HOMEWORK ASSIGNMENTS
245C, SPRING 2019

WILFRID GANGBO

Section 6.5.

Exercise 41. Assume $1 < p \leq \infty$ and let q be such that $p^{-1} + q^{-1} = 1$. When $p = \infty$ we further assume that μ is σ-finite. Assume T is a bounded operator on L^p be such that if $f, q \in L^p \cap L^q$ then

$$\int_X Tfg\mu(dx) = \int_X fTg\mu(dx).$$

Set $M := \|T\|_{L^p}$.

a) If $A \subset X$ is measurable and $\mu(A) < \infty$ then $\chi_A \in L^p \cap L^q$. For $f \in L^p \cap L^q$, we have

$$\int_X T(\chi_A) f d\mu = \int_X \chi_A Tf d\mu \leq \|\chi_A\|_{L^q} \|Tf\|_{L^p}.$$

As $L^p \cap L^q$ is dense in L^p, it holds that

$$\sup_{\|f\|_{L^p} \leq 1} \Big\{ \int_X T(\chi_A) f d\mu \Big\} \leq M \|\chi_A\|_{L^q}.$$

Since $\mu(\{x : \chi_A(x) \neq 0\}) = \mu(A) < \infty$, we apply Theorem 6.14 to conclude that $T(\chi_A) \in L^q$ and

$$\|T(\chi_A)\|_{L^q} \leq M \|\chi_A\|_{L^q}.$$

b) If $\{A_i\}_{i=1}^n$ are pairwise disjoint sets of finite measure and $\{a_i\}_{i=1}^n \subset \mathbb{C}$ then $g := \sum_{i=1}^n a_i \chi_{A_i} \in L^q$ and so, by a), $Tg \in L^q$. From the parallel density arguments in a), we have

$$\sup_{\|f\|_{L^p} \leq 1} \Big\{ \int_X Tgf d\mu \Big\} = \sup_{\|f\|_{L^p} \leq 1} \Big\{ \int_X gTf d\mu \Big\} \leq \sup_{\|f\|_{L^p} \leq 1} \Big\{ \|g\|_{L^q} \|Tf\|_{L^p} \Big\} \leq M \|g\|_{L^q}$$

As $\mu(\{x : g(x) \neq 0\}) = \sum_{i=1}^n \mu(A_i) < \infty$, we apply Theorem 6.14 to conclude that $T(g) \in L^q$ and

$$\|T(g)\|_{L^q} \leq M \|g\|_{L^q}.$$

c) If $(g_n)_n \subset L^q$ is a sequence of simple function converging to $g \in L^q$ then by b), $(Tg_n)_n \subset L^q$ is a Cauchy sequence and so, it admits a limit which we denote as Tg. One checks that Tg is independent of the sequence $(g_n)_n \subset L^q$ and T is linear.

d) Assume $f_p, f_p^* \in L^p$, $f_q, f_q^* \in L^q$ and $f = f_p + f_q = f_p^* + f_q^*$. We have $f_p^* - f_p = f_q - f_q^* \in L^p \cap L^q$. Thus, $T(f_p^* - f_p) = T(f_q - f_q^*)$ which proves that $T(f_p) + T(f_q) = T(f_p^*) + T(f_q^*)$. This shows that $T(f) := T(f_p) + T(f_q)$ is well–defined.

e) We have $T : L^p + L^q \to L^p + L^q$ is linear and has bounded restrictions $T : L^p \to L^p$ and $T : L^q \to L^q$. We use the Riesz-Thorin Interpolation Theorem to conclude.

Exercise 45. Let c_n be the volume of the unit ball in $\mathbb{R}^n$ and let $0 < \alpha < n$. Set

$K(x,y) = |x-y|^{-\alpha}$ for $x, y \in \mathbb{R}^n$ and let $\mathcal{F}$ be the set of $f : \mathbb{R}^n \to \mathbb{R}$ which a measurable and such that $K(x,\cdot)f \in L^1$ for $x \in \mathbb{R}^n$. For $f \in \mathcal{F}$, define

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y)f(y)dy.$$

We have

$$\lambda_{K(x,\cdot)}(t) = \mathcal{L}^n\Big\{y \in \mathbb{R}^n \mid |x-y| < t^{\frac{-1}{\alpha}}\Big\} = c_n t^{\frac{-n}{\alpha}}$$

Similarly, $\lambda_{K(\cdot,y)}(t) \leq c_n t^{\frac{-n}{\alpha}}$. We conclude that

$$[K(x,\cdot)]_{\frac{n}{\alpha}}, \quad [K(\cdot,y)]_{\frac{n}{\alpha}} \leq c_n.$$

Thus, $L^p \subset \mathcal{F}$ for any $1 \leq p < \infty$. From Theorem 6.36, the operator T is weak type $(1, \frac{n}{\alpha})$. Choose r such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{n}.$$

Then T is strong type (p, r). Note,

$$\frac{1}{r} = \frac{1}{p} + \frac{\alpha}{n} - 1 = \frac{1}{p} - \frac{n-\alpha}{n}.$$

Section 8.1.

Exercise 4. Let $f \in L^\infty$ and assume

$$\lim_{y \to 0} \|\tau_y f - f\|_\infty = 0.$$

For $\delta > 0$ we set

$$e(\delta) := \sup_{|y| \leq \delta} \|\tau_y f - f\|_\infty.$$

We choose a measurable set $N_y \subset \mathbb{R}^d$ such that $\mathbb{R}^d \setminus N_y$ is of null measure and

$$\|\tau_y f - f\|_\infty = \sup_{x \in N_y} |\tau_y f(x) - f(x)|.$$

Note that

$$|f(z-y) - f(z)| \leq e(|y|) \quad \forall y \in \mathbb{R}^d, \quad \forall z \in N_y \qquad \lim_{\delta \to 0} e(\delta) = 0. \tag{0.1}$$

1. If $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$ then

$$A_r f(x_2) - A_r f(x_1) = \frac{1}{c_d r^d} \int_{B_r(0)} \Big(f(x_2+u) - f(x_1+u)\Big) du = \frac{1}{c_d r^d} \int_{E(r)} \Big(f(x_2+u) - f(x_1+u)\Big) du.$$

where

$$E(r) := B_r(0) \cap \{x_2 + u \in N_{x_2 - x_1}\}$$

But by (0.1), if $u \in E(r)$

$$|f(x_2+u) - f(x_1+u)| \leq e(|x_2 - x_1|)$$

Thus,

$$|A_r f(x_2) - A_r f(x_1)| \leq e(|x_2 - x_1|). \tag{0.2}$$

This proves that $(A_r f)_r$ is uniformly continuous.

2. Let $\mathbb{L}$ be the set of Lebesgue point for f. Recall $\mathbb{L}$ is a set of full measure. If $x \in \mathbb{L}$ then $(A_r f(x))_r$ converges to $f(x)$ as r tend to 0. By (0.2)

$$|f(x_2) - f(x_1)| \leq e(|x_2 - x_1|) \qquad \forall x_1, x_2 \in \mathbb{L}. \tag{0.3}$$

If $(z_n)_n, (\bar{z}_n)_n \subset \mathbb{L}$ converge to z then by (0.3), $(f(z_n))_n$ and $(f(\bar{z}_n))_n$ are Cauchy sequence and so, they converge. One uses again (0.3) to check that they have the same limit which we denote as $g(x)$. We have g coincides with f on $\mathbb{L}$,

$$|g(x_2) - g(x_1)| \leq e(|x_2 - x_1|),$$

and so, g is uniformly continuous.

Department of Mathematics, UCLA, Los Angeles, CA 90095
E-mail address: wgangbo@math.ucla.edu