HOMEWORK ASSIGNMENTS
245C, SPRING 2026

WILFRID GANGBO

1. Homework #1; Due Wednesday 15 April

In this homework, $(\Omega, \mathcal{M}, \mu)$ is an outer measure space and $f : \Omega \to [-\infty, +\infty]$ is measurable function which assume finite values μ–almost everywhere.

Exercise 1.1. *Let $g \in C(\mathbb{R}^d)$ be such that for all $a \in \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that*

$$g(x) \geq g(a) + (x - a, p) \qquad \forall x \in \mathbb{R}^d.$$

Show that g is convex.

Exercise 1.2 (∗)**.** *Let $g \in C^2(\mathbb{R}^d)$. Show that g is convex if and only if $\nabla^2 g \geq 0$.*

Exercise 1.3. *Let $p \in [1, +\infty)$.*

(i) *Let $g : \Omega \to [-\infty, +\infty]$ be measurable functions which assume finite values μ–almost everywhere. Show that if $c \in \mathbb{R}$ then $[cf]_p = |c|[f]_p$, and $[f + g]_p \leq 2\big([f]_p^p + [g]_p^p\big)^{1/p}$.*
(ii) *Deduce that* weak$(L^p(\Omega))$ *is a vector space.*
(iii) *Show that the "balls" $\{g \in \text{weak}(L^p(\Omega)) : [g - f]_p < r\}_{\{f \in \text{weak}(L^p(\Omega)), r>0\}}$ generate a topology on* weak$(L^p(\Omega))$ *that makes* weak$(L^p(\Omega))$ *into a topological vector space.*

Exercise 1.4 (∗)**.** *Let $p \in [1, +\infty)$.*

(i) *Show that if $f \in \text{weak}(L^p(\Omega))$ and $\mu\big(\{x \in \Omega : \ f(x) \neq 0\}\big) < +\infty$ then $f \in L^q(\Omega)$ for any $q \in [1, p)$.*
(ii) *Show that if $f \in \text{weak}(L^p(\Omega)) \cap L^\infty(\Omega)$ then $f \in L^q(\Omega)$ for any $q \in (p, +\infty)$.*

Exercise 1.5 (∗)**.** *Let $E(A) = \{|f| > A\}$ and set*

$$h_A = f\chi_{\Omega \setminus E(A)} + A\,\text{sgn}(f)\chi_{E(A)}, \quad g = f - h_A.$$

Show that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \qquad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if} \quad \alpha < A \\ 0 & \text{if} \quad \alpha \geq A \end{cases}$$

Exercise 1.6 (∗)**.** *Let $p \in (0, +\infty)$. Show that $f \in L^p(\Omega)$ if and only if $\sum_{k=-\infty}^{+\infty} 2^{pk}\lambda_f(2^k) < +\infty$.*

Exercise 1.7. *Let $p \in (0, +\infty)$. Show that if $f \in L^p(\Omega)$ then $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0$.*

2. Homework #2; Due Friday 01 May

Notation. Given a closed ball $B_r(a)$ of center a and radius $r>0$, we denote by $\hat{B}_r(a)$ the ball of center a and radius $5r$.

Exercise 2.1. *Let $f:\mathbb{R}^d\to[-\infty,+\infty]$ be a $\mathcal{L}^d$–measurable function. Show that if $f\in L^\infty(\mathbb{R}^d)$ then $\|f\|_\infty=\inf_{\alpha>0}\big\{\alpha:\ \lambda_f(\alpha)=0\big\}$.*

Exercise 2.2 (∗)**.** *Let N be sets of finite Lebesgue measure.*

(i) *Show that if $\mathcal{L}^d(N)=0$ then for every $\delta,\epsilon>0$, there exists a family $\{B_j\}_{i=1}^\infty$ of disjoint non degenerate closed balls such that* $\operatorname{diam}(B_i)<\delta$,

$$N\subset\cup_{i=1}^\infty\hat{B}_i\quad\textit{and}\quad\sum_{i=1}^\infty\mathcal{L}^d(\hat{B}_i)<\epsilon.$$

(ii) *Show that if $A\subset\mathbb{R}^d$ is a set of finite Lebesgue measure then for every $\delta,\epsilon>0$ there exists a family $\{C_j\}_{j=1}^\infty$ of disjoint non-degenerate closed balls such that* $\operatorname{diam}(C_j)<\delta$ *and there exists a family $\{B_j\}_{i=1}^\infty$ of disjoint non degenerate closed balls such that* $\operatorname{diam}(B_i)<\delta$ *and*

$$A\subset\big(\cup_{j=1}^\infty C_j\big)\cup\big(\cup_{i=1}^\infty\hat{B}_i\big),\qquad\sum_{j=1}^\infty\mathcal{L}^d(C_j)+5^d\sum_{i=1}^\infty\mathcal{L}^d(B_i)\le\mathcal{L}^d(A)+2\epsilon.$$

Exercise 2.3 (∗)**.** *Let $f:\mathbb{R}^d\to\mathbb{R}^d$ be an l–Lipschitz function. Show that if $A\subset\mathbb{R}^d$ then $\mathcal{L}^d\big(f(A)\big)\le l^d\mathcal{L}^d(A)$.*

Exercise 2.4 (∗)**.** *Let $O\subset\mathbb{R}^d$ be an open set and let $f\in C^1(O,\mathbb{R}^d)$. Let $a\in O$ and denote by $B_r(a)$ the closed ball of radius r, centered at a and by $D_r(a)$ the interior of $B_r(a)$. Set*

$$L(x):=f(a)+\nabla f(a)(x-a).$$

Show that if $\det(\nabla f)(a)>0$ *then for $r>0$ small enough*

$$f\big(B_r(a)\big)\subset L\Big(B_{r+o(r)}(a)\Big)\quad\textit{and}\quad L\Big(B_r(a)\Big)\subset f\Big(B_{r+o(r)}(a)\Big).$$

Hint. By the inverse function theorem, there exists $r_0>0$ such that $B_{r_0}(a)\subset O$, $f(D_r(a))$ is open, $f:D_{r_0}(a)\to f(D_{r_0}(a))$ is a bijection with a continuous inverse.

Exercise 2.5 (∗)**.** *Let $f:\mathbb{R}^d\to\mathbb{R}^d$ be a l–Lipschitz function. We learned that if $A\subset\mathbb{R}^d$ is $\mathcal{L}^d$–measurable then $y\to\mathcal{H}^0\big(f^{-1}(y)\cap A\big)$ is a $\mathcal{L}^d$–measurable function and so, we can define*

$$\bar{\mu}(A)=\int_{\mathbb{R}^d}\mathcal{H}^0\big(f^{-1}(y)\cap A\big)dy.$$

Show that $\bar{\mu}$ can be extended to an outer measure μ which is a Radon measure on $\mathbb{R}^d$. Show that every $\mathcal{L}^d$–measurable set is μ–measurable.

Exercise 2.6 (∗)**.** *Let $f:\mathbb{R}^d\to\mathbb{R}^d$ be as in Exercise 2.4 and let μ be as in Exercise 2.5.*

(i) *Show that $\mu<<\mathcal{L}^d$ (μ is absolutely continuous with respect to $\mathcal{L}^d$).*
(ii) *Show that if $a\in\mathbb{R}^d$ is such that* $\det(\nabla f(a))>0$, *then*

$$\lim_{r\to0}\frac{\mathcal{L}^d\big[f\big(B_r(a)\big)\big]}{r^d}=\lim_{r\to0}\frac{\mathcal{L}^d\big[L\big(B_r(a)\big)\big]}{r^d}.$$

3. Homework #3: Due Friday 15 May

Exercise 3.1 (∗). *Show that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with the semi–norms $\|\cdot\|_{(N,\beta)}$ is a Fréchet space.*

Exercise 3.2. *Let $(X,\mathcal{T})$ be a topological space and let $B \subset X$.*
(i) Show that $\mathrm{int}(B)$ is the union of the open sets contained in B.
(ii) Show the $\bar{B}$ is the intersection of the closed sets containing B.

Exercise 3.3. *Let $(X,\mathcal{T})$ be a topological space and let $B \subset X$. Show that $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}(J) \in \mathcal{T}$ for every open set $J \subset \mathbb{R}$.*

Exercise 3.4. *Let $(X,\mathcal{T})$ be a topological space and let $K \subset X$.*
(i) Show that if K is closed and X is compact, then K is compact.
(ii) Show that if X is a Hausdorff space and K is compact then K is closed.

Hint. (i) Use the fact that if $\{O_i\}_{i\in I}$ is an open cover of K, then $\{K^c\} \cup \{O_i\}_{i\in I}$ is an open cover of X. (ii) Show that for every $x \notin K$ there exist two disjoint open sets U, V such that $x \in U \subset K^c$ and $K \subset V$.

Exercise 3.5 (∗). *Let X be an LCH space and let μ be a Radon measure on X.*
(i) Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Show that $\mu(N) = 0$. The complement of N, denoted by $\mathrm{spt}(\mu)$ is called the support of μ.
(ii) Show that $x \in \mathrm{spt}(\mu)$ if and only if $\int_X f d\mu > 0$ for every $f \in C_c(X,[0,1])$ such that $f(x) > 0$.

Exercise 3.6 (∗). *Let X be an LCH space and let μ be a Radon measure on X. Show that μ is inner regular on every σ–finite set.*

Exercise 3.7 (∗). *Let $X = \mathbb{N}$ with the discrete topology. Show that $C_0(X)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$.*

Exercise 3.8. *Let $p \in [1,+\infty)$ and suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that*

$$\lim_{y\to 0} \left\| \frac{\tau_{-y} f - f}{y} - h \right\|_p = 0,$$

we call h the strong L^p–derivative of f. If $f \in L^p(\mathbb{R}^d)$, L^p–partial derivatives of f are defined similarly.

Suppose that p and q are conjugate exponents, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and $\partial_j f$, the strong L^p–partial derivatives of f exists. Show that the ordinary derivative $\frac{\partial}{\partial x_j}(f * g)$ exist and equal $(\partial_j f) * g$.

Exercise 3.9 (∗). *Let $p \in [1,+\infty)$ and suppose that $f \in L^p(\mathbb{R})$. Show that the following are equivalent*

(i) *The strong L^p–derivative of f exists, call it h.*
(ii) *f is absolutely continuous on every bounded interval (perhaps after modification on a null set) and its pointwise derivative f' is in L^p, in which case $h = f'$.*

For "only if" use exercise 3.8 with $g \in C_c(\mathbb{R})$ such that $\int_{\mathbb{R}} g = 1$, $f * g_t \to f$ and $(f * g_t)' \to h$ as $t \to 0$. For "if", write

$$\frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y}\int_0^y \big(f'(x+t) - f'(x)\big)dt.$$

4. Homework #4: Due Friday 29 May

Exercise 4.1. *Let $O \subset \mathbb{R}^d$ be an open set and let $u \in C^2(O)$ be a harmonic function in the sense that $\triangle u = 0$ on O. Show that if $r > 0$, $x \in O$ and $B_r(x) \subset O$ then if ν is the surface measure ((d − 1)-Hausdorff dimensional measure) then*

$$u(x) = \frac{1}{\nu(\partial B_r(x))} \int_{\partial B_r(x)} u d\nu = \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} u dy$$

Hint. Set

$$\phi(r) = \frac{1}{\nu(\partial B_r(x))} \int_{\partial B_r(x)} u d\nu.$$

Show that

$$\phi'(r) = \frac{1}{\nu(\partial B_1(0))} \int_{\partial B_1(0)} \nabla u(x + rw) \cdot w \nu(dw) = 0.$$

Use the change of variables formula

$$\int_{B_r(x)} u dy = \int_0^r \Big(\int_{\partial B_s(x)} u d\nu \Big) ds$$

Exercise 4.2 (∗)**.** *Let $O \subset \mathbb{R}^d$ and let $u \in C^2(O)$ be a harmonic function. Show that $u \in C^\infty(O)$.*

Hint. Let $(\varrho_\epsilon)_\epsilon$ be the standard mollifiers. Use Exercise 4.1 to show that $\varrho_\epsilon * u = u$.

Exercise 4.3. *Assume that $D \subset \mathbb{C}$ is an open set and $f : D \to \mathbb{C}$ is differentiable on D. Show that $u : (x, y) \to \mathrm{Re}(f(x + iy))$ and $v : (x, y) \to \mathrm{Im}(f(x + iy))$ are differentiable on D and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Conclude that if u and v are of class C^2 then they are harmonic functions.

Exercise 4.4. *If $f \in C^\infty$ show that $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β if and only if $\partial^\alpha(x^\beta f)$ is bounded for all multi-indices α, β.*

Exercise 4.5 (∗)**.** *Suppose that Σ is a σ–algebra and (X, Σ, μ) is a measure space. Suppose that $-\infty < a < b < +\infty$ and $f : X \times [a, b] \to \mathbb{R}$ is such that $f(\cdot, t) \in L^1(\mu)$ for each $t \in [a, b]$. Let*

$$F(t) := \int_X f(x, t)\mu(dx)$$

(i) *Suppose there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all $x \in X$ and $t \in [a, b]$. Show that if $\lim_{t \to t_0} f(x, t) = f(x, t_0)$ for every $x \in X$ then*

$$\lim_{t \to t_0} F(t) = F(t_0).$$

(ii) *Suppose that $\partial f / \partial t$ exists and there exists $h \in L^1(X)$ such that $|(\partial f / \partial t)(t, x)| \leq h(x)$ for all $x \in X$ and $t \in [a, b]$. Show that F is differentiable and*

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)\mu(dx).$$

Exercise 4.6 (∗)**.** *Suppose that $f \in L^1$, $g \in C^k$ and $\partial^\alpha g$ is bounded for $|\alpha| \leq k$. Show that $f * g \in C^k$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.*

Exercise 4.7 (∗)**.** *Suppose that $0 < p < q < r \leq \infty$.*

(i) *Show that* $L^q \subset L^p + L^r$.
(ii) *Let* $\lambda \in (0,1)$ *be defined by* $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$. *Show that* $L^p \cap L^r \subset L^q$ *and* $\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$

Proof. (i) Let $f \in L^q$ and set

$$E := \{|f| > 1\}, \quad g := f\chi_E, \quad h := f\chi_{E^c}.$$

We have $f = g + h$, $|g|^p = |f|^p \chi_E \le |f|^q \chi_E$ and $|h|^r = |f|^r \chi_{E^c} \le |f|^q \chi_{E^c}$.

(ii) We only consider the case $r < +\infty$ since the other case is rather straightforward to treat. We have

$$\|f\|_q^q = \int_\Omega |f|^{\lambda q} |f|^{(1-\lambda)q} d\mu \le \big\| |f|^{\lambda q} \big\|_{\frac{p}{\lambda q}} \big\| |f|^{(1-\lambda)q} \big\|_{\frac{r}{(1-\lambda)q}} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q}$$

□

Exercise 4.8. *Let* $f(x) = 1/2 - x$ *on* $[0,1)$ *and extend* f *periodically to* $\mathbb{R}$.

(i) *Show that*

$$\hat{f}(0) = 0, \quad \textit{and} \quad \hat{f}(k) = \frac{1}{2\pi i k}, \quad \textit{if} \quad k \neq 0.$$

(ii) *Show that (use Parseval inequality)*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Exercise 4.9 ((∗) Wirtinger's inequality)**.** *Show that if* $f \in C^1([a,b])$ *is such that* $f(a) = f(b) = 0$ *then*

$$\int_a^b f^2(x)dx \le \frac{(b-a)^2}{\pi^2} \int_a^b (f'(x))^2 dx$$

Hint: It suffices to prove the result when $a = 0$ *and* $b = 1/2$. *Extend* f *to* $[-0.5, 0.5]$ *by setting* $f(-x) = -f(x)$ *and then extend* f *periodically on* $\mathbb{R}$. *Check that* $f \in C^1(\mathbb{T})$ *and apply Parseval inequality.*

5. Homework #5

Exercise 5.1. *For* $\alpha \in (0,1]$ *let* $\Lambda_\alpha(\mathbb{T})$ *be the space of Hölder continuous functions on* $\mathbb{T}$ *of exponent* α. *Let* $p, q \in (1, +\infty)$ *be such that* $1/p + 1/q = 1$.

(i) *Show that if* f *is periodic and absolutely continuous on* $\mathbb{R}$, *and* $f' \in L^p(\mathbb{T})$ *then* $f \in \Lambda_{1/q}(\mathbb{T})$, *but* f *need not lie in* $\Lambda_\alpha(\mathbb{T})$ *for any* $\alpha > 1/q$. *(Hint:* $f(b) - f(a) = \int_a^b f'(t)dt$.
(ii) *Show that if* $\alpha < 1$, $\Lambda_\alpha(\mathbb{T})$ *contains functions that are not of bounded variations and hence are not absolutely continuous.*

Recall that if $f(x) = e^{-\pi a |x|^2}$ then

$$\hat{f}(\xi) = a^{-\frac{d}{2}} e^{-\pi \frac{|\xi|^2}{a}}. \tag{5.1}$$

Exercise 5.2. *The aim of this exercise is to show that the Fourier transform of* $e^{-2\pi|\xi|}$ *on* $\mathbb{R}^d$ *is*

$$\phi(x) = \frac{\Gamma\big(\frac{1}{2}(d+1)\big)}{\pi^{(d+1)/2}} \frac{1}{(1+|x|^2)^{(d+1)/2}}.$$

(i) *Show that if* $\beta \geq 0$ *then*

$$\pi e^{-\beta} = \int_{\mathbb{R}} \frac{e^{-it\beta}}{1+t^2} dt.$$

(ii) *Show that if* $\beta \geq 0$ *then (use* (5.1)*)*

$$e^{-\beta} = \int_0^{+\infty} \frac{1}{\sqrt{\pi s}} e^{-s} e^{-\frac{\beta^2}{4s}} ds.$$

(iii) *Compute the Fourier transform of of* $x \to e^{-2\pi|\xi|}$ *is* ϕ *(Set* $\beta = 2\pi|\xi|$ *where* $\xi \in \mathbb{R}^d$ *and use* (5.1)*).*

Exercise 5.3. *Show that if* $f \in L^1(\mathbb{R}^d)$, f *is continuous at* 0 *and* $\hat{f} \geq 0$ *then* $\hat{f} \in L^1(\mathbb{R}^d)$.

Department of Mathematics, UCLA, Los Angeles, CA 90095
Email address: wgangbo@math.ucla.edu