HOMEWORK ASSIGNMENTS: MATH 245 B

WILFRID GANGBO

Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

You are allowed to use Exercise N to solve Exercise $N+1$ even if you failed to answer correctly Exercise N; but not allowed to go the other way around. In Exercise N, you are allowed to use question k to solve question $k+1$ even if you failed to answer correctly question k.

Below, X is a non empty set.

Definition 1. *We call $\mathcal{M} \subset 2^X$ a monotone class if for any $(A_n)_{n=1}^{\infty}, (B_n)_{n=1}^{\infty} \subset \mathcal{M}$ such that $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$ for all n, we have $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$ and $\cap_{n=1}^{\infty} B_n \in \mathcal{M}$.*

1. Homework #1: Due on Friday 14 January

Exercise 1.1. *Let $\mathcal{S}$ be a semialgebra on X and let $\mathcal{A}$ be the collection of subsets of X which consist of finite unions of elements of $\mathcal{S}$. Show that $\mathcal{A}$ is an algebra on X.*

Exercise 1.2 $(*)$**.** *Suppose $\mathcal{S}$ is a semialgebra on X and μ is a finitely additive function on $\mathcal{S}$. Suppose $E_1, \cdots, E_n \in \mathcal{S}$ are disjoint, $F_1, \cdots, F_m \in \mathcal{S}$ are disjoint and $\cup_{i=1}^{n} E_i \in \mathcal{S}$.*

(i) Show that if $\cup_{i=1}^{n} E_i = \cup_{j=1}^{m} F_j$ then

$$\sum_{i=1}^{n} \mu(E_i) = \sum_{j=1}^{n} \mu(F_j).$$

(ii) Show that μ extends uniquely to a finitely additive set function on the set $\mathcal{A}$ defined in Exercise 1.1.

Exercise 1.3 $(*)$**.** *Assume $\mathcal{A}$ is an algebra on X and $\mu_0 : \mathcal{A} \to [0, +\infty]$ is such that*

(i) $\mu_0(\emptyset) = 0$,

(ii) $E, F \in \mathcal{A}$ and $E \cap F = \emptyset$ implies $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$.

(iii) If $\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$ is such that $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$ then $\mu_0(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu_0(E_i)$.

Define

$$\mu^*(A) := \inf_{\{E_i\}_{i=1}^{\infty}} \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : \{E_i\}_{i=1}^{\infty} \subset \mathcal{A}, \ \ A \subset \cup_{i=1}^{\infty} E_i \right\}.$$

Show that μ^* is an outer measure on X, μ^* coincides with μ_0 on $\mathcal{A}$ and every element of $\mathcal{A}$ is μ^*–measurable.

Exercise 1.4 $(*)$**.** *Let $\mathcal{S}$ be a semialgebra on X and assume that $\mu_0 : \mathcal{S} \to [0, +\infty]$ is such that $\mu_0(\emptyset) = 0$. Show that if μ_0 is finitely additive and countably sub–additive then so is also its extension (guaranteed by Exercise 1.2) to the algebra $\mathcal{A}$ generated by taking all finite unions of sets from $\mathcal{S}$.*

Exercise 1.5 $(*)$**.** *(Monotone Class Lemma) Let $\mathcal{A}, \mathcal{M} \subset 2^X$ be such that $\mathcal{M}$ is a monotone class and $\mathcal{A}$ is an algebra. Show that if $\mathcal{A} \subset \mathcal{M}$ then $\sigma(\mathcal{A}) \subset \mathcal{M}$.*

Hint: It suffices to show that $m(\mathcal{A})$, the intersection of all monotone classes on X which contain $\mathcal{A}$, is nothing but $\sigma(\mathcal{A})$. To achieve this goal, we could go through the following steps:

(i) Show that if $\mathcal{M}_0$ is a monotone class, so is $\mathcal{M}_0' := \{E^c : E \in \mathcal{M}_0\}$ and apply this to show that $m(\mathcal{A})$ is closed under complement.

(ii) For $E \subset X$, show that $\mathcal{M}_E := \{C \in m(\mathcal{A}) : C \cap E \in m(\mathcal{A})\}$ is a monotone class.

(iii) Show that for any $C \in \mathcal{A}$, we have $m(\mathcal{A}) \subset \mathcal{M}_C$. Deduce that the same conclusions hold for $C \in m(\mathcal{A})$ and show that $m(\mathcal{A})$ is an algebra.

Exercise 1.6. *Let $\mathcal{A} \subset 2^X$ be an algebra and let μ and ν be finite measures on $\sigma(\mathcal{A})$. Show that*

$$\mu = \nu \quad \text{on} \quad \mathcal{A} \qquad \Longrightarrow \qquad \mu = \nu \quad \text{on} \quad \sigma(\mathcal{A}).$$

Show that the result remains true if $\mu(X) = \infty$ and there exists $\{X_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $X = \cup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$.

Hint: Assume $\mathcal{F}$ is a σ-algebra on which μ and ν are finite measures. Apply the monotone class lemma to $\mathcal{M} := \{A \in \mathcal{F} : \mu[A] = \nu[A]\}$.

Department of Mathematics, UCLA, Los Angeles, CA 90095

Email address: wgangbo@math.ucla.edu