SOLUTIONS TO HOMEWORK ASSIGNMENTS 245C, SPRING 2019

WILFRID GANGBO

Section 6.4.

For $f : X \to \mathbb{C}$ and $\alpha > 0$, we set

$$E(\alpha, f) := \{x \in X \mid |f(x)| > \alpha\}$$

Exercise 35. (a) Since

$$E(\alpha, cf) = E\Big(\frac{\alpha}{c}, f\Big)$$

we conclude that $\lambda_{cf}(\alpha) = \lambda_f\Big(\frac{\alpha}{c}\Big)$. Hence, if $0 < p < \infty$

$$[cf]_p^p = \sup_{\alpha>0}\big\{\alpha^p \lambda_{cf}(\alpha)\big\} = c^p \sup_{\alpha>0}\Big\{\Big(\frac{\alpha}{c}\Big)^p \lambda_f\Big(\frac{\alpha}{c}\Big)\Big\} = c^p [f]_p^p.$$

(b) Since $|f+g| \leq |f| + |g|$ we conclude that $E(\alpha, f+g) \subset E(\alpha, |f|+|g|)$. Thus,

$$E(\alpha, f+g) \subset E\Big(\frac{\alpha}{2}, f\Big) \cup E\Big(\frac{\alpha}{2}, g\Big).$$

Thus,

$$\lambda_{f+g}(\alpha) \leq \lambda_f\Big(\frac{\alpha}{2}\Big) + \lambda_g\Big(\frac{\alpha}{2}\Big).$$

We conclude that

$$[f+g]_p^p = \sup_{\alpha>0}\big\{\alpha^p \lambda_{f+g}(\alpha)\big\} \leq \sup_{\alpha>0}\Big\{\alpha^p\Big(\lambda_f\Big(\frac{\alpha}{2}\Big) + \lambda_g\Big(\frac{\alpha}{2}\Big)\Big)\Big\}.$$

Hence,

$$[f+g]_p^p \leq 2^p \sup_{\alpha>0}\Big\{\Big(\frac{\alpha}{2}\Big)^p\Big(\lambda_f\Big(\frac{\alpha}{2}\Big) + \lambda_g\Big(\frac{\alpha}{2}\Big)\Big)\Big\} \leq 2^p\Big([f]_p^p + [g]_p^p\Big).$$

Exercise 36. (a) Set

$$c_0 := \mu\Big(\{x \in X \mid f(x) \neq 0\}\Big)$$

and assume $c_0 < \infty$. Then $\lambda_f \leq c_0$. Let $q \in (0,p)$ and assume $f \in L^p$. We have

$$\|f\|_{L^q}^q = q\int_0^1 \alpha^{q-1}\lambda_f(\alpha)d\alpha + q\int_1^\infty \alpha^{q-1}\lambda_f(\alpha)d\alpha \leq qc_0\int_0^1 \alpha^{q-1}d\alpha + q[f]_p^p\int_1^\infty \alpha^{q-p-1}d\alpha.$$

Integrating, we conclude

$$\|f\|_{L^q}^q \leq c_0 + \frac{q}{p-q}[f]_p^p.$$

(b) Assume $f \in L^\infty \cap L^p(\mu, \text{weak})$. Note $\lambda_f \equiv 0$ on $\big[b, \infty\big)$ where we have set $b := \|f\|_{L^\infty}$. If $q > p$ then $f \in L^q$ because

$$\|f\|_{L^q}^q = q\int_0^b \alpha^{q-p-1}\alpha^p\lambda_f(\alpha)d\alpha \leq q[f]_p^p\int_0^b \alpha^{q-p-1}d\alpha = \frac{qb^{q-p}}{q-p}[f]_p^p$$

Exercise 37. For $A>0$ we define $\phi_A:\mathbb{C}\to\mathbb{C}$ as $\phi_A(z)=z$ if $|z|<A$ and $\phi_A(z)=\frac{z}{|z|}A$ if $|z|\geq A$. Note that ϕ_A is continuous in $\mathbb{C}$ and has its range contained in the ball centered at the origin in $\mathbb{C}$ that has radius A. Given a measurable function $f:X\to\mathbb{C}$, we set

$$h_A:=\phi_A\circ f,\quad g_A:=f-h_A.$$

Note that $|h_A|\leq A$ and so, $\lambda_{h_A}\equiv 0$ on $[A,\infty)$.

Claim 1. If $\alpha\in(0,A)$ then $\lambda_f(\alpha)=\lambda_{h_A}(\alpha)$.

Proof of Claim 1. To prove the claim, it suffices to show that $E(\alpha,h_A)=E(\alpha,f)$. As $|h_A|\leq|f|$, this means it suffices to show that $E(\alpha,f)\subset E(\alpha,h_A)$.

Assume on the contrary $x\in E(\alpha,f)$ and $x\notin E(\alpha,h_A)$. Then $|h_A(x)|\leq\alpha<A$ and so, $f(x)=h_A(x)$. Thus, $|f(x)|=|h_A(x)|\leq\alpha$ which contradicts that fact that $x\in E(\alpha,f)$. We have shown that $E(\alpha,f)\subset E(\alpha,h_A)$ which concludes the proof of the claim.

Claim 2. We claim that if $\alpha>0$ then $\lambda_{g_A}(\alpha)=\lambda_f(A+\alpha)$.

Proof of Claim 2. To prove the claim, it suffices $E(\alpha,g_A)=E(A+\alpha,f)$.

Note $g_A\equiv 0$ on $\{x\in X\mid |f(x)|\leq A\}$. However, if $|f(x)|>A$ then

$$g_A(x)=f(x)\Big(1-\frac{A}{|f(x)|}\Big)=f(x)\Big(\frac{|f(x)|-A}{|f(x)|}\Big)$$

and so,

$$|g_A(x)|=|f(x)|-A>0.$$

Thus, if $\alpha>0$

$$E(\alpha,g_A)=\Big\{x\in X\mid |f(x)|-A>\alpha\Big\}=\Big\{x\in X\mid |f(x)|>\alpha+A\Big\}=E(A+\alpha,f).$$

Exercise 38. Assume $f:X\to\mathbb{C}$ is measurable.

Case 1. We assume $p\geq 1$. For any $k\in\mathbb{Z}$ and any $\alpha\in\big[2^k,2^{k+1}\big]$ we have $\lambda_f(\alpha)\in\Big[\lambda_f\big(2^{k+1}\big),\lambda_f\big(2^k\big)\Big]$ and so, since $p-1\geq 0$ we obtain

$$2^{k(p-1)}\lambda_f\big(2^{k+1}\big)\leq\alpha^{(p-1)}\lambda_f(\alpha)\leq 2^{(k+1)(p-1)}\lambda_f\big(2^k\big).\tag{0.1}$$

Thus,

$$\Big(2^{k+1}-2^k\Big)2^{k(p-1)}\lambda_f\big(2^{k+1}\big)\leq\int_{2^k}^{2^{k+1}}\alpha^{(p-1)}\lambda_f(\alpha)d\alpha\leq\Big(2^{k+1}-2^k\Big)2^{(k+1)(p-1)}\lambda_f\big(2^k\big).$$

This reads off

$$2^{-p}2^{(k+1)p}\lambda_f\big(2^{k+1}\big)\leq\int_{2^k}^{2^{k+1}}\alpha^{(p-1)}\lambda_f(\alpha)d\alpha\leq 2^{(p-1)}2^{kp}\lambda_f\big(2^k\big).$$

We conclude that

$$2^{-p}p\sum_{k\in\mathbb{Z}}2^{kp}\lambda_f\big(2^k\big)\leq\|f\|_{L^p}^p\leq p2^{(p-1)}\sum_{k\in\mathbb{Z}}2^{kp}\lambda_f\big(2^k\big).$$

Case 2. We assume $0<p<1$. In this case, instead of (0.2) we have

$$2^{(k+1)(p-1)}\lambda_f\big(2^{k+1}\big)\leq\alpha^{(p-1)}\lambda_f(\alpha)\leq 2^{k(p-1)}\lambda_f\big(2^k\big).\tag{0.2}$$

Thus,

$$2^{-1}2^{(k+1)p}\lambda_f\big(2^{k+1}\big)\leq\int_{2^k}^{2^{k+1}}\alpha^{(p-1)}\lambda_f(\alpha)d\alpha\leq 2^{kp}\lambda_f\big(2^k\big).$$

As above, we conclude that $\|f\|_{L^p}^p<\infty$ if and only if $\sum_{k\in\mathbb{Z}}2^{kp}\lambda_f\big(2^k\big)<\infty$.

Exercise 40. Monotone rearrangement of functions are well studied in the literature. If μ is a Borel measure on $\mathbb{R}$, one defines

$$M_\mu(x) = \mu(-\infty, x].$$

Note that M_μ is monotone nondecreasing and maps $\mathbb{R}$ into $[0,\infty]$. If we denote as N_μ the generalized inverse of M_μ, one shows that N_μ pushes $\mathcal{L}^1_{(0,1)}$ forward to μ.

Department of Mathematics, UCLA, Los Angeles, CA 90095
E-mail address: wgangbo@math.ucla.edu