrho_2) = \int_\Omega |X_{\varrho_1}(m) - X_{\varrho_2}(m)|^2\, \mathrm{d}m = \|X_{\varrho_1} - X_{\varrho_2}\|^2_{\mathscr{L}^2(\Omega)}. \tag{2.7}$$

The map $\varrho \mapsto X_\varrho$ is an isometry between $\mathscr{P}_2(\mathbb{R})$ and $\mathscr{K}$, where $\mathscr{K} \subset \mathscr{L}^2(\Omega)$ is the set of nondecreasing functions. Without loss of generality, we may consider precise representatives of nondecreasing functions only, which are defined everywhere.

2.2. **Some Tools of Convex Analysis for $\mathscr{K}$.** Let $\mathscr{K}$ be the collection of right-continuous nondecreasing functions in $\mathscr{L}^2(\Omega)$ introduced in (1.31). Then one can check that $\mathscr{K}$ is a closed convex cone in the Hilbert space $\mathscr{L}^2(\Omega)$.

*Metric Projection and Indicator Function.* It is well-known that the metric projection onto a nonempty closed convex set of an Hilbert space is a well defined Lipschitz map (see e.g. [25]): we denote it by $\mathsf{P}_{\mathscr{K}}: \mathscr{L}^2(\Omega) \longrightarrow \mathscr{K}$. For all $X \in \mathscr{L}^2(\Omega)$ it is characterized by

$$Y = \mathsf{P}_{\mathscr{K}}(X) \quad \Longleftrightarrow \quad Y \in \mathscr{K}, \quad \int_\Omega (X - Y)(\tilde{Y} - Y) \leqslant 0 \quad \text{for all } \tilde{Y} \in \mathscr{K}. \tag{2.8}$$

The projection $\mathsf{P}_{\mathscr{K}}(X)$ admits a more explicit characterization in terms of the convex envelope of the primitive of $X$ [18, Theorem 3.1]:

$$Y = \mathsf{P}_{\mathscr{K}}(X) = \frac{\mathrm{d}^+}{\mathrm{d}m} \mathcal{X}^{**}(m), \quad \mathcal{X}(m) := \int_0^m X(\ell)\,\mathrm{d}\ell, \tag{2.9}$$

where $\frac{\mathrm{d}^+}{\mathrm{d}m}$ denotes the right derivative and

$$\mathcal{X}^{**}(m) := \sup\Big\{a + b\,m \colon a, b \in \mathbb{R}, \quad a + b\,m \leqslant \mathcal{X}(m) \quad \forall\, m \in (0,1)\Big\} \tag{2.10}$$

is the greatest convex and l.s.c. function below $\mathcal{X}$.

Let now $I_{\mathscr{K}}: \mathscr{L}^2(\Omega) \longrightarrow [0, +\infty]$ be the indicator function of $\mathscr{K}$, defined as

$$I_{\mathscr{K}}(X) := \begin{cases} 0 & \text{if } X \in \mathscr{K}, \\ +\infty & \text{otherwise,} \end{cases}$$

which is convex and lower semicontinuous. Its subdifferential $\partial I_{\mathscr{K}}(X)$ is a maximal monotone operator in $\mathscr{L}^2(\Omega)$. In particular, its graph is strongly-weakly closed in $\mathscr{L}^2(\Omega) \times \mathscr{L}^2(\Omega)$.

Notice that $\partial I_{\mathscr{K}}(X) = \varnothing$ for all $X \notin \mathscr{K}$ since then $I_{\mathscr{K}}(X) = \infty$. For $X \in \mathscr{K}$ we find that

$$\partial I_{\mathscr{K}}(X) = \Big\{Z \in \mathscr{L}^2(\Omega) \colon 0 \geqslant \int_\Omega Z(\tilde{X} - X) \text{ for all } \tilde{X} \in \mathscr{K}\Big\}, \tag{2.11}$$

so that $\partial I_{\mathscr{K}}(X)$ coincides with the normal cone $N_X \mathscr{K}$ defined by (1.32).

Now (2.8) implies the following equivalence: For all $X, Y \in \mathscr{L}^2(\Omega)$ we have

$$Y = \mathsf{P}_{\mathscr{K}}(X) \quad \Longleftrightarrow \quad X - Y \in \partial I_{\mathscr{K}}(Y). \tag{2.12}$$

Decomposing $X = Y + Z$ in (2.12) with $Y, Z \in \mathscr{L}^2(\Omega)$, we find that

$$Z \in \partial I_{\mathscr{K}}(Y) \quad \Longleftrightarrow \quad Y = \mathsf{P}_{\mathscr{K}}(Y + Z). \tag{2.13}$$

**Lemma 2.1** (Contraction)**.** *Let $\psi \colon \mathbb{R} \longrightarrow (-\infty, +\infty]$ be a convex, l.s.c. function. Then*

$$\int_\Omega \psi(\mathsf{P}_{\mathscr{K}}(X_1) - \mathsf{P}_{\mathscr{K}}(X_2)) \leqslant \int_\Omega \psi(X_1 - X_2) \quad \textit{all } X_1, X_2 \in \mathscr{L}^2(\Omega).$$

*In particular, the metric projection $\mathsf{P}_{\mathscr{K}}$ is a contraction with respect to the $\mathscr{L}^p(\Omega)$-norm with $p \in [1, \infty]$ and for all $X_1, X_2 \in \mathscr{L}^2(\Omega)$ we can estimate*

$$\|\mathsf{P}_{\mathscr{K}}(X_1) - \mathsf{P}_{\mathscr{K}}(X_2)\|_{\mathscr{L}^p(\Omega)} \leqslant \|X_1 - X_2\|_{\mathscr{L}^p(\Omega)}. \tag{2.14}$$

We refer the reader to Theorem 3.1 in [18] for a proof. Notice that by choosing $X_2 = 0$ in Lemma 2.1, for which $\mathsf{P}_{\mathscr{K}}(X_2) = 0$, we obtain the inequalities

$$\int_\Omega \psi(\mathsf{P}_{\mathscr{K}}(X)) \leqslant \int_\Omega \psi(X) \quad \text{for all } X \in \mathscr{L}^2(\Omega), \tag{2.15}$$

$$\|\mathsf{P}_{\mathscr{K}}(X)\|_{\mathscr{L}^p(\Omega)} \leqslant \|X\|_{\mathscr{L}^p(\Omega)} \quad \text{for all } X \in \mathscr{L}^2(\Omega). \tag{2.16}$$

A similar result holds for the $\mathscr{L}^2$-orthogonal projection $\mathsf{P}_{\mathscr{H}_X}$ onto the closed subspace

$$\mathscr{H}_X := \Big\{U \in \mathscr{L}^2(\Omega) \colon U \text{ is constant in each interval } (\alpha, \beta) \subset \Omega_X\Big\} \tag{2.17}$$

for $X \in \mathscr{K}$ (see (1.33) for the definition of $\Omega_X$). Notice that $\mathsf{P}_{\mathscr{H}_X}(V) = V$ a.e. in $\Omega \setminus \Omega_X$ and

$$\mathsf{P}_{\mathscr{H}_X}(V) = \fint_\alpha^\beta V(s)\,\mathrm{d}s \quad \text{in any maximal interval } (\alpha, \beta) \subset \Omega_X, \tag{2.18}$$

for all $V \in \mathscr{L}^2(\Omega)$. Jensen's inequality then easily yields

**Lemma 2.2** ($\mathscr{H}_X$–Contraction). *Let $\psi : \mathbb{R} \to [0,\infty]$ be a convex l.s.c. function. Then*

$$\int_\Omega \psi(\mathsf{P}_{\mathscr{H}_X}(X))\,\mathrm{d}m \leqslant \int_\Omega \psi(X)\,\mathrm{d}m \quad \textit{for all } X \in \mathscr{K}.$$

For any pair $X, Y \in \mathscr{L}^1(\Omega)$ we say that $Y$ is dominated by $X$ (we write $Y \prec X$) if

$$\int_\Omega \psi(Y)\,\mathrm{d}m \leqslant \int_\Omega \psi(X)\,\mathrm{d}m$$

for all convex, l.s.c. $\psi\colon \mathbb{R} \longrightarrow [0,\infty]$. By (2.15), we have $\mathsf{P}_{\mathscr{K}}(X) \prec X$ for all $X \in \mathscr{L}^2(\Omega)$.

*Normal and Tangent Cones.* Applying [18, Thm. 3.9] we get the following characterization:

**Lemma 2.3.** *Let $X \in \mathscr{K}$ be given. For given $W \in \mathscr{L}^2(\Omega)$ we denote by*

$$\Xi_W(m) := \int_0^m W(s)\,\mathrm{d}s \quad \textit{for all } m \in [0,1],$$

*its primitive. Then $W \in N_X\mathscr{K}$ if and only if $\Xi_W \in \mathcal{N}_X$, where*

$$\mathcal{N}_X := \Big\{ \Xi \in \mathrm{C}([0,1])\colon \Xi \geqslant 0 \textit{ in } [0,1] \textit{ and } \Xi = 0 \textit{ in } \Omega \setminus \Omega_X \Big\}.$$

*In particular, for every $X_1, X_2 \in \mathscr{K}$ we have*

$$\Omega_{X_1} \subset \Omega_{X_2} \quad \Longrightarrow \quad N_{X_1}\mathscr{K} \subset N_{X_2}\mathscr{K}. \tag{2.19}$$

The tangent cone $T_X\mathscr{K}$ to $\mathscr{K}$ at $X \in \mathscr{K}$ can be defined as in (1.18) by

$$T_X\mathscr{K} := \mathrm{cl}\Big\{ \theta(\tilde{X} - X) : \tilde{X} \in \mathscr{K},\ \theta \geq 0 \Big\} \tag{2.20}$$

or, equivalently, as the polar cone of $N_X\mathscr{K}$:

$$T_X\mathscr{K} = \Big\{ U \in \mathscr{L}^2(\Omega)\colon \int_\Omega U(m)W(m)\,\mathrm{d}m \leqslant 0 \text{ for all } W \in N_X\mathscr{K} \Big\}. \tag{2.21}$$

**Lemma 2.4.** *Let $X \in \mathscr{K}$ be given. Then*

$$T_X\mathscr{K} = \Big\{ U \in \mathscr{L}^2(\Omega)\colon U \textit{ is nondecreasing in each interval } (\alpha,\beta) \subset \Omega_X \Big\}.$$

More precisely, the map $U \in T_X\mathscr{K}$ must be nondecreasing up to Lebesgue null sets. We may assume that $U$ is right-continuous in each $(\alpha,\beta) \subset \Omega_X$.

*Proof of Lemma 2.4.* Let $U \in T_X\mathscr{K}$ be given and fix some interval $(\alpha,\beta) \subset \Omega_X$. For all nonnegative $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{spt}\varphi \subset (\alpha,\beta)$ we have $\varphi' \in N_X\mathscr{K}$ because of Lemma 2.3. Then

$$\int_\alpha^\beta U(m)\varphi'(m)\,\mathrm{d}m \leqslant 0 \quad \text{for all nonnegative } \varphi \in \mathscr{D}(\Omega) \text{ with } \operatorname{spt}\varphi \subset (\alpha,\beta);$$

see (2.21). This shows that the distributional derivative of $U$ in $(\alpha,\beta)$ is a nonnegative Radon measure, and so $U$ is nondecreasing in that interval.

Conversely, assume that $U \in \mathscr{L}^2(\Omega)$ is nondecreasing in each interval $(\alpha,\beta)$ that is contained in $\Omega_X$. For any $W \in N_X\mathscr{K}$ we then decompose the integral

$$\int_\Omega U(m)W(m)\,\mathrm{d}m = \int_{\Omega\setminus\Omega_X} U(m)W(m)\,\mathrm{d}m + \sum_n \int_{\alpha_n}^{\beta_n} U(m)W(m)\,\mathrm{d}m, \tag{2.22}$$

where the sum is over all maximal intervals $(\alpha_n,\beta_n) \subset \Omega_X$ (there are at most countably many). Then the first integral on the right-hand side vanishes because $W(m) = 0$ for a.e. $m \in \Omega \setminus \Omega_X$. For each integral in the sum, an approximation argument (see again Lemma 3.10 in [18]) allows us to integrate by parts to obtain

$$\int_{\alpha_n}^{\beta_n} U(m)W(m)\,\mathrm{d}m = -\int_{\alpha_n}^{\beta_n} \Xi_W(s)\gamma(ds),$$

where $\gamma$ is the distributional derivative of $U$ in $(\alpha_n,\beta_n)$. Since $U$ is assumed nondecreasing and $\Xi_W$ is nonnegative, we conclude that $U \in T_X\mathscr{K}$. $\square$

Lemma 2.4 immediately yields the equivalence

$$U \in \mathscr{H}_X \quad \Longleftrightarrow \quad \pm U \ \in T_X\mathscr{K}. \tag{2.23}$$

Observe that if $X_1, X_2 \in \mathscr{K}$ then

$$\Omega_{X_1} \subset \Omega_{X_2} \quad \Longrightarrow \quad \mathscr{H}_{X_2} \subset \mathscr{H}_{X_1}. \tag{2.24}$$

Whenever $U \in \mathscr{H}_X$, then (2.22) equals zero because every term in the sum vanishes since $U$ is constant and $W$ has vanishing average. Thus

$$N_X\mathscr{K} \subset \mathscr{H}_X^\perp \quad \text{for all } X \in \mathscr{K}, \tag{2.25}$$

with $\mathscr{H}_X^\perp$ the orthogonal complement of $\mathscr{H}_X$. In particular, we obtain that

$$Y = \mathsf{P}_{\mathscr{K}}(X) \quad \Longrightarrow \quad Y = \mathsf{P}_{\mathscr{H}_Y}(X), \quad \mathscr{H}_Y \subset \mathscr{H}_X. \tag{2.26}$$

We have in fact a more precise characterization of $\mathscr{H}_X^\perp$ in terms of $N_X\mathscr{K}$: in the following, $\mathscr{I}(\Omega_X)$ denotes the collection of all maximal intervals $(\alpha,\beta)$ (the connected components) of $\Omega_X$.

**Lemma 2.5.** *For every $X \in \mathscr{K}$ we have*

$$\begin{aligned} \mathscr{H}_X^\perp = \Big\{ W \in \mathscr{L}^2(\Omega) \colon W = 0 \ \text{a.e. in } \Omega \setminus \Omega_X,& \\ \int_\alpha^\beta W(m)\,\mathrm{d}m = 0 \quad \text{for every } (\alpha,\beta) \in \mathscr{I}(\Omega_X) \Big\},& \end{aligned} \tag{2.27}$$

*and it is the closed linear subspace of $\mathscr{L}^2(\Omega)$ generated by $N_X\mathscr{K}$. Moreover, it holds*

$$\mathscr{H}_X^\perp = \Big\{ W \in \mathscr{L}^2(\Omega) \colon \int_\Omega W(m)\varphi(X(m))\,\mathrm{d}m = 0 \text{ for all } \varphi \in \mathrm{C}_b(\mathbb{R}) \Big\}. \tag{2.28}$$

*Proof.* Identity (2.27) follows immediately by the definition (2.17) of $\mathscr{H}_X$. Then (2.25) shows that the linear subspace generated by $N_X\mathscr{K}$ is contained in $\mathscr{H}_X^\perp$. To prove the converse inclusion it suffices to check that any $U \in \mathscr{L}^2(\Omega)$ orthogonal to all elements of $\mathscr{K}$ is orthogonal to $\mathscr{H}_X^\perp$, and thus belongs to $\mathscr{H}_X$. This is true since, if $U$ is orthogonal to $N_X\mathscr{K}$, then both $U$ and $-U$ belongs to the polar cone to $N_X\mathscr{K}$ which is $T_X\mathscr{K}$. From (2.23) we then deduce that $U \in \mathscr{H}_X$. Concerning (2.28) we notice that all $U \in \mathscr{H}_X$ can be written as $U = u \circ X$ for some map $u \in \mathscr{L}^2(\mathbb{R},\varrho)$, where $\varrho = X_\#\mathfrak{m}$. Approximating $u$ in $L^2(\mathbb{R},\varrho)$ by functions $\varphi \in \mathrm{C}_b(\mathbb{R})$ we obtain (2.28). □

**Lemma 2.6.** *For any $X \in \mathscr{K}$ and $U \in T_X\mathscr{K}$ we have that*

$$(\mathsf{P}_{\mathscr{H}_X} - \mathrm{id})U \in N_X\mathscr{K}. \tag{2.29}$$

*Proof.* Lemma 3.11 in [18] shows that (2.29) holds if $U \in \mathscr{K}$. Since $\mathsf{P}_{\mathscr{H}_X}X - X = 0$, we obtain (2.29) for $U - X$ and, since $N_X\mathscr{K}$ is a cone, for arbitrary $\theta(U-X)$, $\theta \geq 0$ and $U \in \mathscr{K}$. We now conclude recalling (2.20). □

*Remark* 2.7. If $\mathscr{C}$ is a closed convex subset of $\mathscr{L}^2(\Omega)$ and $X \in L^1((0,T);X)$ with $X(t) \in \mathscr{C}$ for a.e. $t \in (0,T)$ then a simple application of Jensen's inequality yields

$$⨍_0^T X(t)dt \in \mathscr{C}; \quad \text{moreover} \quad \int_0^T X(t)dt \in \mathscr{C} \quad \text{if } \mathscr{C} \text{ is a cone.} \tag{2.30}$$

2.3. **Convex Functions.** In this section we recall some auxiliary results on convex functions. We are interested in functions $\psi \colon \mathbb{R} \longrightarrow [0,\infty)$ that are

$$\text{even, convex, of class } \mathrm{C}^1(\mathbb{R}), \text{ with } \psi(0) = 0, \tag{2.31}$$

and for which the homogeneous *doubling condition* holds:

$$\text{there exists } q \geqslant 1 \text{ such that } \psi(\lambda r) \leqslant \lambda^q \psi(r) \text{ for all } r \in \mathbb{R}, \lambda \geqslant 1. \tag{2.32}$$

Notice that if condition (2.32) holds for $\psi$, then it also holds for the map $r \mapsto \psi^p(r)$, $p \geq 1$, with exponent $pq$. Combining (2.31) and (2.32), we obtain the inequality

$$\psi(r_1 + r_2) \leqslant 2^{q-1}(\psi(r_1) + \psi(r_2)) \quad \text{for all } r_1, r_2 \in \mathbb{R}. \tag{2.33}$$

We will denote by $\Psi\colon \mathscr{L}^1(\Omega) \longrightarrow [0,\infty]$ the associated convex functional

$$\Psi[X] := \int_\Omega \psi(X(m))\,\mathrm{d}m \quad \text{for all } X \in \mathscr{L}^1(\Omega). \tag{2.34}$$

**Lemma 2.8.** *Suppose $\psi\colon \mathbb{R} \longrightarrow [0,\infty)$ satisfies* (2.31)*. Then the doubling condition* (2.32) *holds if and only if $\psi$ has one of the following, equivalent properties:*

$$r\psi'(r) \leqslant q\psi(r) \text{ for all } r > 0; \tag{2.35}$$

$$\text{there exists } C \geqslant 0 \text{ such that } \psi(2r) \leqslant C\psi(r) \text{ for all } r > 0. \tag{2.36}$$

*Proof.* Property (2.36) is a consequence of (2.32), and (2.35) follows from

$$r\psi'(r) = \lim_{\lambda\to 1+} \frac{\psi(\lambda r) - \psi(r)}{\lambda - 1} \leqslant \lim_{\lambda\to 1+} \frac{\lambda^q - 1}{\lambda - 1}\psi(r) = q\psi(r)$$

for all $r > 0$ and $q > 1$. To prove the converse statement, we notice first that since $\psi$ is an even, smooth function, we have that $\psi'(0) = 0$ and so $\psi$ is nonnegative and nondecreasing for all $r > 0$, by convexity. Moreover, if (2.36) holds, then again by convexity we find

$$\psi'(r) \leqslant \frac{\psi(2r) - \psi(r)}{r} \leqslant (C-1)\frac{\psi(r)}{r} \quad \text{for all } r > 0.$$

Thus (2.35) holds with $q := C - 1$, which must not only be a nonnegative number but cannot be smaller than 1. Assuming now that (2.35) is true, we consider the Cauchy problem

$$\eta'(s) = q\frac{\eta(s)}{s} \quad \text{for } s \in [r,\infty), \text{ with } \eta(r) = \psi(r), \tag{2.37}$$

which admits a unique solution $\eta(s) = \psi(r)(r^{-1}s)^q$ for all $s \geqslant r > 0$. A standard comparison estimate for solutions of ordinary differential equation yields

$$\psi(s) \leqslant \eta(s) = \Big(\frac{s}{r}\Big)^q \psi(r) \quad \text{for all } s \geqslant r > 0. \tag{2.38}$$

Since $\psi$ is nondecreasing, we conclude that $q \geqslant 1$ and then (2.32) follows for $r > 0$. By evenness of $\psi$ and since $\psi(0) = 0$, the inequality extends to $r \leqslant 0$ as well. □

**Lemma 2.9.** *Let $p \in [1,\infty)$ be given and suppose that the function $\eta\colon \mathbb{R} \longrightarrow [0,\infty)$ satisfies* (2.31) *and the $p$-coercivity condition*

$$0 < \liminf_{r\to 0+} \frac{\eta(r)}{r^p} \quad \textit{and} \quad \lim_{r\to\infty} \frac{\eta(r)}{r^p} = \infty. \tag{2.39}$$

*For every $q > p$, there exists a map $\psi\colon \mathbb{R} \longrightarrow [0,\infty)$ satisfying* (2.31)/(2.32) *with*

$$\psi(r) \leqslant \eta(r) \text{ for all } r \in \mathbb{R} \quad \textit{and} \quad \lim_{r\to\infty} \frac{\psi(r)}{r^p} = \infty. \tag{2.40}$$

*Proof.* By [20, Lemma 3.7] it is not restrictive to assume that $\eta$ is of the form $\bar\eta^2$ for a suitable convex function $\bar\eta$ with superlinear growth, and so we may just consider the case $p = 1$ (see the remark following (2.32)). By convolution, we can assume that $\eta$ is smooth in the open interval $(0,\infty)$, with $\delta := \inf_{r>0} \eta'(r) > 0$.

We then choose $q > 1$ and we set $\psi(r) := \delta r^q/q$ for all $r \in [0,1]$, so that

$$\psi(r) \leqslant \eta(r) \text{ in } [0,1] \quad \text{and} \quad \psi'(1) = \delta \leqslant \eta'(1).$$

For $r \geqslant 1$ we define $\psi$ to be the solution of the Cauchy problem

$$\psi'(r) = \min\Big\{\eta'(r), q\frac{\psi(r)}{r}\Big\} \quad \text{for } r \in [1,\infty), \text{ with } \psi(1) = \delta/q. \tag{2.41}$$

Then $\psi(r) \leqslant \eta(r)$ for all $r \geqslant 0$ and $\psi$ satisfies (2.35) of the previous lemma.

To prove that $\psi$ also satisfies (2.31), notice first that $\psi'(0) = 0$ since $q > 1$, and that $\psi'$ is continuous at $r = 1$. Hence $\psi$ can be extended to an even $\mathrm{C}^1(\mathbb{R})$-function. In order to check that $\psi'$ is nondecreasing, let us first observe that if a continuous function $\beta$ is nondecreasing in each connected component of an open set $A \subset \mathbb{R}$ that is dense in $[1,+\infty)$, then $\beta$ is nondecreasing in $[1,+\infty)$. We apply this observation to $\beta := \psi'$ and we set $A := A_0 \cup A_1$, where

$$A_0 := \Big\{ r \in (1,\infty) \colon q\frac{\psi(r)}{r} < \eta'(r) \Big\},$$
$$A_1 := \text{interior of } \Big\{ r \in (1,\infty) \colon \eta'(r) \leqslant q\frac{\psi(r)}{r} \Big\}.$$

In each connected component of $A_0$, the function $\psi$ solves the differential equation $\psi'(r) = q\psi(r)/r$, and so $\psi$ is of the form $cr^q$ for some constant $c > 0$. Therefore $\psi'$ is nondecreasing in $A_0$. On the other hand, in each connected component of $A_1$, we have that $\psi'(r) = \eta'(r)$ and $\eta'$ is nondecreasing, by assumption. Finally, notice that $\psi$ is nondecreasing on the interval $[0,1]$ since $\psi(r) = \delta r^q/q$ there. We can now apply Lemma 2.8 to conclude that $\psi$ has the doubling property (2.32).

It only remains to prove the second statement in (2.40). Since $\eta$ has superlinear growth, its derivative $\eta'(r) \longrightarrow \infty$ as $r \to \infty$. Assume now that $\psi(r)/r$ remains bounded as $r \to \infty$. Then there exists a number $r_1 \geqslant 1$ such that

$$\psi'(r) = q\frac{\psi(r)}{r} \quad \text{for all } r \in [r_1,\infty);$$

see (2.41). But this implies that $\psi'(r) = cr^q + c_0$ for all $r \in [r_1,\infty)$ and suitable constant $c > 0$ and therefore is unbounded as $r \to \infty$. This is a contradiction. □

**Lemma 2.10** (Compactness in $\mathscr{K}$)**.** *Let $\Psi$ be the integral functional defined in* (2.34) *corresponding to an even, convex function $\psi \colon \mathbb{R} \longrightarrow [0,\infty)$ with*

$$\lim_{|r|\to\infty} \frac{\psi(r)}{|r|^2} = \infty. \tag{2.42}$$

*Then the sublevels $\mathscr{K}(\Psi,\alpha) := \{X \in \mathscr{K} : \Psi[X] \leqslant \alpha\}$ of $\Psi$ is compact in $\mathscr{L}^2(\Omega)$ for all $\alpha \geq 0$.*

*Proof.* Because of (2.42), the $\mathscr{L}^2(\Omega)$-norm of elements of $\mathscr{K}(\Psi,\alpha)$ is bounded by some constant $A$ that depends on $\alpha$ and $\psi$ only. By monotonicity, we find that

$$X(w) \leqslant \frac{1}{1-w}\int_w^1 X(m)\,\mathrm{d}m \leqslant \Big(\frac{1}{1-w}\int_w^1 |X(m)|^2\,\mathrm{d}m\Big)^{1/2} \leqslant \frac{A}{(1-w)^{1/2}} \quad \text{for all } w \in \Omega,$$

for all $X \in \mathscr{K}(\Psi,\alpha)$. Analogously, we obtain a lower bound

$$X(w) \geqslant -\frac{A}{w^{1/2}} \quad \text{for all } w \in \Omega.$$

Any sequence $\{X_n\}$ in $\mathscr{K}(\Psi,\alpha)$ is therefore uniformly bounded in each interval $[\delta, 1-\delta]$ where $\delta > 0$. Applying Helly's theorem and a standard diagonal argument we can find a subsequence (still denoted by $\{X_n\}$) that converges pointwise to an element $X \in \mathscr{K}$. Since $\psi$ satisfies (2.42), the sequence $\{|X_n - X|^2\}$ is uniformly integrable and thus $X_n \longrightarrow X$ in $\mathscr{L}^2(\Omega)$. □

## 3. Lagrangian solutions

As explained in the Introduction, when studying system (1.1), one is lead to consider solutions to the Cauchy problem for the first-order differential inclusion in $\mathscr{L}^2(\Omega)$

$$\dot X(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar V + \int_0^t F[X(s)]\,\mathrm{d}s \quad \text{for a.e. } t \geqslant 0, \quad X(0) = \lim_{t\downarrow 0} X(t) = \bar X, \tag{3.1}$$

and, possibly, satisfying further properties.

Before discussing (3.1), we will state below the precise assumptions on the force operator $F$; examples, covering the case of (1.2) or (1.3), are detailed in Section 6.

3.1. **The force operator** $F$**.** Let us first recall the link of the map $F\colon \mathscr{K} \to \mathscr{L}^2(\Omega)$ with the force distribution $f\colon \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ in (1.1): as in (1.35) we will assume that

$$\int_{\mathbb{R}} \psi(x)\, f[\varrho](\mathrm{d}x) = \int_{\Omega} \psi(X_\varrho(m))F[X](m)\,\mathrm{d}m \quad \text{for all } \psi \in \mathscr{D}(\mathbb{R}) \text{ and } \varrho \in \mathscr{P}(\mathbb{R}), \tag{3.2}$$

where $X_\varrho \in \mathscr{K}$ and $(X_\varrho)_\#\mathfrak{m} = \varrho$. Recalling (2.28) one immediately sees that $F[X]$ is uniquely characterized by (3.2) only when $\mathscr{H}_X^\perp = \{0\}$ or, equivalently, when $\mathscr{H}_X = \mathscr{L}^2(\Omega)$, i.e., $\Omega_X = \varnothing$. Tthis is precisely the case when $X$ is (essentially) strictly increasing.

One could, of course, always take the orthogonal projection of $F[X]$ onto $\mathscr{H}_X$ in order to characterize it starting from (3.2). This procedure, however, could lead to a discontinuous operator which would be hard to treat by the theory of first order differential inclusions. This happens, e.g., for the (attractive or repulsive) Euler-Poisson system. We thus prefer to allow for a greater flexibility in the choice of $F$ complying with (3.2), asking that it is everywhere defined on $\mathscr{K}$ and satisfies suitable boundedness and continuity properties.

**Definition 3.1** (Boundedness)**.** An operator $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is *bounded* if there exists a constant $C \geqslant 0$ such that

$$\|F[X]\|_{\mathscr{L}^2(\Omega)} \leqslant C_2\Big(1 + \|X\|_{\mathscr{L}^2(\Omega)}\Big) \quad \text{for all } X \in \mathscr{K}. \tag{3.3}$$

We say that $F$ is pointwise linearly bounded if there exists a constant $C_{\mathsf{p}} \geqslant 0$ such that

$$\big|F[X](m)\big| \leqslant C_{\mathsf{p}}\Big(1 + |X(m)| + \|X\|_{\mathscr{L}^1(\Omega)}\Big) \quad \text{for a.e. } m \in \Omega \text{ and all } X \in \mathscr{K}. \tag{3.4}$$

Notice that if $F$ is pointwise linearly bounded, then $F$ is bounded and satisfies (3.3) with the constant $C_2 := 2C_{\mathsf{p}}$. Let us recall that a modulus of continuity is a concave continuous function $\omega\colon [0,\infty) \longrightarrow [0,\infty)$ with the property that $0 = \omega(0) < \omega(r)$ for all $r > 0$.

**Definition 3.2** (Uniform continuity)**.** We say that an operator $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is uniformly continuous if there exists a modulus of continuity $\omega$ with the property that

$$\big\|F[X_1] - F[X_2]\big\|_{\mathscr{L}^2(\Omega)} \leqslant \omega\Big(\|X_1 - X_2\|_{\mathscr{L}^2(\Omega)}\Big) \quad \text{for all } X_1, X_2 \in \mathscr{K}. \tag{3.5}$$

We say that $F$ is Lipschitz continuous if it is uniformly continuous and (3.5) holds with $\omega(r) = Lr$ for all $r \geqslant 0$, where $L \geqslant 0$ is some constant.

Notice that if $F$ is uniformly continuous then it is also bounded. Whenever a uniformly continuous $F$ is defined by (3.2) on the convex subset $\mathscr{K}_{si}$ of all the strictly increasing maps and satisfies (3.5) in $\mathscr{K}_{si}$, then it admits a unique extension to $\mathscr{K}$ preserving the continuity property (3.5) and the compatibility condition (3.2).

As we observed at the beginning of this section, a last property of $F$ which will play a crucial role concerns its behaviour on the subset $\Omega_X$ where the map $X$ is constant. Since the force functional determines the change in velocity, in the framework of sticky evolution it would be natural to assume $F[X] \in \mathscr{H}_X$ for every $X \in \mathscr{K}$. We shall see that a weaker property is still sufficient to preserve the sticky condition: it will turn particularly useful when the attractive Euler-Poisson equation will be considered.

**Definition 3.3** (Sticking)**.** The map $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is called sticking if

$$F[X] - \mathsf{P}_{\mathscr{H}_X}(F[X]) \in \partial I_{\mathscr{K}}(X) \quad \text{for all } X \in \mathscr{K}.$$

3.2. **Lagrangian Solutions.** Let us start by giving a suitable notion of solutions to (3.1).

**Definition 3.4** (Lagrangian solutions to the differential inclusion (3.1))**.** Let $F : \mathscr{K} \longrightarrow L^2(\Omega)$ be a uniformly continuous operator and let $\bar{X} \in \mathscr{K}$ and $\bar{V} \in \mathscr{H} = \mathscr{L}^2(\Omega)$ be given. A Lagrangian solution to (3.1) with initial data $(\bar{X}, \bar{V})$ is a curve $X \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty); \mathscr{K})$ satisfying $X(0) = \bar{X}$ and (3.1) for a.e. $t \in (0,\infty)$.

By introducing the new variable

$$Y(t) := \bar{V} + \int_0^t F[X(s)]\,\mathrm{d}s$$

we immediately see that (3.1) is equivalent to the evolution system

$$\begin{cases} \dot{X}(t) + \partial I_{\mathscr{K}}(X(t)) \ni Y(t), \\ \qquad\qquad \dot{Y}(t) = F[X(t)], \end{cases} \quad \text{for a.e. } t \geqslant 0, \quad (X(0), Y(0)) = (\bar{X}, \bar{V}), \tag{3.6}$$

Notice that the continuity of $F$ yields $Y \in \mathrm{C}^1([0,\infty); \mathscr{L}^2(\Omega))$.

**Theorem 3.5.** *Let $F : \mathscr{L}^2(\Omega) \to \mathscr{K}$ be a uniformly continuous operator and let $(X, Y)$ be a Lagrangian solution to* (3.6)*. Then the following properties hold:*

- Right-Derivative:

$$\textit{The right-derivative } V := \frac{\mathrm{d}^+}{\mathrm{d}t} X \textit{ exists for all } t \geqslant 0. \tag{3.7}$$

- Minimal Selection:

$$V(t) = \Big( Y(t) - \partial I_{\mathscr{K}}(X(t)) \Big)^{\circ} \quad \textit{for all } t \geqslant 0, \tag{3.8}$$

  *where $A^\circ$ denotes the unique element of minimal norm in a closed convex set $A \subset \mathscr{L}^2(\Omega)$. In particular, if we replace $\dot{X}(t)$ by $V(t)$ then* (3.1) *and* (3.6) *hold for all $t \geqslant 0$.*
- Projection on the tangent cone:

$$V(t) = \mathsf{P}_{T_{X(t)}\mathscr{K}}\big(Y(t)\big) \quad \textit{for all } t \geqslant 0. \tag{3.9}$$

- Continuity of the velocity:

$$V \textit{ is right-continuous for all } t \geqslant 0; \tag{3.10}$$

  *in particular, we have*

$$\lim_{t \downarrow 0} V(t) = \bar{V} \quad \textit{if and only if} \quad \bar{V} \in T_{\bar{X}}\mathscr{K}. \tag{3.11}$$

  *If $\mathcal{T}^0 \subset (0,\infty)$ is the subset of all times at which the map $s \mapsto \|V(s)\|_{\mathscr{L}^2(\Omega)}$ is continuous, then $(0,\infty) \setminus \mathcal{T}^0$ is negligible and $V$ is continuous, $X$ is differentiable in $\mathscr{L}^2(\Omega)$ at every point of $\mathcal{T}^0$. Setting $\varrho_t := X(t)_\# \mathfrak{m}$ there exists a unique map $v_t \in L^2(\mathbb{R}, \varrho_t)$ such that*

$$\dot{X}(t) = V(t) = \mathsf{P}_{\mathscr{H}_{X(t)}}(Y(t)) = v_t \circ X_t \in \mathscr{H}_{X(t)} \quad \textit{for every } t \in \mathcal{T}^0. \tag{3.12}$$

- Solution to (1.1): *If moreover $F$ is linked to $f$ by* (3.2)*, $\bar{\varrho} = \bar{X}_\# \boldsymbol{m}$ and $\bar{V} = \bar{v} \circ \bar{X}$, then the couple $(\varrho, v)$ defined as above is a distributional solution to* (1.1) *such that*

$$\lim_{t \downarrow 0} \varrho(t,\cdot) = \bar{\varrho} \textit{ in } \mathscr{P}_2(\mathbb{R}), \quad \lim_{t \downarrow 0} \varrho(t,\cdot) v(t,\cdot) = \bar{\varrho}\,\bar{v} \textit{ in } \mathscr{M}(\mathbb{R}). \tag{3.13}$$

*Proof.* (3.7), (3.8), and (3.10) are consequence of the general theory of [9], Theorem 3.5; (3.9) follows immediately by (3.8) since $V(t) \in T_{X(t)}\mathscr{K}$ by (3.7) and $V(t) + N_{X(t)}\mathscr{K} \ni Y(t)$.

Concerning (3.12) we can apply the Remark 3.9 (but see also Remark 3.4) of [9], which shows that at each differentiability point $t$ of $X$, its derivative is the projection of 0 onto the affine space generated by $Y(t) - \partial I_{\mathscr{K}}(X(t))$, i.e., the orthogonal projection of $Y(t)$ onto the orthogonal complement of the space generated by $\partial I_{\mathscr{K}}(X(t))$. Recalling Lemma 2.5 we get (3.12).

To prove the last statement, we use the crucial information of (3.12) that $V(t) \in \mathscr{H}_{X(t)}$ for a.e. $t > 0$, a fact that may have been noticed for the first time in [12]. The projected velocities

$$V^*(t) = P_{\mathscr{H}_{X(t)}} V(t) \tag{3.14}$$

coincide with $V(t)$ for every $t \in \mathcal{T}^0$, where $\mathcal{T}^0$ is a set of full measure in $(0,\infty)$. Since any element $V$ of $\mathscr{H}_{X(t)}$ can be written as $v \circ X$ for a suitable Borel map $v \in \mathscr{L}^2(\Omega)$ we deduce that there exists a Borel map $v : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that $v(t,\cdot) \in \mathscr{L}^2(\mathbb{R}, \varrho(t,\cdot))$ and

$$V^*(t,\cdot) = v(t, X(t,\cdot)) \quad \text{for a.e. } x \in \Omega \text{ and all } t \geqslant 0. \tag{3.15}$$

From Equation (3.14) we also have $V(t) = v(t, X(t))$ for $t \in \mathcal{T}^0$.

We then argue as follows: For all test functions $\varphi \in \mathscr{D}((0,T)\times\mathbb{R})$ we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \Big( \partial_t\varphi(t,x)v(t,x) + \partial_x\varphi(t,x)v^2(t,x) \Big)\, \varrho(t,dx)\,\mathrm{d}t \\
&\quad = \int_0^\infty \int_\Omega \Big( \partial_t\varphi(t,X(t,m)) + \partial_x\varphi(t,X(t,m))v(t,X(t,m)) \Big) V(t,m)\,\mathrm{d}m\,\mathrm{d}t \\
&\quad = \int_0^\infty \int_\Omega \Big( \partial_t\varphi(t,X(t,m)) + \partial_x\varphi(t,X(t,m))v(t,X(t,m)) \Big) Y(t,m)\,\mathrm{d}m\,\mathrm{d}t \\
&\quad = \int_0^\infty \int_\Omega \Big( \frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t,X(t,m))] \Big) Y(t,m)\,\mathrm{d}m\,\mathrm{d}t \\
&\quad = -\int_0^\infty \int_\Omega \varphi(t,X(t,m))F[X(t,\cdot)](m)\,\mathrm{d}m\,\mathrm{d}t
\end{aligned} \tag{3.16}$$

Applying formula (3.2) in (3.16), we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \Big( \partial_t\varphi(t,x)v(t,x) + \partial_x\varphi(t,x)v^2(t,x) \Big)\, \varrho(t,dx)\,\mathrm{d}t \\
&\quad = -\int_0^1 \int_{\mathbb{R}} \varphi(t,x)\, f[\varrho(t,\cdot)](\mathrm{d}x)\,\mathrm{d}t,
\end{aligned} \tag{3.17}$$

which yields the momentum equation in (1.1) in distributional sense. An (even easier) analogous argument holds for the continuity equation. This shows that the pair $(\varrho, v)$ defined by (3.14) and (3.15) is a solution of (1.1). The first limit of (3.13) follows since $\lim_{t\downarrow 0} X(t) = \bar{X}$ in $\mathscr{L}^2(\Omega)$ and $\bar{X}_\#\mathfrak{m} =: \bar{\varrho}$. Concerning the second limit of (3.13), we have to show that

$$\int_\Omega \varphi\bar{v}(x)\,\bar{\varrho}(\mathrm{d}x) = \lim_{t\to 0}\int_\Omega \varphi v(t,x)\,\varrho(t,dx) \quad \text{for every } \varphi \in \mathrm{C}_b(\mathbb{R}). \tag{3.18}$$

Since $\bar{V} = \bar{v}\circ\bar{X}$ we have $\bar{V} \in \mathscr{H}_{\bar{X}} \subset T_{\bar{X}}\mathscr{K}$, so that $\lim_{t\downarrow 0} V(t) = \bar{V}$ in $\mathscr{L}^2(\Omega)$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\bar{v}(x)\,\bar{\varrho}(\mathrm{d}x) &= \int_\Omega \varphi(\bar{X})\bar{V}\,\mathrm{d}m = \lim_{t\downarrow 0}\int \varphi(X(t))V(t)\,\mathrm{d}m \\
&= \lim_{t\downarrow 0}\int \varphi(X(t))V^*(t)\,\mathrm{d}m = \lim_{t\downarrow 0}\int \varphi(x)v(t,x)\,\varrho(t,dx),
\end{aligned}$$

where we used the fact that $V(t) - V^*(t)$ is perpendicular to $\mathscr{H}_{X(t)}$. □

As we already observed in the previous proof, notice that (3.11) surely holds if $\bar{V} \in \mathscr{H}_{\bar{X}}$.

3.3. **Existence, uniqueness, and stability of Lagrangian solutions for Lipschitz forces.** Applying the general results of [9], we can now prove the following result.

**Theorem 3.6.** *Suppose that $F : \mathscr{K} \to \mathscr{L}^2(\Omega)$ is Lipschitz. Then for every $(\bar{X},\bar{V}) \in \mathscr{K}\times\mathscr{L}^2(\Omega)$ there exists a unique Lagrangian solution $X$ to (3.1), and for every $T \geqslant 0$ there exists a constant $C_T \geqslant 0$ independent of the initial data such that for every $t\in[0,T]$ we have*

$$\|X(t)\|_{\mathscr{L}^2(\Omega)} + \|V(t)\|_{\mathscr{L}^2(\Omega)} \leqslant C_T\Big(1 + \|\bar{X}\|_{\mathscr{L}^2(\Omega)} + \|\bar{V}\|_{\mathscr{L}^2(\Omega)}\Big). \tag{3.19}$$

*Moreover, for any $T \geqslant 0$ there exists a constant $C_T \geqslant 0$ with the following property: For any pair of Lagrangian solutions $X_i$ with initial data $(\bar{X}_i,\bar{V}_i)$, $i = 1,2$, for all $t\in[0,T]$ we have*

$$\|X_1(t) - X_2(t)\|_{\mathscr{L}^2(\Omega)} \leqslant C_T\Big(\|\bar{X}_1 - \bar{X}_2\|_{\mathscr{L}^2(\Omega)} + \|\bar{V}_1 - \bar{V}_2\|_{\mathscr{L}^2(\Omega)}\Big) \tag{3.20}$$

$$\begin{aligned}
\int_0^T \|V_1(t) - V_2(t)\|^2_{\mathscr{L}^2(\Omega)}\,\mathrm{d}t \leqslant C_T \sum_{i=1\ldots 2}&\Big(\|\bar{X}_i\|_{\mathscr{L}^2(\Omega)} + \|\bar{V}_i\|_{\mathscr{L}^2(\Omega)}\Big) \\
&\times\Big(\|\bar{X}_1 - \bar{X}_2\|_{\mathscr{L}^2(\Omega)} + \|\bar{V}_1 - \bar{V}_2\|_{\mathscr{L}^2(\Omega)}\Big).
\end{aligned} \tag{3.21}$$

*Proof.* Recalling the formulation (3.6), we introduce the Hilbert space $H := \mathscr{L}^2(\Omega) \times \mathscr{L}^2(\Omega)$ and the (multivalued) operator $A(X,Y) := (\partial I_{\mathscr{K}}(X) - Y, F[X])$. It is easy to check that $A$ is a Lipschitz perturbation of the subdifferential of the proper, convex, and l.s.c. functional $\Phi(X,Y) := I_{\mathscr{K}}(X)$. Thus existence, uniqueness, and the estimates (3.19), (3.20) follow by [9, Theorem 3.17].

The same estimate also yields for the second component $Y_i(t) = \bar V_i + \int_0^t F[X_i(s)]\,\mathrm{d}s$ that

$$\|Y_1(t) - Y_2(t)\|_{\mathscr{L}^2(\Omega)} \leqslant C_T\Big(\|\bar X_1 - \bar X_2\|_{\mathscr{L}^2(\Omega)} + \|\bar V_1 - \bar V_2\|_{\mathscr{L}^2(\Omega)}\Big) \tag{3.22}$$

and, by the boundedness of $F$, that

$$\int_0^T \|\dot Y_i(t)\|^2_{\mathscr{L}^2(\Omega)}\,\mathrm{d}t \leqslant C_T^2\Big(1 + \|\bar X_i\|^2_{\mathscr{L}^2(\Omega)} + \|\bar V_i\|^2_{\mathscr{L}^2(\Omega)}\Big).$$

Applying Theorem 2 in [21] to the first equation of (3.6), we obtain (3.21). □

A straightforward application of the previous theorem reveals that Lagrangian solutions are stable if the operator $F$ is Lipschitz continuous: a sequence of Lagrangian solutions with strongly converging initial data converges to another Lagrangian solution.

3.4. **Sticky Lagrangian solutions and the semigroup property.** We consider here an important class of Lagrangian solutions.

**Definition 3.7** (Sticky Lagrangian solutions)**.** We say that a Lagrangian solution $X$ is *sticky* if

$$\text{for any } t_1 \leqslant t_2 \text{ we have } \Omega_{X(t_1)} \subset \Omega_{X(t_2)}. \tag{3.23}$$

By (2.19) and (2.24), any sticky Lagrangian solution satisfies the monotonicity condition

$$\partial I_{\mathscr{K}}(X(t_1)) \subset \partial I_{\mathscr{K}}(X(t_2)), \quad \mathscr{H}_{X(t_2)} \subset \mathscr{H}_{X(t_1)} \quad \text{for any } t_1 \leqslant t_2. \tag{3.24}$$

**Proposition 3.8** (Projection formula)**.** *If $X$ is a sticky Lagrangian solution, then*

$$V(t) \in \mathscr{H}_{X(t)} \quad \textit{for all times } t \geqslant 0 \tag{3.25}$$

*and $(X,V)$ satisfy*

$$X(t) = \mathsf{P}_{\mathscr{K}}\Big(\bar X + \int_0^t Y(s)\,\mathrm{d}s\Big) = \mathsf{P}_{\mathscr{K}}\Big(\bar X + t\bar V + \int_0^t (t-s)F[X(s)]\,\mathrm{d}s\Big), \tag{3.26}$$

$$V(t) = \mathsf{P}_{\mathscr{H}_{X(t)}}\big(Y(t)\big) = \mathsf{P}_{\mathscr{H}_{X(t)}}\Big(\bar V + \int_0^t F[X(s)]\,\mathrm{d}s\Big). \tag{3.27}$$

*Proof.* Property (3.25) follows easily from (3.10) and (3.12), thanks to the monotonicity property (3.24). Equation (3.27) then follows from (3.6) (where $\dot X$ is replaced by $V$) and (2.25).

In order to prove (3.26), for any $s \geqslant 0$ we observe that

$$\Xi(s) := Y(s) - V(s) \in \partial I_{\mathscr{K}}(X(s)) \subset \partial I_{\mathscr{K}}(X(t)) \quad \text{if } t \geqslant s, \tag{3.28}$$

and we integrate (3.28) w.r.t. $s$ from 0 to $t$ to obtain

$$\int_0^t \Xi(s)\,\mathrm{d}s = \bar X - X(t) + \int_0^t Y(s)\,\mathrm{d}s \in \partial I_{\mathscr{K}}(X(t))$$

for a.e. $t \geqslant 0$. Recalling (2.12), we get (3.26). □

**Lemma 3.9** (Concatenation property)**.** *Consider Lagrangian solutions $X_1, X_2$ with initial data $(\bar X_1, \bar V_1)$ and $(\bar X_2, \bar V_2)$ respectively, and let us suppose that*

$$\Omega_{\bar X_2} \subset \Omega_{X_2(t)} \quad \textit{for every } t \geq 0. \tag{3.29}$$

*If for some $\tau > 0$ we have*

$$\bar X_2 = X_1(\tau), \quad Y_1(\tau) - \bar V_2 = \bar\Xi_2 \in \partial I_{\mathscr{K}}(\bar X_2), \tag{3.30}$$

*then the curve*

$$\tilde X := \begin{cases} X_1(t) & \textit{if } 0 \leqslant t \leqslant \tau, \\ X_2(t-\tau) & \textit{if } t \geqslant \tau, \end{cases} \tag{3.31}$$

*is a Lagrangian solution with initial data $(\bar X_1, \bar V_1)$. In particular, if $X_1, X_2$ are sticky Lagrangian solutions, then $\tilde X$ is also sticky.*

Notice that

$$\text{the choice } \bar V_2 := V(\tau_1) \text{ always satisfies (3.30).} \tag{3.32}$$

*Proof.* It is easy to check that

$$\tilde Y(t) := \begin{cases} Y_1(t) & \text{if } 0 \leqslant t \leqslant \tau, \\ Y_1(\tau) - \bar V_2 + Y_2(t-\tau) & \text{if } t \geqslant \tau, \end{cases}$$

is Lipschitz continuous and satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\tilde Y(t) = F[\tilde X(t)]$ for a.e. $t \in (0,\infty)$. We must check that $\tilde X$ satisfies the first differential inclusion of (3.6) for $t \geqslant \tau$ w.r.t. $\tilde Y$. By definition of $\tilde X$, we have

$$\begin{aligned} \tilde Y(t) - \frac{\mathrm{d}}{\mathrm{d}t}\tilde X(t) &= Y_1(\tau) - \bar V_2 + Y_2(t-\tau) - V_2(t-\tau) \\ &= \bar\Xi_2 + \Xi_2(t-\tau) \in \partial I_{\mathscr K}(X_2(t-\tau)) = \partial I_{\mathscr K}(\tilde X(t)) \end{aligned}$$

for $t \geqslant \tau$ since by (3.30) and (3.29) it holds that $\bar\Xi_2 \in \partial I_{\mathscr K}(X_2(t-\tau))$. □

It would not be difficult to show that Lagrangian solutions in general do not have the sticky nor the concatenation property. The next result shows that these properties are related.

**Theorem 3.10** (Semigroup property)**.** *If the force operator $F$ is Lipschitz and*

$$\Omega_{\bar X} \subset \Omega_{X(t)} \quad \textit{for every Lagrangian solution } X \textit{ starting from } (\bar X, \bar V) \in \mathscr K \times \mathscr H_{\bar X}, \tag{3.33}$$

*then every Lagrangian solution $X$ with initial data $(\bar X, \bar V) \in \mathscr K \times \mathscr H_{\bar X}$ is sticky and satisfies the following semigroup property: for every $\tau > 0$ the curve $\tilde X(t) := X(t-\tau)$ is the unique Lagrangian solution with initial data $(X(\tau), V(\tau))$. In particular, for all $t \geqslant t_1 \geq 0$ we have*

$$X(t) = \mathsf P_{\mathscr K}\bigg(X(t_1) + (t-t_1)V(t_1) + \int_{t_1}^t (t-s)F[X(s)]\,\mathrm{d}s\bigg) \tag{3.34}$$

$$V(t) = \mathsf P_{\mathscr H_{X(t)}}\bigg(V(t_1) + \int_{t_1}^t F[X(s)]\,\mathrm{d}s\bigg). \tag{3.35}$$

*Proof.* Let $\mathcal T^0 \in [0,\infty)$ be as in (3.12) (so that $(0,\infty)\setminus\mathcal T^0$ is negligible). For every $\tau \in \mathcal T^0$ consider the Lagrangian solution $X_2$ with initial datum $(X(\tau), V(\tau))$. By the concatenation property (with the choice (3.32)), the map $\tilde X$ defined as in (3.31) (with $X_1 := X$) is a Lagrangian solution and therefore coincides with $X$, since $F$ is Lipschitz. Then (3.33) yields that

$$\Omega_{X(\tau)} \subset \Omega_{X(t)} \quad \text{for every } 0 \leqslant \tau < t, \quad \tau \in \mathcal T^0 \cup \{0\}. \tag{3.36}$$

Let us now fix $s > 0$ and consider a sequence $h_n \downarrow 0$ such that

$$\frac{1}{h_n}\big(X(s) - X(s-h_n)\big) \rightharpoonup V_- \quad \text{in } L^2(\Omega).$$

Since $T_{X(s)}\mathscr K$ is a closed convex cone, it is also weakly closed, so that by its very definition we have $-V_- \in T_{X(s)}\mathscr K$. We set $\Xi(t) := Y(t) - \dot X(t) \in \partial I_{\mathscr K}(X(t))$ thanks to the differential inclusion of (3.6). An integration in time from $s - h_n$ to $s$ and (3.36) then yields

$$⨍_{s-h_n}^s Y(r)\,\mathrm{d}r - \frac{1}{h_n}\big(X(s) - X(s-h_n)\big) = ⨍_{s-h_n}^s \Xi(r)\,\mathrm{d}r \in \partial I_{\mathscr K}(X(s)).$$

Passing to the limit as $n \to \infty$, we obtain

$$\Xi_- := Y(s) - V_- \in \partial I_{\mathscr K}(X(s)).$$

Therefore, by (2.25), we have

$$\bar V := \mathsf P_{X_{X(s)}}(V_-) = \mathsf P_{X_{X(s)}}(Y(s)).$$

Since

$$Y(s) - \bar V = Y - V_- + \big(-\bar V - (-V_-)\big) = \Xi_- + \big(-\bar V - (-V_-)\big) \in \partial I_{\mathscr K}(X(s))$$

by (2.29) and the fact that $-V_- \in T_{X(s)}\mathscr{K}$, we can apply the concatenation property as before, joining at the time $s$ the Lagrangian solution $X_1 := X$ with the Lagrangian solution $X_2$ arising from the initial data $\bar{X} := X(s)$ and $\bar{V}$. The uniqueness theorem shows that this map coincides with $X$ and therefore (3.29) yields $\Omega_{X(s)} \subset \Omega_{X(t)}$ for every $t > s$. In particular, we have $V(t) \in X_{X(t)}$ for every $t \geq 0$ so that a further application of the concatenation Lemma 3.9 yields the semigroup property. (3.34) and (3.35) follow then by the corresponding (3.26) and (3.27). □

We conclude this section with our main result concerning the existence of sticky Lagrangian solution. The proof will require a careful analysis of the discrete particle models and therefore will be postponed at the end of Section 5; see Remark 5.4.

**Theorem 3.11** (Sticking forces yields sticky Lagrangian solutions)**.** *If the force operator $F$ is Lipschitz continuous and sticking (according to Definition 3.3), then every Lagrangian solution to* (3.1) *with initial data $\bar{X} \in \mathscr{K}$ and $\bar{V} \in \mathscr{H}_{\bar{X}}$ is sticky.*

3.5. **Lagrangian solutions for continuous force fields.** Using Schauder's fixed point theorem, it is possible to extend the existence result of Theorem 3.6 to the case of (uniformly) continuous force operators.

**Theorem 3.12.** *Suppose that $F : \mathscr{K} \to \mathscr{L}^2(\Omega)$ satisfies the pointwise linear bound* (3.4) *and is uniformly continuous according to* (3.5)*. Then, for every initial data $(\bar{X}, \bar{V}) \in \mathscr{K} \times \mathscr{L}^2(\Omega)$, there exists a Lagrangian solution $(X, Y)$ of* (3.6)*. For any $T \geqslant 0$ there exists a constant $C_T \geqslant 0$ such that any Lagrangian solution $X$ with velocity $V := \frac{\mathrm{d}^+}{\mathrm{d}t}X$ satisfies*

$$\|X(t)\|_{\mathscr{L}^2(\Omega)} + \|V(t)\|_{\mathscr{L}^2(\Omega)} \leqslant C_T\Big(1 + \|\bar{X}\|_{\mathscr{L}^2(\Omega)} + \|\bar{V}\|_{\mathscr{L}^2(\Omega)}\Big) \tag{3.37}$$

*for all $t \in [0, T]$. If $\psi\colon \mathbb{R} \longrightarrow [0, +\infty)$ is an integrand satisfying* (2.31) *and* (2.32) *for some $q \geqslant 1$, then there exists a constant $C_{q,T} \geqslant 0$ such that*

$$\Psi[X(t)] + \Psi[V(t)] \leqslant C_{q,T}\Big(1 + \Psi[\bar{X}] + \Psi[\bar{V}]\Big) \tag{3.38}$$

*for all $t \in [0, T]$, with functional $\Psi$ defined in* (2.34)*.*

We omit the details of the *proof* but we state the related uniform a priori bounds that will turn to be useful in view of the next applications.

**Lemma 3.13** (A priori bounds)**.** *Let $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ be pointwise linearly bounded with constant $C_{\mathsf{p}} > 0$ (see* (3.4)*)). Let $\psi\colon \mathbb{R} \longrightarrow [0, +\infty)$ be an integrand satisfying* (2.31), (2.32) *for some $q \geqslant 1$. Let $X \in \mathrm{Lip}(0, T; \mathscr{L}^2(\Omega))$ and $Z, W \in \mathscr{L}^\infty(0, T; \mathscr{L}^2(\Omega))$ be such that $X(0) = \bar{X}$ and*

$$V(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar{V} + \int_0^t Z(s)\,\mathrm{d}s \quad \textit{for all } t \in [0, T), \tag{3.39}$$

*where*

$$V(t) = \frac{\mathrm{d}^+ X}{\mathrm{d}t}(t) \quad \textit{and} \quad Z(s) \prec F[W(s)].$$

*For a.e. $t \in (0, T)$, we then have*

$$\Psi(V(t)) \leqslant \Psi\Big(\bar{V} + \int_0^t Z(s) ds\Big), \tag{3.40}$$

$$\Psi(V(t)) \leqslant 2^{q-1}\bigg(\Psi(\bar{V}) + (3t)^{q-1}C_{\mathsf{p}}^q\Big(\psi(1)t + 2\int_0^t \Psi(W(s)) ds\Big)\bigg), \tag{3.41}$$

$$\Psi(X(t)) \leqslant 2^{q-1}\bigg(\Psi(\bar{X}) + T^q\Lambda(T) + (6C_{\mathsf{p}}T^2)^q \int_0^t \Psi(W(s)) ds\bigg). \tag{3.42}$$

*Proof.* Recalling Theorem 3.5 and (3.12), equation (3.39) yields

$$V(t) = P_{\mathscr{H}_{X(t)}}\bigg(\bar{V} + \int_0^t Z(s) ds\bigg) \quad \text{for every } t \in \mathcal{T}^0,$$

where $\mathcal{T}^0$ has full measure in $(0,T)$. Hence, by Lemma 2.2, we have

$$\Psi(V(t)) = \Psi\Big(P_{\mathscr{H}_{X(t)}}\Big(\bar V + \int_0^t Z(s)\,ds\Big)\Big) \leqslant \Psi\Big(\bar V + \int_0^t Z(s)\,ds\Big),$$

which proves (3.40). Using (2.33) and Jensen's inequality, we obtain

$$\Psi(V(t)) \leqslant 2^{q-1}\Big(\Psi(\bar V) + \Psi\Big(\int_0^t Z(s)\,ds\Big)\Big) \leqslant 2^{q-1}\Big(\Psi(\bar V) + t^{q-1}\int_0^t \Psi(Z(s))\,ds\Big). \tag{3.43}$$

We use the fact that $F$ is linearly bounded, that $\psi$ is even, and $\psi(\|W\|_1) \leqslant \Psi(W)$, by Jensen's inequality, to obtain that for all $s \geqslant 0$ it holds

$$\Psi(Z(s)) \leqslant \Psi[F(W(s))] \leqslant \Psi[C_{\mathsf{p}}(1+|W(s)|+\|W\|_1)] \leqslant 3^{q-1}C_{\mathsf{p}}^q\Big(\psi(1)+2\Psi[W(s)]\Big). \tag{3.44}$$

The first inequality in (3.44) was obtained via Lemma 2.2. We now combine (3.43), (3.44) to get (3.41). From (3.41), we then obtain

$$\begin{aligned}
\int_0^t \Psi(V(s))\,ds &\leqslant 2^{q-1}\Big(t\Psi(\bar V) + (3T)^{q-1}C_{\mathsf{p}}^q\Big(\psi(1)\frac{t^2}{2} + 2\int_0^t\int_0^s \Psi(W(l))\,dl\Big)\Big)\\
&= \Lambda(t) + 2(6T)^{q-1}C_{\mathsf{p}}^q\int_0^t (t-l)\Psi(W(l))\,dl\\
&\leqslant \Lambda(t) + (6TC_{\mathsf{p}})^q\int_0^t \Psi(W(l))\,dl,
\end{aligned} \tag{3.45}$$

where

$$\Lambda(t) := 2^{q-1}\Big(t\Psi(\bar V) + (3T)^{q-1}C_{\mathsf{p}}^q\psi(1)\frac{t^2}{2}\Big).$$

We have

$$\Psi(X(t)) = \Psi\Big(\bar X + \int_0^t V(s)\,ds\Big) \leqslant 2^{q-1}\Big(\Psi(\bar X) + t^{q-1}\int_0^t \Psi(V(s))\,ds\Big),$$

where have used (2.33) and then Jensen's inequality. This, together with (3.45) yields (3.42). □

## 4. The semigroup property and generalized Lagrangian solutions

We have seen that Lagrangian solutions may fail to satisfy the semigroup property in the natural phase space for the variables $(X,V)$, $V=\dot X$ (stated in Proposition (3.10) for sticky Lagrangian solutions). In fact, the formulation given by the system (3.6) shows that the natural variables for the semigroup property are the couple $(X,Y)$. This motivates an alternate notion of solution (still linked to (1.1)) that tries to recover a mild semigroup property, at the price of losing uniqueness with respect to the initial data. Recall that for any $X\in\mathscr{K}$ the orthogonal projection $\mathsf{P}_{\mathscr{H}_X}$ onto the closed subspace $\mathscr{H}_X \subset \mathscr{L}^2(\Omega)$ leaves the given function unchanged in $\Omega\setminus\Omega_X$ and replaces it with its average in every maximal interval $(\alpha,\beta)\subset\Omega_X$; see (2.18). As a consequence, the function $\mathsf{P}_{\mathscr{H}_X}(F[X])$ is constant wherever $X$ is.

**Definition 4.1.** A *generalized solution* to (3.1) is a curve $X\in\mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathscr{K})$ such that

(1) *Differential inclusion:*

$$\dot X(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar V + \int_0^t Z(s)\,\mathrm{d}s \quad \text{for a.e. } t\in(0,\infty), \tag{4.1}$$

for some map $Z\in\mathscr{L}^\infty_{\mathrm{loc}}([0,\infty);\mathscr{L}^2(\Omega))$ with

$$Z - F[X(t)] \in \mathscr{H}_{X(t)}^\perp \quad \text{and} \quad Z \prec F[X(t)] \quad \text{for a.e. } t\in(0,\infty). \tag{4.2}$$

(2) *Semigroup property:* For all $t\geqslant t_1\geqslant 0$ the right derivative $V=\frac{\mathrm{d}^+}{\mathrm{d}t}X$ satisfies

$$V(t) + \partial I_{\mathscr{K}}(X(t)) \ni V(t_1) + \int_{t_1}^t Z(s)\,\mathrm{d}s. \tag{4.3}$$

(3) *Projection formula:* For all $t \geqslant t_1 \geqslant 0$ we have

$$X(t_2) = \mathsf{P}_{\mathscr{K}}\left(X(t_1) + (t_2 - t_1)V(t_1) + \int_{t_1}^{t_2} (t_2 - s)Z(s)\,\mathrm{d}s\right). \tag{4.4}$$

Note that for generalized Lagrangian solutions the semigroup property and (4.4) are part of the definition, while for sticky Lagrangian solutions (3.34) and (3.35) are consequences of the monotonicity property (3.23). The obvious choice in (4.2) is $Z(t) := F[X(t)]$ for all times $t \geqslant 0$, which also shows that any sticky Lagrangian solution is a generalized Lagrangian solution.

*Remark* 4.2. If one is ultimately interested only in the existence of solutions to the conservation law (1.1), then for this purpose any $Z$ satisfying (4.2) is sufficient. In fact, we proved in Theorem 3.5 that if the force functional $F[X]$ is induced by an Eulerian force field $f[\varrho]$, so that (3.2) holds whenever $X \in \mathscr{K}$ and $X_\#(\mathfrak{m}) = \varrho$, then any strong Lagrangian solution yields a solution of the conservation law (1.1). The same argument works for generalized Lagrangian solutions. Because of (4.2), we have that $\mathsf{P}_{\mathscr{H}_{X(t)}}(Z(t)) = \mathsf{P}_{\mathscr{H}_{X(t)}}(F[X(t)])$. On the other hand, it holds

$$\int_\Omega \varphi(X(m))F[X](m)\,\mathrm{d}m = \int_\Omega \varphi(X(m))\mathsf{P}_{\mathscr{H}_X}(F[X])(m)\,\mathrm{d}m$$

for all $\varphi \in \mathscr{D}(\mathbb{R})$, with a similar formula for $Z$ in place of $F[X]$. Then the argument on page 19 can be adapted to prove the claim; see in particular (3.17).

Since $X$ is everywhere right differentiable, we have $V(t) \in T_{X(t)}\mathscr{K}$ for all $t \geqslant 0$. Then

$$V(t) = \mathsf{P}_{T_{X(t)}\mathscr{K}}\left(V(t_1) + \int_{t_1}^{t_2} F[X(s)]\,\mathrm{d}s\right) \quad \text{for every } t \geqslant t_1 \geqslant 0 \tag{4.5}$$

because of (4.3). This also yields

$$V(t) = \mathsf{P}_{\mathscr{H}_{X(t)}}\left(V(t_1) + \int_{t_1}^{t_2} F[X(s)]\,\mathrm{d}s\right) \quad \text{for a.e. } t \geqslant t_1 \geqslant 0. \tag{4.6}$$

By introducing the new variable

$$Y(t) := \bar{V} + \int_0^t Z(s)\,\mathrm{d}s,$$

we easily see that (4.1) is equivalent to the evolution system

$$\begin{cases} \dot{X}(t) + \partial I_{\mathscr{K}}(X(t)) \ni Y(t), \\ \dot{Y}(t) = Z(t), \\ Z(t) - F[X(t)] \in \mathscr{H}_{X(t)}^\perp, \\ Z(t) \prec F[X(t)], \end{cases} \quad \text{for a.e. } t \geqslant 0, \quad (X(0), Y(0)) = (\bar{X}, \bar{V}); \tag{4.7}$$

notice that the last two conditions include the case of (3.1) where $Z(t) = F[X(t)]$.

4.1. **Stability of generalized Lagrangian solutions.** In this section, we will prove a stability result for generalized Lagrangian solutions. Instead of relying on a semigroup estimate, strong compactness now follows from an argument based on Helly's theorem (recall Lemma 2.10) and on the closure properties of the map $X \mapsto \mathsf{P}_{\mathscr{H}_X}(F[X])$ for $X \in \mathscr{K}$.

**Lemma 4.3.** *Consider* $\{(X_n, Z_n, F_n)\} \subset \mathscr{L}^2(\Omega)$ *with*

$$X_n \in \mathscr{K}, \quad Z_n - F_n \in \mathscr{H}_{X_n}^\perp,$$

*If* $X_n \longrightarrow X$ *strongly and* $(Z_n, F_n) \rightharpoonup (Z, F)$ *weakly in* $\mathscr{L}^2(\Omega)$*, then*

$$Z - F \in \mathscr{H}_X^\perp, \tag{4.8}$$

$$Z \prec F \quad \textit{if } F_n \longrightarrow F \textit{ strongly in } \mathscr{L}^1(\Omega) \textit{ and } \quad Z_n \prec F_n. \tag{4.9}$$

*Analogously, for any* $T > 0$ *consider* $\{(X_n, Z_n, F_n)\} \subset \mathscr{L}^2((0,T), \mathscr{L}^2(\Omega))$ *with*

$$X_n(t) \in \mathscr{K}, \quad Z_n(t) - F_n(t) \in \mathscr{H}_{X_n(t)}^\perp, \quad Z_n(t) \prec F_n(t) \quad \textit{for a.e. } t \in (0,T).$$

*If $X_n \longrightarrow X$ strongly and $(Z_n, F_n) \rightharpoonup (Z, F)$ weakly in $\mathscr{L}^2((0,T), \mathscr{L}^2(\Omega))$, then*

$$Z(t) - F(t) \in \mathscr{H}^{\perp}_{X(t)} \quad a.e.,$$

$$Z(t) \prec F(t) \quad a.e. \quad \text{if } F_n \longrightarrow F \text{ strongly in } \mathscr{L}^1((0,T), \mathscr{L}^1(\Omega)),\ Z_n(t) \prec F_n(t) \quad a.e.$$

*Proof.* By assumption, we know that

$$\int_\Omega Z_n(m)\varphi(X_n(m))\,\mathrm{d}m = \int_\Omega F_n(m)\varphi(X_n(m))\,\mathrm{d}m \tag{4.10}$$

for every $\varphi \in \mathrm{C}_b(\mathbb{R})$. Passing to the limit in (4.10) we get

$$\int_\Omega Z(m)\varphi(X(m))\,\mathrm{d}m = \int_\Omega F(m)\varphi(X(m))\,\mathrm{d}m,$$

which yields (4.8) since the set $\{\varphi \circ X \colon \varphi \in \mathrm{C}_b(\mathbb{R})\}$ is dense in $\mathscr{H}_X$.

In order to prove (4.9) we pass to the limit in the inequality

$$\int_\Omega \psi(Z_n(m))\,\mathrm{d}m \leqslant \int_\Omega \psi(F_n(m))\,\mathrm{d}m$$

for arbitrary convex functions $\psi \colon \mathbb{R} \longrightarrow \mathbb{R}$ with linear growth, noticing that

$$\begin{aligned} \int_\Omega \psi(Z(m))\,\mathrm{d}m &\leqslant \liminf_{n\to\infty} \int_\Omega \psi(Z_n(m))\,\mathrm{d}m, \\ \int_\Omega \psi(F(m))\,\mathrm{d}m &= \lim_{n\to\infty} \int_\Omega \psi(F_n(m))\,\mathrm{d}m. \end{aligned} \tag{4.11}$$

The corresponding inequality for convex functions $\psi$ with arbitrary growth at infinity can be obtained from (4.11) by monotone approximation.

The time-dependent result follows by applying Ioffe's Theorem. □

**Theorem 4.4** (Stability of Generalized Lagrangian Solutions)**.** *Suppose that $F \colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is pointwise linearly bounded and uniformly continuous. Then we have the following result:*

*(1) Every sequence $\{X_n\}$ of generalized Lagrangian solutions with initial data*

$$\bar{X}_n \in \mathscr{K} \quad \text{and} \quad \bar{V}_n \in \mathscr{H}_{\bar{X}_n}$$

*converging strongly in $\mathscr{L}^2(\Omega)$ to $\bar{X} \in \mathscr{K}$ and $\bar{V} \in \mathscr{H}_{\bar{X}}$, admits a subsequence (still denoted by $\{X_n\}$) such that*

*(a) $X_n(t) \longrightarrow X(t)$ in $\mathscr{L}^2(\Omega)$ uniformly on compact time intervals.*

*(b) For any $T > 0$ $V_n \longrightarrow V$ in $\mathscr{L}^2((0,T), \mathscr{L}^2(\Omega))$.*

*(2) If $\{X_n\}$ is a sequence of generalized solutions satisfying the two convergence conditions above, then the limit function $X$ is a generalized Lagrangian solution.*

*Proof.* Since $(\bar{X}_n, \bar{V}_n) \longrightarrow (\bar{X}, \bar{V})$ strongly in $\mathscr{L}^2(\Omega)$ we can find a convex function $\psi$ satisfying (2.31) and $\lim_{r\to\infty} \psi(r)/r^2 = \infty$ with $[\bar{X}_n] + \Psi[\bar{V}_n] \leqslant C$ for all $n$. Here $\Psi$ denotes the functional (2.34) induced by $\psi$. By Lemma 2.9, it is not restrictive to assume that $\psi$ satisfies (2.32). The estimates of Lemma 3.13 (with $W := X$) and Gronwall lemma yields

$$\Psi[X_n(t)] + \Psi[V_n(t)] \leqslant C_T \quad \text{for all } t \in [0,T] \text{ and all } n. \tag{4.12}$$

By Lemma 2.10, it follows that the $X_n$ take values in a fixed compact subset of $\mathscr{L}^2(\Omega)$ and $X_n$ are uniformly Lipschitz continuous in $\mathscr{L}^2(\Omega)$. Recall that pointwise linearly bounded operators $F$ are also bounded. We can then apply Ascoli-Arzelà theorem to obtain a convergent subsequence, which we still denote by $\{X_n\}$ for simplicity. The convergence is uniform in each compact time interval and the limit function $X$ satisfies the same Lipschitz bound.

Consider now the sequence $\{Z_n\}$ of functions given by Definition 4.1. Since $Z_n(t) \prec F[X_n(t)]$ for a.e. $t$ and $F$ is bounded, (4.12) implies that the $Z_n$ are uniformly bounded in $\mathscr{L}^\infty((0,T), \mathscr{L}^2(\Omega))$ for all $T > 0$. Extracting another subsequence if necessary, we may therefore assume that

$$Z_n \rightharpoonup Z \quad \text{weak* in } \mathscr{L}^\infty((0,T), \mathscr{L}^2(\Omega)).$$

On the other hand, by uniform continuity of $F$ we have that

$$F_n := F[X_n] \longrightarrow F[X] =: F \quad \text{strongly in } \mathscr{L}^2((0,T),\mathscr{L}^2(\Omega)).$$

Lemma 4.3 then shows that $Z$ satisfies (4.2).

The uniform bound on $Z_n$ implies that the maps

$$Y_n(t) := \bar{V}_n + \int_0^t Z_n(s)\,\mathrm{d}s \quad \text{for all } t \geqslant 0$$

are uniformly Lipschitz continuous in each time interval $[0,T]$ with values in $\mathscr{L}^2(\Omega)$. Starting from (4.1) and applying standard stability results for differential inclusions (cf. Theorem 3.4 in [9], here the strong convergence of $X_n$ is crucial), we obtain that $X$ solves

$$\dot{X}(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar{V} + \int_0^t Z(s)\,\mathrm{d}s \quad \text{for a.e. } t \geqslant 0. \tag{4.13}$$

In particular, the map $X$ is right-differentiable in $\mathscr{L}^2(\Omega)$ for each $t \geqslant 0$, with right-continuous right-derivative $V$; see Propositions 3.3 and 3.4 in [9]. Therefore (4.13) holds for all $t \geqslant 0$ if $\dot{X}(t)$ is replaced by $V(t)$. We may also assume that

$$V_n \rightharpoonup V \quad \text{weak* in } \mathscr{L}^\infty((0,T),\mathscr{L}^2(\Omega))$$

for all $T > 0$ (extracting another subsequence if necessary). To show that $V_n \longrightarrow V$ strongly in $\mathscr{L}^2((0,T),\mathscr{L}^2(\Omega))$, we multiply the differential inclusion by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((T-t)X_n(t)\Big) = (T-t)V_n(t) - X_n(t)$$

and integrate in time over $(0,T)\times\Omega$. Now notice that since $X_n(t), V_n(t) \in \mathscr{H}_{X_n(t)}$ and since $\partial I_{\mathscr{K}}(X) \subset \mathscr{H}_X^\perp$ for all $X \in \mathscr{K}$, the subdifferential terms vanish after integration over $\Omega$. Integrating by parts in the force term, we obtain

$$\begin{aligned}\int_0^T (T-t)\|V_n(t)\|^2_{\mathscr{L}^2(\Omega)}\,\mathrm{d}t &- \int_0^T \bigg(\int_\Omega V_n(t,m)X_n(t,m)\,\mathrm{d}m\bigg)\,\mathrm{d}t \\ = -T\int_\Omega \bar{V}_n(m)\bar{X}_n(m)\,\mathrm{d}m &- \int_0^T (T-t)\bigg(\int_\Omega Z_n(t,m)X_n(t,m)\,\mathrm{d}m\bigg)\,\mathrm{d}t.\end{aligned}$$

A similar identity holds in the limit. Since the sequence $\{X_n\}$ converges strongly and the sequence $\{(V_n, Z_n)\}$ converges weakly, we can pass to the limit and get

$$\lim_{n\to\infty}\int_0^T (T-t)\|V_n(t)\|^2_{\mathscr{L}^2(\Omega)}\,\mathrm{d}t = \int_0^T (T-t)\|V(t)\|^2_{\mathscr{L}^2(\Omega)}\,\mathrm{d}t$$

for every $T > 0$. This, together with (4.12) yields the desired strong convergence. Therefore there exists an $\mathcal{L}^1$-negligible set $N \subset [0,\infty)$ such that (up to extraction of a subsequence if necessary) $V_n(t) \longrightarrow V(t)$ in $\mathscr{L}^2(\Omega)$ for every $t \in [0,\infty)\setminus N$. We can then pass to the limit in (4.3) written for $(X_n, V_n)$ and obtain the corresponding inclusion for $(X,V)$ in $(t_1,\infty)$ for all $t_1 \in [0,\infty)\setminus N$. Since $V$ is right-continuous, formula (4.3) eventually holds for all $t_1 \geqslant 0$.

Identity (4.4) follows by the same argument, passing to the limit for $t_1 \in [0,\infty)\setminus N$ and recalling that, by (2.8), if $K_n = \mathsf{P}_{\mathscr{K}}(H_n)$ and $K_n \longrightarrow K$, $H_n \rightharpoonup H$ in $\mathscr{L}^2(\Omega)$, then $K = \mathsf{P}_{\mathscr{K}}(H)$. □

We conclude this section with the main existence result for generalized Lagrangian solutions. As for sticky evolutions, its proof relies on the discrete particle approach we will study in the next section, see Remark 5.3.

**Theorem 4.5** (Existence of generalized Lagrangian solutions)**.** *Assume that the force functional $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is pointwise linearly bounded and uniformly continuous. Then for every couple $\bar{X} \in \mathscr{K}$ and $\bar{V} \in \mathscr{H}_{\bar{X}}$ there exists a generalized Lagrangian solution with initial data $(\bar{X}, \bar{V})$.*

## 5. Dynamics of Discrete Particles

We discussed in the Introduction that the conservation law (1.1) formally admits particular solutions for which the density consists of finite linear combinations of Dirac measures; see (1.6) above. In this section, we will reformulate these solutions in the Lagrangian framework and will prove their global existence. For every $N \in \mathbb{N}$ let us introduce the convex sets

$$\mathbb{M}^N := \Big\{ \boldsymbol{m} \in \mathbb{R}^N : m_i > 0 \text{ and } \sum_{i=1}^N m_i = 1 \Big\}, \quad \mathbb{K}^N := \Big\{ \boldsymbol{x} \in \mathbb{R}^N : x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N \Big\}.$$

For all times $t \geqslant 0$, a discrete solution to (1.1) of the form (1.6) is therefore determined by a unique number $N \in \mathbb{N}$ and a vector $(\boldsymbol{m}, \boldsymbol{x}, \boldsymbol{v}) \in \mathbb{M}^N \times \mathbb{K}^N \times \mathbb{R}^N$. To find a Lagrangian representation of (1.6) we consider a partition of $\Omega$ given by

$$0 =: w_0 < w_1 < \ldots < w_N := 1 \quad \text{where} \quad w_i := \sum_{j=1}^i m_j \tag{5.1}$$

for $i = 1, \ldots, N-1$. Writing $W_i := [w_{i-1}, w_i)$ we define functions

$$X := \sum_{i=1}^N x_i \mathbb{1}_{W_i} \quad \text{and} \quad V := \sum_{i=1}^N v_i \mathbb{1}_{W_i}, \tag{5.2}$$

the (finite dimensional) Hilbert space

$$\mathscr{H}_{\boldsymbol{m}} := \Big\{ X = \sum_{i=1}^N x_i \mathbb{1}_{W_i} : \boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N \Big\} \subset \mathscr{L}^2(\Omega) \tag{5.3}$$

and its closed convex cone

$$\mathscr{K}_{\boldsymbol{m}} := \Big\{ X = \sum_{i=1}^N x_i \mathbb{1}_{W_i} : \boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{K}^N \Big\} \subset \mathscr{K} \subset \mathscr{L}^2(\Omega) \tag{5.4}$$

Then clearly $X \in \mathscr{K}_{\boldsymbol{m}} \subset \mathscr{K}$ and $V \in \mathscr{H}_X$, and we easily have

$$\varrho = X_\# \mathfrak{m} = \sum_{i=1}^N m_i \, \delta_{x_i}, \quad V = v \circ X, \quad (\varrho v) = \sum_{i=1}^N m_i v_i \, \delta_{x_i}. \tag{5.5}$$

### 5.1. **Discrete Lagrangian solutions.**

We can reproduce at the discrete level the same approach we followed in Section 3. We can introduce the projected forces

$$F_{\boldsymbol{m}}[X] := \mathsf{P}_{\mathscr{H}_{\boldsymbol{m}}}(F[X]) = \sum_{i=1}^N a_{\boldsymbol{m},i} \mathbb{1}_{W_i}, \quad a_{\boldsymbol{m},i} = \fint_{W_i} F[X(t)](m) \, \mathrm{d}m, \tag{5.6}$$

which satisfy the analogue of (3.2),

$$\int_{\mathbb{R}} \psi(x) f[\varrho](\mathrm{d}x) = \int_\Omega \psi(X) F_{\boldsymbol{m}} \, \mathrm{d}m \quad \text{if } X \in \mathscr{H}_{\boldsymbol{m}}, \ \varrho = X_\# \mathfrak{m} \text{ as in (5.5)}, \tag{5.7}$$

and we can solve the differential inclusion

$$\dot{X}(t) + \partial I_{\mathscr{K}_{\boldsymbol{m}}}(X) = \bar{V} + \int_0^t F_{\boldsymbol{m}}[X(s)] \, \mathrm{d}s, \quad X(0) = \bar{X}, \tag{5.8}$$

for given initial data $(\bar{X}, \bar{V}) \in \mathscr{K}_{\boldsymbol{m}} \times \mathscr{H}_{\bar{X}}$.

Introducing $Y(t) := \bar{V} + \int_0^t F_{\boldsymbol{m}}[X(s)] \, ds = \sum_{i=1}^N y_i(t) \mathbb{1}_{W_i}$, we end up with the system

$$\begin{cases} \dot{X}(t) + \partial I_{\mathscr{K}_{\boldsymbol{m}}}(X(t)) \ni Y(t), \\ \qquad\qquad\qquad \dot{Y}(t) = F_{\boldsymbol{m}}[X(t)], \end{cases} \quad \text{for } t \geqslant 0, \quad (X(0), Y(0)) = (\bar{X}, \bar{V}), \tag{5.9}$$

which is equivalent to (1.27).

If $F$ is Lipschitz continuous, then $F_{\boldsymbol{m}} : \mathscr{K}_{\boldsymbol{m}} \to \mathscr{H}_{\boldsymbol{m}}$ is also Lipschitz and the analogous statements of Theorems 3.5 and 3.6 hold at this discrete level. In particular, as in (3.9), we have

$$V(t) = \frac{\mathrm{d}^+}{\mathrm{d}t} X(t) = \mathsf{P}_{T_{X(t)}\mathscr{K}_{\boldsymbol{m}}}(Y(t)). \tag{5.10}$$

The discrete analogue of Lemma 2.4 thus justifies condition (1.15), which we introduced in the simplified situation of a collision of two particles.

Let us now consider a sequence $X_n$ of discrete Lagrangian solutions of (5.8) corresponding to initial data $(\bar{X}_n, \bar{V}_n) \in \mathscr{K}_{\boldsymbol{m}_n} \times \mathscr{H}_{\boldsymbol{m}_n}$ strongly converging to $(\bar{X}, \bar{V}) \in \mathscr{K} \times \mathscr{L}^2(\Omega)$. We want to show that $X_n \longrightarrow X$ locally uniformly in $\mathrm{C}([0,\infty); \mathscr{L}^2(\Omega))$, where $X$ is the Lagrangian solution associated to $(\bar{X}, \bar{V})$. To make the analysis simpler, we will assume that the distributions of masses $\boldsymbol{m}_n$ give raise by (5.1) to sufficiently fine partitions of the interval $(0,1)$, i.e., we have that

$$\text{for every } K \in \mathscr{K} \text{ there exist } K_n \in \mathscr{K}_{\boldsymbol{m}_n} \text{ such that } K_n \longrightarrow K \quad \text{in } \mathscr{L}^2(\Omega). \tag{5.11}$$

Since $\mathscr{K}_{\boldsymbol{m}_n} \subset \mathscr{K}$, assumption (5.11) is equivalent to the Mosco-convergence of the sequence $\mathscr{K}_{\boldsymbol{m}_n}$ to $\mathscr{K}$ in the Hilbert space $\mathscr{L}^2(\Omega)$ [1, Section 3.3.2]. By first approximating $\mathrm{C}^1([0,1])$-functions (which belong to $\mathscr{K} - \mathscr{K}$) and then applying a density argument, it is not difficult to show that (5.11) implies a similar property for the closed subspaces $\mathscr{H}_{\boldsymbol{m}_n}$ in $\mathscr{L}^2(\Omega)$, i.e., we have that

$$\text{for every } H \in \mathscr{L}^2(\Omega) \text{ there exist } H_n \in \mathscr{H}_{\boldsymbol{m}_n} \text{ such that } H_n \longrightarrow H \quad \text{in } \mathscr{L}^2(\Omega). \tag{5.12}$$

Both (5.11) and (5.12) surely hold if, for example, we have

$$\lim_{n\to\infty} \|\boldsymbol{m}_n\|_\infty = 0,$$

where for a generic $\boldsymbol{m} = (m_1, \cdots, m_N) \in \mathbb{M}^N$ we set $\|\boldsymbol{m}\|_\infty = \sup_i m_i$.

It is not surprising that we have the following approximation result.

**Theorem 5.1** (Convergence of discrete Lagrangian solutions). *Let $F : \mathscr{K} \to \mathscr{L}^2(\Omega)$ be Lipschitz continuous and pointwise linearly bounded, let $\boldsymbol{m}_n \in \mathbb{M}^{N_n}$ be a sequence satisfying* (5.11)*, and let $X_n \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty); \mathscr{K}_{\boldsymbol{m}_n})$ be a sequence of discrete Lagrangian solutions with initial data $(\bar{X}_n, \bar{V}_n) \in \mathscr{K}_{\boldsymbol{m}_n} \times \mathscr{H}_{\boldsymbol{m}_n}$ strongly converging to $(\bar{X}, \bar{V}) \in \mathscr{K} \times \mathscr{L}^2(\Omega)$. Then $X_n \longrightarrow X$ locally uniformly in $\mathrm{C}([0,\infty); \mathscr{L}^2(\Omega))$, where $X$ is the unique Lagrangian solution with data $(\bar{X}, \bar{V})$.*

*Proof.* We cannot directly apply the stability estimates of Theorem 3.6 since the discrete Lagrangian solutions are associated to convex sets $\mathscr{K}_{\boldsymbol{m}_n}$ depending on $n$. Therefore we combine the compactness argument of the proof of Theorem 4.4 and a classical stability result for differential inclusion [1, Theorem 3.74] generated by a Mosco-converging sequence of convex sets.

In fact, we can choose a convex and superquadratic functional $\psi$ satisfying (2.31) such that

$$\Psi[\bar{X}_n] + \Psi[\bar{V}_n] \leqslant C.$$

Then the estimates of Lemma 3.13 (which can be extended to the discrete case) yield

$$\Psi[X_n(t)] + \Psi[V_n(t)] \leqslant C_T \quad \text{for all } t \in [0,T] \text{ and all } n.$$

Arguing as in the proof of Theorem 4.4 we can find a subsequence (still denoted by $X_n$) locally uniformly converging to a limit $X \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty); \mathscr{L}^2(\Omega))$ that takes its value in $\mathscr{K}$. We get that $F_{\boldsymbol{m}_n}[X_n] \longrightarrow F[X]$ in $L^2_{\mathrm{loc}}([0,\infty); \mathscr{L}^2(\Omega))$ since for every time $t \geqslant 0$ it holds

$$\big\|F_{\boldsymbol{m}_n}[X_n] - F[X]\big\|_{\mathscr{L}^2(\Omega)} \leqslant C\big\|X_n - X\big\|_{\mathscr{L}^2(\Omega)} + \big\|F_{\boldsymbol{m}_n}[X] - F[X]\big\|_{\mathscr{L}^2(\Omega)}$$

and $F_{\boldsymbol{m}_n}[X] = \mathsf{P}_{\mathscr{H}_{\boldsymbol{m}_n}}\big(F[X]\big) \longrightarrow F[X]$ in $\mathscr{L}^2(\Omega)$, by (5.12). It follows that

$$Y_n(t) = \bar{V}_n + \int_0^t F_{\boldsymbol{m}_n}[X_n(s)]\,\mathrm{d}s \longrightarrow Y(t) := \bar{V} + \int_0^t F[X(s)]\,\mathrm{d}s$$

locally uniformly in $\mathrm{C}([0,\infty); \mathscr{L}^2(\Omega))$. We then apply [1, Theorem 3.74] to show that the limit $X$ also satisfies the differential inclusion $\dot{X} + \partial I_{\mathscr{K}}(X) \ni Y$ and therefore it is a Lagrangian solution associated to $(\bar{X}, \bar{V})$. Since the limit is uniquely determined (by Theorem 3.6) we conclude that the whole sequence $X_n$ converges to $X$. □

5.2. **A sticky evolution dynamic for discrete particles.** In this section, we will describe a different discrete procedure to construct evolution of a finite number of particles. In the general case, this approach will lead to generalized Lagrangian solutions; when $F$ is sticking, we will obtain a sticky evolution which in fact will coincide with the construction in the previous section.

We already explained the basic idea in the introductory section 1.1: At the discrete level, a collision between two or more particles at some time $t'$ corresponds to the impact of the vector $\boldsymbol{x}$ with the boundary $\partial\mathbb{K}^N$ (equivalently, of the Lagrangian parametrization $X$ with the boundary of $\mathscr{K}_{\boldsymbol{m}}$ in $\mathscr{H}_{\boldsymbol{m}}$). In this case, we relabel the particles and consider the evolution for $t \geqslant t'$ in a reduced convex cone attached to the new configuration, up to the next collision.

Here is a precise description of the evolution. Assume without loss of generality that $\bar{X}$ does not belong to the boundary of $\mathscr{K}_{\bar{\boldsymbol{m}}}$ in $\mathscr{H}_{\bar{\boldsymbol{m}}}$. On the time interval $[t_0, t_1)$, where $t_0 := 0$ and $t_1 > 0$ is to be determined so that $X(t)$ does not touch the boundary of $\partial\mathscr{K}_{\bar{\boldsymbol{m}}}$ in $[t_0, t_1)$, we obtain

$$X(t,\cdot) = \sum_{i=1}^{N} x_i(t)\mathbb{1}_{W_i} \quad \text{and} \quad V(t,\cdot) = \sum_{i=1}^{N} v_i(t)\mathbb{1}_{W_i} \tag{5.13}$$

by solving the system

$$\dot{X}(t) = V(t), \quad \dot{V}(t) = \mathsf{P}_{\mathscr{H}_{\bar{\boldsymbol{m}}}}(F[X(t)]). \tag{5.14}$$

Since $\mathscr{H}_{\bar{\boldsymbol{m}}} = \mathscr{H}_{X(t)}$ in $[t_0, t_1)$, we notice that the projection onto $\mathscr{H}_{\bar{\boldsymbol{m}}}$ returns a function that is piecewise constant on the same partition on which $(X, V)$ is constant. Hence (5.14) is equivalent to the system of the form (1.16), which is well-defined. The time $t_1$ is taken as the smallest $t > 0$ for which $X(t)$ hits the boundary of $\mathscr{K}_{\bar{\boldsymbol{m}}}$ in $\mathscr{H}_{\bar{\boldsymbol{m}}}$. At time $t_1$ we can find an integer $N' < N$ and compute a new state vector $(\bar{\boldsymbol{m}}', \bar{\boldsymbol{x}}', \bar{\boldsymbol{v}}') \in \mathscr{M}^{N'} \times \mathscr{K}^{N'} \times \mathbb{R}^{N'}$. On the interval $[t_1, t_2)$, with $t_2 > t_1$ to be determined, we obtain $X(t,\cdot), V(t,\cdot)$ as in (5.13) by solving (5.14) with $\bar{\boldsymbol{m}}$ replaced by $\bar{\boldsymbol{m}}'$, the initial condition $(\boldsymbol{x}', \boldsymbol{v}')(t_1) := (\bar{\boldsymbol{x}}', \bar{\boldsymbol{v}}')$, and the new subdivision $W_j' := [w_{j-1}', w_j')$ defined as in (5.1). Continuing in the same fashion, we obtain $K \in \mathbb{N}$, a sequence of "collision times"

$$0 =: t_0 < t_1 < \ldots < t_{K-1} < t_K := \infty,$$

and a pair of functions $(X, V)$ such that

$$\mathscr{H}_{X(t)} = \mathscr{H}_{X(t_{k-1})}, \quad \dot{X}(t) = V(t), \quad \dot{V}(t) = \mathsf{P}_{\mathscr{H}_{X(t)}}(F[X(t)]) \tag{5.15}$$

for all $t \in [t_{k-1}, t_k)$ and $k = 1, \ldots, K$. At collision times the space $\mathscr{H}_{X(t_k)}$ is strictly smaller than $\mathscr{H}_{X(t)}$ for all $t < t_k$, which implies that $K \leqslant N$. We have

$$X(t_k+) = X(t_k-), \quad V(t_k+) = \mathsf{P}_{\mathscr{H}_{X(t_k)}}(V(t_k-)). \tag{5.16}$$

It is easy to check that the monotonicity condition (3.23) is satisfied.

5.3. **Sticky and generalized Lagrangian solutions for discrete particles.** The next theorem shows that by using the algorithm described in the previous section we can obtain a generalized Lagrangian solution in the original cone $\mathscr{K}$ starting from the discrete initial data $(\bar{X}, \bar{V})$. When $F$ is sticking, this coincides with the unique sticky Lagrangian solution.

**Theorem 5.2** (Generalized and sticky Lagrangian solutions for discrete particles)**.** *Suppose that $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ is uniformly continuous. Consider functions $(\bar{X}, \bar{V})$ of the form* (5.2) *for some $N \in \mathbb{N}$ and $(\bar{\boldsymbol{m}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{v}}) \in \mathbb{M}^N \times \mathbb{K}^N \times \mathbb{R}^N$. Then we have the following result:*

*(1) The curve $(X, V)$ described by the previous section is a generalized Lagrangian solution to* (3.1) *with initial data $(\bar{X}, \bar{V})$.*
*(2) If $F$ is sticking, then $(X, V)$ is a sticky Lagrangian solution.*

*Proof.* Let us first prove that the map $t \mapsto X(t)$ is a generalized Lagrangian solution with respect to the choice $Z(t) := \mathsf{P}_{\mathscr{H}_{X(t)}}(F[X(t)])$ for all $t \geqslant 0$. The fact that $V$ is the right-derivative of $X$ follows immediately from the construction. In order to prove (4.3), it is not restrictive to assume $t_1 = 0$. We argue by induction on the collision times. In the first interval $[t_0, t_1)$ inclusion (4.3) is satisfied by taking the null selection in the subdifferential $\partial I_{\mathscr{K}}(X(t))$.

Assume now that (4.3) is satisfied in $[t_{k-1}, t_k)$ for some $k$. Then

$$\begin{aligned}\dot X(t) &= V(t_k+) + \int_{t_k}^t \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\,\mathrm{d}s \\ &= \Big(V(t_k+) - V(t_k-)\Big) + V(t_k-) + \int_{t_k}^t \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\,\mathrm{d}s \end{aligned} \tag{5.17}$$

for any $t \in [t_k, t_{k+1})$, by (5.15). By induction assumption, we have that

$$V(t_k-) + \xi = \bar V + \int_0^{t_k} \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\,\mathrm{d}s \tag{5.18}$$

for some $\xi \in \partial I_{\mathscr{K}}(X(t_k))$. Combining (5.17) and (5.18), we obtain

$$\dot X(t) + \xi + \Big(V(t_k-) - V(t_k+)\Big) = \bar V + \int_0^t \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\,\mathrm{d}s$$

Because of (5.15), we have that

$$V(t_k-) = \lim_{h\to 0+} h^{-1}\Big(X(t_k) - X(t_k - h)\Big).$$

Using (5.16), we then obtain

$$\begin{aligned} &V(t_k-) - V(t_k+) \\ &\quad = V(t_k-) - \mathsf{P}_{\mathscr{H}_{X(t_k)}}(V(t_k-)) \\ &\quad = \lim_{h\to 0+} h^{-1}\Big(X(t_k) - X(t_k-h) - \mathsf{P}_{\mathscr{H}_{X(t_k)}}\big(X(t_k) - X(t_k-h)\big)\Big) \\ &\quad = \lim_{h\to 0+} h^{-1}\Big(\mathsf{P}_{\mathscr{H}_{X(t_k)}}\big(X(t_k-h)\big) - X(t_k-h)\Big). \end{aligned}$$

We now use Lemmas 2.4 and 2.6 and conclude that $V(t_k-) - V(t_k+) \in \partial I_{\mathscr{K}}(X(t_k))$, noticing that $N_X\mathscr{K} = \partial I_{\mathscr{K}}(X)$ for all $X \in \mathscr{K}$. Property (3.23) implies the monotonicity of the subdifferentials, which are closed convex cones. This yields

$$\xi + \Big(V(t_k-) - V(t_k+)\Big) \in \partial I_{\mathscr{K}}(X(t))$$

for all $t \in [t_k, t_{k+1})$. Identities (4.4) and (4.6) can be proved as in Proposition 3.8. We conclude that $X$ is a generalized Lagrangian solution.

We now show that if $F$ is sticking, then (3.1) holds. Because of (3.23), we have $X(t) \in \mathscr{H}_{X(s)}$ for all $s \leqslant t$. Then Definition 3.3, the fact that $\Omega_{X(s)} \subset \Omega_{X(t)}$, and (2.19) yield

$$\int_0^t \Big(F[X(s)] - \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\Big)\,\mathrm{d}s \in \partial I_{\mathscr{K}}(X(t)). \tag{5.19}$$

Adding (5.19) to either side of

$$V(t) + \partial I_{\mathscr{K}}(X(t)) \ni \bar V + \int_0^t \mathsf{P}_{\mathscr{H}_{X(s)}}(F[X(s)])\,\mathrm{d}s,$$

we obtain (3.1). Therefore $X$ is a sticky Lagrangian solution. $\square$

We already know that any (even a generalized) Lagrangian solution induces a solution of the conservation law (1.1). Since for each time $t \geqslant 0$ the transport map $X(t,\cdot)$ is piecewise constant, the corresponding solution is in fact a discrete particle solution: the density/momentum is of the form (1.6).

*Remark* 5.3. Notice that piecewise constant functions as in (5.2) are dense in $\mathscr{L}^2(\Omega)$, so we can approximate any given initial data and then combine the existence result 5.2 with the stability Theorem 4.4 to get the proof of Theorem 4.5.

*Remark* 5.4. The proof of Theorem 3.11 follows by a similar approximation argument. By Theorem 3.10, it suffices to show that any Lagrangian solution $X$ with $(\bar{X},\bar{V})\in\mathscr{K}\times\mathscr{H}_{\bar{X}}$ satisfies

$$\Omega_{\bar{X}}\subset\Omega_{X(t)}\quad\text{for all } t\geqslant 0.$$

That is, if $\bar{X}$ is constant on some interval $(\alpha,\beta)\subset\Omega$, then $X(t)$ remains constant on $(\alpha,\beta)$ for all $t\geqslant 0$. We approximate $(\bar{X},\bar{V})$ by a sequence $(\bar{X}_n,\bar{V}_n)$ of the form (5.2) such that $\bar{X}_n$ is constant on $(\alpha,\beta)$. Since this property is preserved by the discrete Lagrangian solution constructed in Theorem 5.2, the stability estimates of Theorem 3.6 show that the limit function $X(t)$ is still constant on $(\alpha,\beta)$.

## 6. Global Existence in Eulerian coordinates

Theorems 3.6, 3.11, 3.12, and 4.5 of the previous sections immediately translate into global existence results for weak solutions of the Euler system of conservation laws (1.1). Before stating some of the related results, let us explore in more detail the relation between the force functionals $f[\varrho]$ in (1.1) and their reformulation in the Lagrangian framework.

6.1. **The Eulerian description of the force field.** Let us first introduce the space

$$\mathscr{T}_2(\mathbb{R}):=\Big\{(\varrho,v)\colon\,\varrho\in\mathscr{P}_2(\mathbb{R}),v\in\mathscr{L}^2(\mathbb{R},\varrho)\Big\}.$$

For all $(\varrho_i,v_i)\in\mathscr{T}_2(\mathbb{R})$ with $i=1,2$, we then define (see also [4, §7.2], [18, §2])

$$D_2^2\Big((\varrho_1,v_1),(\varrho_2,v_2)\Big):=W_2^2(\varrho_1,\varrho_2)+U_2^2\Big((\varrho_1,v_1),(\varrho_2,v_2)\Big),$$

where $W_2$ is the Wasserstein distance and $U_2$ denotes the semi-distance

$$\begin{aligned}U_2^2\Big((\varrho_1,v_1),(\varrho_2,v_2)\Big)&:=\int_{\mathbb{R}\times\mathbb{R}}|v_1(x)-v_2(y)|^2\,\boldsymbol{\varrho}(dx,dy)\\&=\int_\Omega|v_1(X_{\varrho_1}(m))-v_2(X_{\varrho_2}(m))|^2\,\mathrm{d}m.\end{aligned}\tag{6.1}$$

Here $\boldsymbol{\varrho}\in\Gamma_{\mathrm{opt}}(\varrho_1,\varrho_2)$ is the unique optimal transport map between the measures $\varrho_1$ and $\varrho_2$. It can be expressed in terms of the transport maps defined in (2.4); see (2.6). The sequence $\{(\varrho_n,v_n)\}$ converges to $(\varrho,v)$ in the metric space $(\mathscr{T}_2(\mathbb{R}),D_2)$ if and only if $W_2(\varrho_n,\varrho)\longrightarrow 0$, if $\varrho_n v_n\rightharpoonup\varrho v$ weak* in $\mathscr{M}(\mathbb{R})$, and if

$$\int_{\mathbb{R}}|v_n|^2\,\varrho_n\longrightarrow\int_{\mathbb{R}}|v|^2\,\varrho.$$

We refer the reader to [18, Prop. 2.1] and [5] for details (see in particular Definition 5.4.3).

We consider continuous maps $f\colon\mathscr{P}_2(\mathbb{R})\longrightarrow\mathscr{M}(\mathbb{R})$ (with respect to the Wasserstein topology in $\mathscr{P}_2(\mathbb{R})$ and the weak* topology on $\mathscr{M}(\mathbb{R})$ induced by $\mathrm{C}_b(\mathbb{R})$) such that $f[\varrho]$ is absolutely continuous with respect to $\varrho\in\mathscr{P}_2(\mathbb{R})$. In that case, we let $f_\varrho$ is the Radon-Nikodym-derivative of $f[\varrho]$ with respect to $\varrho$, and we assume that $f_\varrho\in\mathscr{L}^2(\mathbb{R},\varrho)$. That is, the operator $f$ can be written as

$$f[\varrho]=f_\varrho\,\varrho,\quad f_\varrho\in\mathscr{L}^2(\mathbb{R},\varrho).\tag{6.2}$$

**Definition 6.1** (Boundedness)**.** We say that a map $f\colon\mathscr{P}_2(\mathbb{R})\longrightarrow\mathscr{M}(\mathbb{R})$ as in (6.2) is bounded if there exists a constant $C\geqslant 0$ such that

$$\|f_\varrho\|^2_{\mathscr{L}^2(\mathbb{R},\varrho)}\leqslant C\Big(1+\int_{\mathbb{R}}|x|^2\,\mathrm{d}\varrho\Big)\quad\text{for all }\varrho\in\mathscr{P}_2(\mathbb{R}).$$

We say that $f$ is pointwise linearly bounded if there exists a $C_{\mathsf{p}}\geqslant 0$ such that

$$|f_\varrho(x)|\leqslant C_{\mathsf{p}}\Big(1+|x|+\int_{\mathbb{R}}|x|\,\mathrm{d}\varrho\Big)\quad\text{for a.e. }x\in\mathbb{R}\text{ and all }\varrho\in\mathscr{P}_2(\mathbb{R}).$$

**Definition 6.2** (Uniform continuity I)**.** We say that a map $f\colon\mathscr{P}_2(\mathbb{R})\longrightarrow\mathscr{M}(\mathbb{R})$ as in (6.2) is uniformly continuous if there exists a modulus of continuity $\omega$ such that

$$U_2\Big((\varrho_1,f_{\varrho_1}),(\varrho_2,f_{\varrho_2})\Big)\leqslant\omega\Big(W_2(\varrho_1,\varrho_2)\Big)\quad\text{for all }\varrho_1,\varrho_2\in\mathscr{P}_2(\mathbb{R}).\tag{6.3}$$

In the case $\omega(r)=Lr$ for some constant $L\geqslant 0$ and all $r\geqslant 0$, we say that $f$ is Lipschitz continuous.

As discussed in Section 2.1, there is a one-to-one correspondence between measures $\varrho \in \mathscr{P}_2(\mathbb{R})$ and optimal transport maps $X \in \mathscr{L}^2(\Omega)$, given by

$$X \in \mathscr{K} \quad \text{and} \quad X_\# \mathfrak{m} = \varrho. \tag{6.4}$$

We now want to construct a functional $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ such that

$$\int_{\mathbb{R}} \varphi(x)\, f[\varrho](\mathrm{d}x) = \int_\Omega \varphi(X(m)) F[X](m)\, \mathrm{d}m \quad \text{for all } \varphi \in \mathrm{C}_b(\mathbb{R}), \tag{6.5}$$

whenever $(X, \varrho)$ are related by (6.4). One possible choice is to set

$$F[X] := f_\varrho \circ X \quad \text{for all } (X, \varrho) \text{ satisfying (6.4)}, \tag{6.6}$$

which implies $F[X] \in \mathscr{L}^2(\Omega)$. Then the boundedness and continuity assumptions on the functional $f$ in Definitions 6.1 and 6.2 translate immediately into the corresponding properties for $F$ in Definitions 3.1 and 3.2. It can be useful, however, to also consider different choices for $F$.

**Definition 6.3** (Uniform continuity II)**.** We call map $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ as in (6.2) *densely* uniformly continuous if (6.3) holds for measures that are absolutely continuous with respect to $\mathcal{L}^1$ with bounded densities. We define dense Lipschitz continuity similarly.

**Lemma 6.4.** *If $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ is densely uniformly continuous, then there exists a unique uniformly continuous map $F\colon \mathscr{K} \longrightarrow \mathscr{L}^2(\Omega)$ such that* (6.5) *holds for all $(X, \varrho)$ satisfying* (6.4).

Note that (6.5) implies that $f_\varrho \circ X = \mathsf{P}_{\mathscr{H}_X}(F[X])$ for all $(X, \varrho)$ with (6.4).

*Proof.* We denote by $\mathscr{K}_{\mathrm{reg}}$ the dense subset of $\mathscr{K}$ whose elements are $\mathrm{C}^1(\bar{\Omega})$-maps with strictly (thus uniformly) positive derivatives. For all $X \in \mathscr{K}_{\mathrm{reg}}$ the push-forward $\varrho := X_\# \mathfrak{m}$ is absolutely continuous with respect to $\mathcal{L}^1$ and has a bounded density. We can then define

$$F[X] := f_\varrho \circ X \quad \text{for all } X \in \mathscr{K}_{\mathrm{reg}}. \tag{6.7}$$

Applying definition (6.1) and (6.3) we obtain

$$\|F[X_1] - F[X_2]\|_{\mathscr{L}^2(\Omega)} \leqslant \omega\Big(\|X_1 - X_2\|_{\mathscr{L}^2(\Omega)}\Big) \quad \text{for all } X_1, X_2 \in \mathscr{K}_{\mathrm{reg}}.$$

Then $F$ can be extended to all of $\mathscr{K}$ by density. One can check that this functional satisfies (6.7), therefore it is uniquely determined by $f$. ☐

**Definition 6.5** (Sticking)**.** Let $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ be densely uniformly continuous and let $F$ be the functional from Lemma 6.4. We say that $f$ is sticking if $F$ is sticking.

6.2. **Existence results and examples.** We state here a simple example of possible applications of the previous Lagrangian results. We omit the details of the proof, which can be derived by the fine structure properties and the a priori estimates we obtained for the Lagrangian formulation. It is worth noticing that all the solutions can be obtained as a suitable limit of discrete particle evolutions. The first statement in the following result follows from Theorem 4.5, the second one by Theorem 3.6, and Theorems 3.10 and 3.11 yield the last assertion.

**Theorem 6.6** (Global Existence)**.** *Let us fix $\bar{\varrho} \in \mathscr{P}_2(\mathbb{R})$ and $\bar{v} \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho})$.*

*(1) Suppose that the force functional $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ is pointwise linearly bounded and densely uniformly continuous. Then there exists a solution $(\varrho, v)$ of the conservation law* (1.1) *with initial data $(\bar{\varrho}, \bar{v})$.*

*(2) If $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ is densely Lipschitz continuous, then there exists a stable selection of a solution $(\varrho, v)$ of* (1.1) *with respect to the initial data $(\bar{\varrho}, \bar{v})$ in $(\mathscr{T}_2(\mathbb{R}), D_2)$.*

*(3) If $f\colon \mathscr{P}_2(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R})$ is densely Lipschitz continuous and sticking, then there exists a stable sticky solution $(\varrho, v)$ of* (1.1) *with initial data $(\bar{\varrho}, \bar{v})$. Then $S_t : (\bar{\varrho}, \bar{v}) \mapsto \varrho(t, \cdot), v(t, \cdot))$ is a semigroup in $(\mathscr{T}_2(\mathbb{R}), D_2)$.*

We finish the paper by giving a number of examples of force functionals.

*Example* 6.7. Let $v\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$|v(x)| \leqslant C_v(1+|x|) \quad \text{for all } x \in \mathbb{R}, \tag{6.8}$$

with $C_v \geqslant 0$ some constant. Then the operator defined by

$$f[\varrho] := \varrho v \quad \text{for all } \varrho \in \mathscr{P}_2(\mathbb{R})$$

is pointwise linearly bounded. It is Lipschitz continuous provided $v$ is a Lipschitz function. Note that $f[\varrho]$ is the Wasserstein differential of the potential energy (see [5])

$$\mathscr{V}[\varrho] := \int_{\mathbb{R}} V(x)\,\varrho(\mathrm{d}x) \quad \text{where } v = V'.$$

*Example* 6.8. Let $w\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying (6.8). Then

$$f[\varrho] := \varrho(w \star \varrho) = \varrho\bigg(\int_{\mathbb{R}} w(\cdot - y)\,\varrho(\mathrm{d}y)\bigg) \quad \text{for all } \varrho \in \mathscr{P}_2(\mathbb{R})$$

is pointwise linearly bounded, since

$$\big|(w\star\varrho)(x)\big| \leqslant C_w \int_{\mathbb{R}} (1+|x-y|)\,\varrho(\mathrm{d}y) \leqslant C_w\Big(1+|x|+\int_{\mathbb{R}} |y|\,\varrho(\mathrm{d}y)\Big)$$

It is Lipschitz if $w$ is a Lipschitz function. Writing $f_\varrho := w \star \varrho$ for all $\varrho \in \mathscr{P}_2(\mathbb{R})$, we have

$$\begin{aligned}
|f_{\varrho_1}(x) - f_{\varrho_2}(y)| &= \bigg|\int_{\mathbb{R}} w(x-x')\,\varrho_1(\mathrm{d}x') - \int_{\mathbb{R}} w(y-y')\,\varrho_2(\mathrm{d}y')\bigg| \\
&= \bigg|\int_{\mathbb{R}\times\mathbb{R}} \Big(w(x-x') - w(y-y')\Big)\,\boldsymbol{\varrho}(\mathrm{d}x',\mathrm{d}y')\bigg| \\
&\leqslant L\bigg(|x-y| + \int_{\mathbb{R}\times\mathbb{R}} |x'-y'|\,\boldsymbol{\varrho}(\mathrm{d}x',\mathrm{d}y')\bigg),
\end{aligned}$$

where $L \geqslant 0$ is the Lipschitz constant of $w$ and $\boldsymbol{\varrho} \in \Gamma_{\mathrm{opt}}(\varrho_1, \varrho_2)$. This implies

$$U_2\Big((\varrho_1, f_{\varrho_1}), (\varrho_2, f_{\varrho_2})\Big) \leqslant 4LW_2(\varrho_1, \varrho_2) \quad \text{for all } \varrho_1, \varrho_2 \in \mathscr{P}_2(\mathbb{R}).$$

Note that $f[\varrho]$ is the Wasserstein differential of the interaction energy (see [5])

$$\mathscr{W}[\varrho] = \int_{\mathbb{R}\times\mathbb{R}} W(x-y)\,\varrho(\mathrm{d}x)\,\varrho(\mathrm{d}y) \quad \text{where } w = W'.$$

*Example* 6.9. Let us consider the previous example with the Borel function

$$w(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases},$$

which corresponds to $W(x) := |x|$. To show that $f[\varrho]$ is continuous, note that

$$f_\varrho(x) = m_\varrho(x) + M_\varrho(x) - 1 \quad \text{for all } x \in \mathbb{R},$$

where $m_\varrho(x) := \varrho\big((-\infty, x)\big)$ and $M_\varrho(x) := \varrho\big((-\infty, x]\big)$ as in (2.3). Up to rescaling and adding constants, the function $f_\varrho$ is the precise representative of the cumulative distribution function of the measure $\varrho$. For convenience, we define

$$\tilde{f}_\varrho(x) := f_\varrho(x) + 1 \quad \text{for all } x \in \mathbb{R}, \quad \tilde{f}[\varrho] := \tilde{f}_\varrho\,\varrho.$$

We now introduce the sets

$$J_\varrho := \Big\{x \in \mathbb{R}\colon \varrho\big(\{x\}\big) > 0\Big\} \quad \text{and} \quad \mathcal{J}_\varrho := \bigcup_{x \in J_\varrho} \big(m_\varrho(x), M_\varrho(x)\big).$$

Note that $J_\varrho$ is at most countable. If $X_\varrho$ is defined by (2.4), then

$$\begin{aligned}
X_\varrho(m) = x \quad &\text{for all } m \in [m_\varrho(x), M_\varrho(x)] \text{ and } x \in \mathcal{J}_\varrho, \\
\tilde{f}_\varrho(X_\varrho(m)) = 2m \quad &\text{for all } m \in \Omega \setminus \mathcal{J}_\varrho.
\end{aligned}$$

For any $\varphi \in \mathrm{C}_b(\mathbb{R})$ we have

$$\begin{aligned}
\int_\Omega \varphi(x)\,\tilde f[\varrho](\mathrm{d}x) &= \int_\Omega \varphi(X_\varrho(m))\tilde f_\varrho(X_\varrho(m))\,\mathrm{d}m \\
&= \int_{\Omega\setminus\mathcal{J}_\varrho} \varphi(X_\varrho(m))\tilde f_\varrho(X_\varrho(m))\,\mathrm{d}m + \sum_{x\in J_\varrho}\int_{m_\varrho(x)}^{M_\varrho(x)} \varphi(X_\varrho(m))\tilde f_\varrho(X_\varrho(m))\,\mathrm{d}m \\
&= 2\int_{\Omega\setminus\mathcal{J}_\varrho} \varphi(X_\varrho(m))\,m\,\mathrm{d}m + \sum_{x\in J_\varrho}\Big(M_\varrho^2(x)-m_\varrho^2(x)\Big)\varphi(x) \\
&= 2\int_{\Omega\setminus\mathcal{J}_\varrho} \varphi(X_\varrho(m))\,m\,\mathrm{d}m + 2\sum_{x\in J_\varrho}\int_{m_\varrho(x)}^{M_\varrho(x)} \varphi(x)\,m\,\mathrm{d}m \\
&= 2\int_\Omega \varphi(X_\varrho(m))\,m\,\mathrm{d}m.
\end{aligned}$$

It follows that

$$\int_\mathbb{R} \varphi(x) f_\varrho(x)\,\varrho(\mathrm{d}x) = \int_\Omega \varphi(X_\varrho(m))\,(2m-1)\,\mathrm{d}m$$

Then the map $f$ is pointwise linearly bounded because $|f_\varrho(x)| \leqslant 1$ for all $x \in \mathbb{R}$. It is continuous since $\varrho_n \longrightarrow \varrho$ in $\mathscr{P}_2(\mathbb{R})$ implies that $X_{\varrho_n} \longrightarrow X_\varrho$ in $\mathscr{L}^2(\Omega)$. It is densely Lipschitz continuous since the associated functional $F$ is given by

$$F[X](m) := 2m-1 \quad \text{for all } m \in \Omega, \tag{6.9}$$

which does not even depend on $X \in \mathscr{K}$ anymore.

*Example* 6.10. For $\sigma \in \mathscr{L}^\infty(\mathbb{R})$ let $q_\varrho$ be the solution of (recall (1.3))

$$-\partial^2_{xx} q_\varrho = \lambda\big(\varrho - \sigma\big). \tag{6.10}$$

Then $q_\varrho$ is locally Lipschitz continuous and its (opposite) derivative $a_\varrho := -\partial_x q_\varrho$ is locally of bounded variation. Choosing its precise representative we then define

$$f[\varrho] := \varrho a_\varrho \quad \text{for all } \varrho \in \mathscr{P}_2(\mathbb{R}). \tag{6.11}$$

Setting $Q_\sigma(x) := \int_0^x \sigma(y)\,\mathrm{d}y$ it is not difficult to check that

$$a_\varrho(x) = -\lambda\Big(\frac{1}{2}\big(m_\varrho(x) + M_\varrho(x) - 1\big) - Q_\sigma(x)\Big) \quad \text{for all } x \in \mathbb{R},$$

so that the associated operator $F$ is given by

$$F[X](m) = -\lambda\Big(\frac{1}{2}(2m-1) - Q_\sigma(X(m))\Big) \quad \text{for all } m \in \Omega.$$

This corresponds to the Euler-Poisson system discussed in the Introduction. For simplicity, let us consider consider the case when $\sigma$ vanishes.

*Sticky solutions for the attractive Euler-Poisson system.* In the attractive case (when $\lambda > 0$) the functional $F[X]$ is sticking: Let $\Omega_X$ be defined by (1.33) and $(\alpha, \beta) \subset \Omega_X$ be a maximal interval. Then $\mathsf{P}_{\mathscr{H}_X}(F[X])$ is constant in $(\alpha, \beta)$ and equal to its average over the interval. We define

$$\Xi(m) := \int_\alpha^m \Big(F[X](m) - \mathsf{P}_{\mathscr{H}_X}(F[X])(m)\Big)\,\mathrm{d}m \quad \text{for all } m \in (\alpha, \beta). \tag{6.12}$$

Then $\Xi(\alpha) = \Xi(\beta) = 0$. Since $\lambda > 0$ and $\Xi$ is concave we have $\Xi(m) \geqslant 0$ in $(\alpha, \beta)$. By Lemma 2.3, we conclude that the functional $F$ is sticking. Sticky Lagrangian solutions to the Euler-Poisson system (1.1) (thus obtained as limit of sticky particle dynamics) are therefore unique and in fact form a semigroup in the metric space $(\mathscr{T}_2(\mathbb{R}), D_2)$, by Theorems 3.6, 3.10, and 3.11.

We can then apply the representation formula (3.26) and (2.9) to obtain the following result.

**Theorem 6.11** (Representation formula for attractive Euler-Poisson system)**.** *The unique sticky Lagrangian solution of the Euler-Poisson system ($\lambda \geqslant 0$) corresponding to initial data $(\bar\varrho, \bar v)$ with $\bar\varrho = \bar X_\# \mathfrak{m}$, $\bar X \in \mathscr{K}$, and $\bar V = \bar v \circ \bar X$, can be obtained by the formula*

$$\varrho(t,\cdot) = X(t,\cdot)_\# \mathfrak{m}, \quad X(t,m) = \frac{\partial}{\partial m}\mathcal{X}^{**}(t,m) \tag{6.13}$$

*where $\mathcal{X}^{**}(t,m)$ is the convex envelope (w.r.t. $m$, see (2.10)) of*

$$\mathcal{X}(t,m) := \int_0^m \Big(\bar X(\ell) + t\bar V(\ell) - \lambda\frac{t^2}{4}\big(2\ell - 1\big)\Big)\,\mathrm{d}\ell \tag{6.14}$$

Notice that when $\lambda = 0$ we find the sticky particle solution of [18].

*Lagrangian solutions for the repulsive Euler-Poisson system.* In the repulsive case $\lambda < 0$ the function $\Xi$ defined in (6.12) is convex and vanishes at the endpoints of $(\alpha, \beta)$, thus $\Xi(m) \leqslant 0$ for all $m \in (\alpha, \beta)$, and the map $F$ does not satisfies the sticking condition. In this case (6.13)-(6.14) may be different from the solution given by Theorem 3.6.

Here is a simple example for $\lambda = -2$: Consider the initial data

$$\bar X(m) := m - 1/2, \quad \bar V(m) := -\operatorname{sign}(m - 1/2), \tag{6.15}$$

for which (6.14) yields

$$\mathcal{X}(t,m) = \frac{1}{2}(1+t^2)(m-1/2)^2 - t|m-1/2| - c(t), \quad c(t) := \frac{1}{8}(1+t^2-4t). \tag{6.16}$$

It is easy to check that

$$\mathcal{X}^{**}(t,m) = \begin{cases} \mathcal{X}(t,m) & \text{if } |m-1/2| \geq \delta(t), \\ -\frac{t^2}{2(1+t^2)} - c(t) & \text{if } |m-1/2| \leq \delta(t), \end{cases} \qquad \text{where} \quad \delta(t) := \frac{t}{1+t^2}, \tag{6.17}$$

so that $X(t,\cdot)$ is the piecewise linear continuous map

$$X(t,m) = \begin{cases} \bar X(t,m) & \text{if } |m-1/2| \geq \delta(t), \\ 0 & \text{if } |m-1/2| \leq \delta(t), \end{cases} \qquad \text{where } \bar X(t,m) := \frac{\partial}{\partial m}\mathcal{X}(t,m).$$

If we introduce

$$Y(t,m) := \frac{\partial}{\partial t}\bar X(t,m) = \frac{\partial^2}{\partial t\,\partial m}\mathcal{X}(t,m) = 2t(m-1/2) - \operatorname{sign}(m-1/2),$$

we obtain (recalling (3.6)) that $X$ is a Lagrangian solution if and only if

$$Y(t,\cdot) - \dot X(t,\cdot) \in \partial I_{\mathscr{K}}(X(t,\cdot)) \quad \text{a.e. in } (0,\infty).$$

By Lemma 2.3, we find the equivalent condition

$$\frac{\partial}{\partial t}\big(\mathcal{X}(t,m) - \mathcal{X}^{**}(t,m)\big) \geq 0 \quad \text{in } -\delta(t) < m < \delta(t), \tag{6.18}$$

which is not compatible with (6.16) and (6.17). To see this, fix e.g. $0 < \delta < 1/2$, $m_* := 1/2 + \delta$, and $t_\pm := \frac{1 \pm \sqrt{1-4\delta^2}}{2\delta}$, so that $\delta(t_\pm) = \delta < \delta(t)$ for every $t \in (t_-, t_+)$. Then we have

$$\mathcal{X}(t_\pm, m_*) - \mathcal{X}^{**}(t_\pm, m_*) = 0, \quad \mathcal{X}(t, m_*) - \mathcal{X}^{**}(t, m_*) > 0 \quad \text{for } t_- < t < t_+,$$

which contradicts (6.18).

## 7. Convergence of the Time Discrete Scheme of Section 1.4

In this section, we establish the convergence of the time discrete scheme of Section 1.4. Since the proof does not substantially differ from the one provided in [8] for order-preserving vibrating strings, we only sketch the main steps. The key point is the non-expansive property of the time-discrete scheme. Indeed, we first observe that the rearrangement operator, even in the periodic case, is non-expansive in $\mathscr{L}^2(\Omega)$. More precisely, we have that

$$\int_0^1 |Y^*(m) - Z^*(m)|^2\,\mathrm{d}m \leqslant \int_0^1 |Y(m) - Z(m)|^2\,\mathrm{d}m$$

for all pairs $(Y,Z)$ of maps such that $Y-\mathrm{id}$ and $Z-\mathrm{id}$ are 1-periodic and square integrable. Next, we see that the harmonic oscillations (1.40) are isometric in phase space for $(X(t,m)-m, V(t,m))$, for each fixed $m$. Let $(X_{\tau,n}, V_{\tau,n})$, $(Y_{\tau,n}, W_{\tau,n})$ be generated by the time-discrete scheme. Then

$$\begin{aligned}
&\|X_{\tau,n+1}-Y_{\tau,n+1}\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}-W_{\tau,n+1}\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad\leqslant \|\hat{X}_{\tau,n+1}-\hat{Y}_{\tau,n+1}\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}-W_{\tau,n+1}\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad= \|X_{\tau,n}-Y_{\tau,n}\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n}-W_{\tau,n}\|^2_{\mathscr{L}^2(\Omega)}.
\end{aligned}$$

Since $(X=\mathrm{id}, V=0)$ is a trivial solution of the scheme, we immediately get

$$\|X_{\tau,n+1}-\mathrm{id}\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}\|^2_{\mathscr{L}^2(\Omega)}\leqslant \|\bar{X}-\mathrm{id}\|^2_{\mathscr{L}^2(\Omega)}+\|\bar{V}\|^2_{\mathscr{L}^2(\Omega)}.$$

Because the scheme is translation invariant in $m$ and (discretely) in $n$, we easily deduce the strong compactness in $\mathrm{C}^0_t(\mathscr{L}^2_m)$ of the discrete solutions, linearly interpolated in time, for each 1-periodic initial condition $(\bar{X}-\mathrm{id}, \bar{V})$, first in $\mathscr{H}^1$ and then (by a density argument, using the non-expansive property of the scheme) in $\mathscr{L}^2$. Let us now examine the consistency of the scheme. To do that, let us compare a solution of the discrete scheme to any smooth test function $m\to(Y(m),W(m))$ where $Y$ is nondecreasing and $(Y(m)-m, W(m))$ is 1-periodic. Since the rearrangement operator is non-expansive and $Y=Y^*$ is nondecreasing, we first get

$$\begin{aligned}
&\|X_{\tau,n+1}-Y\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}-W\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad\leqslant \|\hat{X}_{\tau,n+1}-Y\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}-W\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad=\int_0^1\Big\{\big|(X_{\tau,n}(m)-m)\cos(\tau)+V_{\tau,n}(m)\sin(\tau)-(Y(m)-m)\big|^2\\
&\qquad\qquad+\big|(X_{\tau,n}(m)-m)\sin(\tau)-V_{\tau,n}(m)\cos(\tau)+W(m)\big|^2\Big\}\,\mathrm{d}m.
\end{aligned}$$

One can then check that

$$\begin{aligned}
&\|X_{\tau,n+1}-Y\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n+1}-W\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad\leqslant \|X_{\tau,n}-Y\|^2_{\mathscr{L}^2(\Omega)}+\|V_{\tau,n}-W\|^2_{\mathscr{L}^2(\Omega)}\\
&\quad+2\tau\int_0^1\Big\{\big(X_{\tau,n}(m)-Y(m)\big)V_{\tau,n}(m)-\big(X_{\tau,n}(m)-m\big)\big(V_{\tau,n}(m)-W(m)\big)\Big\}\,\mathrm{d}m+\kappa\tau^2,
\end{aligned}$$

with constant $\kappa$ depending only on the test functions $(Y,W)$ and the initial data $(\bar{X},\bar{V})$. Clearly, this estimate is consistent with the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|X(t,\cdot)-Y\|^2_{\mathscr{L}^2(\Omega)}+\|V(t,\cdot)-W\|^2_{\mathscr{L}^2(\Omega)}\Big\}\\
&\quad\leqslant 2\int_0^1\Big\{\big(X(t,m)-Y(m)\big)V(t,m)-\big(X(t,m)-m\big)\big(V(t,m)-W(m)\big)\Big\}\,\mathrm{d}m,
\end{aligned}\tag{7.1}$$

valid for all pair of 1-periodic functions of form $m\mapsto(Y(m)-m, W(m))$ with $Y$ nondecreasing, which is nothing but the "metric formulation" of (1.36). Indeed, for a.e. $t\geqslant 0$ fixed, by choosing $Y=X(t,\cdot)$ and $W=V(t,\cdot)\pm Z$ for arbitrary 1-periodic $Z\in\mathscr{L}^2(\Omega)$, we find that

$$\dot{V}(t,m)+X(t,m)-m=0;$$

cf. (1.40). On the other hand, by choosing $W=V(t,\cdot)$ and $Y=0$ resp. $Y=2X(t,\cdot)$, we obtain

$$\int_0^1 X(t,m)(\dot{X}(t,m)-V(t,m))\,\mathrm{d}m=0$$

$$\int_0^1 Y(m)(\dot{X}(t,m)-V(t,m))\,\mathrm{d}m\geqslant 0\quad\text{for all } Y \text{ nondecreasing with } Y(m)-m \text{ 1-periodic.}$$

This implies precisely that $-\dot{X}(t,\cdot)+V(t,\cdot)\in\partial I_{\mathscr{K}}(X(t,\cdot))$, which gives (1.36). This concludes the proof of convergence for the time-discrete scheme.

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