Exercise 3.1. Let $(X, T)$ be a topological space and let $B \subset X$.
(i) Show that $\text{int}(B)$ is the union of the open sets contained in $B$.
(ii) Show the $\overline{B}$ is the intersection of the closed sets containing $B$.

Exercise 3.2. Let $(X, T)$ be a topological space and let $B \subset X$. Show that $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}(J) \in T$ for every open set $J \subset \mathbb{R}$.

Exercise 3.3. Let $(X, T)$ be a topological space and let $K \subset X$.
(i) Show that if $K$ is closed and $X$ is compact, then $K$ is compact.
(ii) Show that if $X$ is a Hausdorff space and $K$ is compact then $K$ is closed.
Hint. (i) Use the fact that if $\{O_i\}_{i \in I}$ is an open cover of $K$, then $\{K^c\} \cup \{O_i\}_{i \in I}$ is an open cover of $X$. (ii) Show that for every $x \notin K$ there exist two disjoint open sets $U, V$ such that $x \in U$ and $F \subset V$.

Exercise 3.4 (*). Let $X$ be an LCH space and let $\mu$ be a Radon measure on $X$.
(i) Let $N$ be the union of all open $U \subset X$ such that $\mu(U) = 0$. Show that $\mu(N) = 0$. The complement of $N$, denoted by $\text{spt}(\mu)$ is called the support of $\mu$.
(ii) Show that $x \in \text{spt}(\mu)$ if and only if $\int_X f d\mu > 0$ for every $f \in C_c(X, [0, 1])$ such that $f(x) > 0$.

Exercise 3.5 (*). Let $X$ be an LCH space and let $\mu$ be a Radon measure on $X$. Show that $\mu$ is inner regular on every $\sigma$–finite set.

Exercise 3.6 (*). Let $X = \mathbb{N}$ with the discrete topology. Show that $C_0(X)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$.

Exercise 3.7. Let $p \in [1, +\infty)$ and suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that
$$\lim_{y \to 0} \left\| \frac{\tau_y f - f - h}{y} \right\|_p = 0,$$
we call $h$ the strong $L^p$–derivative of $f$. If $f \in L^p(\mathbb{R}^d)$, $L^p$–partial derivatives of $f$ are defined similarly.
Suppose that $p$ and $q$ are conjugate exponents, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and $\partial_j f$, the strong $L^p$–partial derivatives of $f$ exists. Show that the ordinary derivative $\frac{\partial}{\partial x_j}(f * g)$ exist and equal $(\partial_j f) * g$.

Exercise 3.8 (*). Let $p \in [1, +\infty)$ and suppose that $f \in L^p(\mathbb{R})$. Show that the following are equivalent
(i) The strong $L^p$–derivative of $f$ exists, call it $h$. 

\* \* \*
(ii) $f$ is absolutely continuous on every bounded interval (perhaps after modification on a null set) and its pointwise derivative $f'$ is in $L^p$, in which case $h = f'$.

For “only if” use exercise 3.7 with $g \in C_c(\mathbb{R})$ such that $\int_{\mathbb{R}} g = 1$, $f * g_t \to f$ and $(f * g_t)' \to h$ as $t \to 0$. For “if”, write

$$\frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y} \int_0^y \left( f'(x + t) - f'(x) \right) dt.$$