HOMEWORK ASSIGNMENTS: MATH 245 B

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Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

You are allowed to use Exercise N to solve Exercise $N+1$ even if you failed to answer correctly Exercise N; but not allowed to go the other way around. In Exercise N, you are allowed to use question k to solve question $k+1$ even if you failed to answer correctly question k.

1. Homework #1: Due on Friday 20 January

Below, X is a non empty set.

Definition 1. *We call $\mathcal{M} \subset 2^X$ a monotone class if for any $(A_n)_{n=1}^\infty, (B_n)_{n=1}^\infty \subset \mathcal{M}$ such that $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$ for all n, we have $\cup_{n=1}^\infty A_n \in \mathcal{M}$ and $\cap_{n=1}^\infty B_n \in \mathcal{M}$.*

Exercise 1.1. *Let $\mathcal{S}$ be a semialgebra on X and let $\mathcal{A}$ be the collection of subsets of X which consist of finite unions of elements of $\mathcal{S}$. Show that $\mathcal{A}$ is an algebra on X.*

Exercise 1.2 $(*)$**.** *Suppose $\mathcal{S}$ is a semialgebra on X and μ is a finitely additive function on $\mathcal{S}$. Suppose $E_1, \cdots, E_n \in \mathcal{S}$ are disjoint, $F_1, \cdots, F_m \in \mathcal{S}$ are disjoint and $\cup_{i=1}^n E_i \in \mathcal{S}$.*

(i) Show that if $\cup_{i=1}^n E_i = \cup_{j=1}^m F_j$ then

$$\sum_{i=1}^n \mu(E_i) = \sum_{j=1}^n \mu(F_j).$$

(ii) Show that μ extends uniquely to a finitely additive set function on the set $\mathcal{A}$ defined in Exercise 1.1.

Exercise 1.3 $(*)$**.** *Assume $\mathcal{A}$ is an algebra on X and $\mu_0 : \mathcal{A} \to [0, +\infty]$ is such that*

(i) $\mu_0(\emptyset) = 0$,

(ii) $E, F \in \mathcal{A}$ and $E \cap F = \emptyset$ implies $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$.

(iii) If $\{E_i\}_{i=1}^\infty \subset \mathcal{A}$ is such that $\cup_{i=1}^\infty E_i \in \mathcal{A}$ then $\mu_0(\cup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu_0(E_i)$.

Define

$$\mu^*(A) := \inf_{\{E_i\}_{i=1}^\infty} \left\{ \sum_{i=1}^\infty \mu_0(E_i) : \{E_i\}_{i=1}^\infty \subset \mathcal{A}, \;\; A \subset \cup_{i=1}^\infty E_i \right\}.$$

Show that μ^* is an outer measure on X, μ^* coincides with μ_0 on $\mathcal{A}$ and every element of $\mathcal{A}$ is μ^*–measurable.

Exercise 1.4 (∗)**.** *Let $\mathcal{S}$ be a semialgebra on X and assume that $\mu_0 : \mathcal{S} \to [0, +\infty]$ is such that $\mu_0(\emptyset) = 0$. Show that if μ_0 is finitely additive and countably sub–additive then so is also its extension (guaranteed by Exercise 1.2) to the algebra $\mathcal{A}$ generated by taking all finite unions of sets from $\mathcal{S}$.*

Exercise 1.5 (∗)**.** *(Monotone Class Lemma) Let $\mathcal{A}, \mathcal{M} \subset 2^X$ be such that $\mathcal{M}$ is a monotone class and $\mathcal{A}$ is an algebra. Show that if $\mathcal{A} \subset \mathcal{M}$ then $\sigma(\mathcal{A}) \subset \mathcal{M}$.*

Hint: It suffices to show that $m(\mathcal{A})$, the intersection of all monotone classes on X which contain $\mathcal{A}$, is nothing but $\sigma(\mathcal{A})$. To achieve this goal, we could go through the following steps:

(i) Show that if $\mathcal{M}_0$ is a monotone class, so is $\mathcal{M}_0' := \{E^c : E \in \mathcal{M}_0\}$ and apply this to show that $m(\mathcal{A})$ is closed under complement.

(ii) For $E \subset X$, show that $\mathcal{M}_E := \{C \in m(\mathcal{A}) : C \cap E \in m(\mathcal{A})\}$ is a monotone class.

(iii) Show that for any $C \in \mathcal{A}$, we have $m(\mathcal{A}) \subset \mathcal{M}_C$. Deduce that the same conclusions hold for $C \in m(\mathcal{A})$ and show that $m(\mathcal{A})$ is an algebra.

Exercise 1.6. *Let $\mathcal{A} \subset 2^X$ be an algebra and let μ and ν be finite measures on $\sigma(\mathcal{A})$. Show that*

$$\mu = \nu \quad on \quad \mathcal{A} \qquad \Longrightarrow \qquad \mu = \nu \quad on \quad \sigma(\mathcal{A}).$$

Show that the result remains true if $\mu(X) = \infty$ and there exists $\{X_n\}_{n=1}^\infty \subset \mathcal{A}$ such that $X = \cup_{n=1}^\infty X_n$ and $\mu(X_n) < \infty$.

Hint: Assume $\mathcal{F}$ is a σ-algebra on which μ and ν are finite measures. Apply the monotone class lemma to $\mathcal{M} := \{A \in \mathcal{F} : \mu[A] = \nu[A]\}$.

2. Homework #2: Due on Friday 03 February

If $x \in \mathbb{R}^d$ and $r > 0$, we denote by $B(x, r)$ the closed ball in $\mathbb{R}^d$, centered at x and of radius r.

Definition 2. *Let Σ be a σ–algebra on a set X. We call $\mu : \Sigma \to [-\infty, +\infty]$ a signed measure if $\mu(\emptyset) = 0$ and whenever $(A_k)_k \subset \Sigma$ is a sequence of pairwise disjoint sets, then $\sum_{k=1}^\infty \mu(A_k)$ is defined as an extended value and*

$$\sum_{k=1}^\infty \mu(A_k) = \mu\big(\cup_{k=1}^\infty A_k \big).$$

Definition 3. *Let μ and ν be two measures on Σ, which is a σ–algebra on a set X. We say that $f = d\nu/d\mu$ if $\nu << \mu$ and f is a non–negative μ–measurable function such that*

$$\nu(A) = \int_A f d\mu,$$

for any μ–measurable set $A \subset X$.

Theorem A (Jordan decomposition)**.** *Let Σ be a σ–algebra on a set X and let $\mu : \Sigma \to [-\infty, +\infty]$ be a signed measure. Then there exists two measures μ_1 and μ_2, at least one of which must be finite such that $\mu = \mu_2 - \mu_1$, and $\mu_1 \perp \mu_2$. In fact*

$$\mu_2(E) = \sup_A \Big\{ \mu(A) : A \subset E, A \in \Sigma \Big\}, \quad \mu_1(E) = \sup_A \Big\{ -\mu(A) : A \subset E, A \in \Sigma \Big\}$$

Exercise 2.1 (∗)**.** *Let μ be a Borel (outer) measure on a separable metric space X. Let V be the union of all open sets $O \subset X$ such that $\mu(U) = 0$ ($\operatorname{spt}\mu := X \setminus V$ is called the support of μ).*

(i) Show that X is second countable (has a countable basis of open sets).

(ii) Show that $\mu(X \setminus \operatorname{spt}\mu) = 0$.

(iii) Further assume that $\mu(X) < \infty$. Show that $\operatorname{spt}\mu$ is the smallest closed subset K such that $\mu(K) = \mu(X)$.

Exercise 2.2. *Let $E \subset \mathbb{R}^d$ be a $\mathcal{L}^d$–measurable set of finite measure and let $\mathcal{V}$ be a family of non–degenerate closed balls, which is a fine cover for E. Show that there exists a finite or countably infinite disjoint subcollection $\{U_j\}_{j\in I} \subset \mathcal{V}$ such that*

$$\mathcal{L}^d\Big(E \setminus \cup_{j\in I} U_j\Big) = 0.$$

Hint. Use the Vitali's covering Lemma or its Corollaries.

Exercise 2.3. *Let μ and ν be two Radon (outer) measures on $\mathbb{R}^d$ and fix $\alpha \in (0, \infty)$. For $A \subset \mathbb{R}^d$ (not necessarily measurable), show that the following hold:*

(i) If $A \subset \{x \in \mathbb{R}^d : \underline{D}_\mu \nu(x) \leq \alpha\}$ then $\nu(A) \leq \alpha\mu(A)$.

(ii) If $A \subset \{x \in \mathbb{R}^d : \overline{D}_\mu \nu(x) \geq \alpha\}$ then $\nu(A) \geq \alpha\mu(A)$.

Exercise 2.4 (∗)**.** *Let $\mathcal{L}$ be the class of Lebesgue measurable sets in $\mathbb{R}^d$ and let μ be the Lebesgue measure on $\mathbb{R}^d$. Let $(\nu_j)_j$ be a monotone sequence of Radon measures on $\mathbb{R}^d$ such that $\nu(E) := \lim_{j\to\infty} \nu_j(E)$ is a Radon measure on $\mathbb{R}^d$. Show that*

$$D_\mu \nu = \lim_{j\to\infty} D_\mu \nu_j \quad a.e.$$

Exercise 2.5. *Let μ be the Lebesgue measure on $\mathbb{R}^d$ and let $f \in L^p(\mathbb{R}^d, \mu)$ for some $p \in [1, \infty)$. Show that*

$$\lim_{B\to x} \frac{1}{\mu(B)} \int_B |f - f(x)|^p \mu(dy) = 0, \textit{for } \mu \textit{ a.e. } x \in \mathbb{R}^d.$$

Here, $B \to x$ means that $x \in B$ and the radius of B tends to 0.

Exercise 2.6 (∗)**.** *Let $a < b$ be real numbers and let $F : [a, b] \to \mathbb{R}$ be a continuous function.*

(i) Show that the set A of points of differentiability of F is a Borel set.

(ii) Suppose that F is monotone nondecreasing and μ_F is the associated Lebesgue–Stieltjes measure. Let $B_\infty := \{F' = +\infty\}$. Show that for any Borel set E

$$\mu_F(E) = \int_E F'(t)dt + \mu_F(E \cap B_\infty).$$

Similar results can be obtained when F is of bounded variations and μ_F is a signed measure.

Exercise 2.7. *Let $(\nu_j)_j$ be a sequence of Borel measures on a metric space Ω and let μ be a Borel measure on Ω. Show that if $\nu_j \perp \mu$ for all j then $\sum_{j=1}^\infty \nu_j \perp \mu$.*

Exercise 2.8 (∗)**.** *Let $X = [0, 1]$, Σ is the σ algebra of Borel subset of X, $m := \mathcal{L}^1|_X$ is the Lebesgue measure on X and μ is the counting measure.*

(i) Show that $m << \mu$ but there is no f such that $f = dm/d\mu$.

(ii) Show that μ has not Lebesgue decomposition with respect to m.

Exercise 2.9. *Let μ be a Radon measure on $\mathbb{R}^d$ and let $r > 0$. Show that $x \to \mu\big(B(x,r)\big)$ is upper semicontinuous in the sense that $\limsup_{y\to x} \mu\big(B(y,r)\big) \leq \mu\big(B(x,r)\big)$.*

Exercise 2.10 $(*)$**.** *Let μ and ν be σ–finite Borel measures on $\mathbb{R}^d$, which are Borel–regular and such that $\nu << \mu$. Show that if $\lambda := \mu + \nu$ and $f = d\nu/d\lambda$, then $0 \leq f < 1$ μ a.e. and $d\nu/d\mu = f/(1-f)$.*

Exercise 2.11 $(*)$**.** *Let $(\mathbb{R}^d, \Sigma, \mu)$ be a finite Borel regular measure space. Let $\mathcal{N}$ be a sub-σ–algebra of Σ and let $\nu := \mu|_{\mathcal{N}}$. Show that if $f \in L^1(\mu)$, then there exists a unique $g \in L^1(\mathcal{N})$ (which implies g is ν–measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$. In probability theory, g is called the conditional expectation of f on $\mathcal{N}$ and is denoted by $\mathbb{E}(f|\mathcal{N})$.*

The results remain true if $(\mathbb{R}^d, \Sigma, \mu)$ is replaced by any σ–finite measure space (X, Σ, μ).

3. Homework #3: Due on Friday 17 February

If X is a metric space, we denote by $C_b(X)$ the set of continuous and bounded functions $f : X \to \mathbb{R}$. We denote by $C_0(X)$ the set of $f \in C(X)$ which vanish at infinity. This means that for every $\epsilon > 0$, $\{|f| \geq \epsilon\}$ is compact. We denote by $\mathcal{P}(X)$ the set of Borel probability measure on X.

Definition 4. *Let X be a separable metric space.*
(i) Let $\mu^\pm$ be measures on X such that $\mu^+ \perp \mu^-$ and either $\mu^+(X) < +\infty$ or $\mu^-(X) < +\infty$. We call $\mu := \mu^+ - \mu^-$ a signed measure and call $|\mu| := \mu^+ + \mu^-$ the total variation of μ. We call μ^+ the positive variation of μ and call μ^- the negative variation of μ. The Jordan decomposition theorem asserts that $\mu^\pm$ are uniquely determined by μ.
(ii) If in addition $\mu^\pm$ are Radon measures, we call μ a signed Radon measure. We denote by $M(X,\mathbb{R})$ the set of signed Radon measures on X such that $\mu^\pm(X) < +\infty$. We write $\|\mu\| = |\mu|(X)$.
(iii) We say that $(\mu_n)_{n=1}^\infty \subset M(X,\mathbb{R})$ converges vaguely to $\mu_\infty \in M(X,\mathbb{R})$ if

$$\lim_{n\to\infty} \int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx), \qquad \forall f \in C_0(X)$$

Definition 5. *Let X be a separable metric space, let $\mu \in \mathcal{P}(X)$ and let $(\mu_n)_n$ be a sequence in $\mathcal{P}(X)$.*
(i) We say that $(\mu_n)_n$ converges narrowly to μ if

$$\lim_{n\to\infty} \int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx), \qquad \forall f \in C_b(X)$$

(ii) We say that $(\operatorname{spt}\mu_n)_n$ converges to $\operatorname{spt}\mu$ in the sense of Kuratowski if for all $x \in \operatorname{spt}\mu$ there exists $x_n \in \operatorname{spt}\mu_n$ such that $\lim_{n\to\infty} x_n = x$.

Exercise 3.1. *Suppose $(\mu_n)_n$ is a sequence of Radon measures on $\mathbb{R}^d$ and μ is a Radon measure on $\mathbb{R}^d$. Show that the following are equivalent:*
(i) $(\mu_n)_n$ converges weakly to μ.
(ii) For every compact set $K \subset \mathbb{R}^d$ and every open set $O \subset \mathbb{R}^d$, we have

$$\liminf_n \mu_n(O) \geq \mu(O), \qquad \limsup_n \mu_n(K) \leq \mu(K).$$

(iii) For every bounded Borel set $B \subset \mathbb{R}^d$ such that $\mu(\partial B) = 0$, we have $\lim_n \mu_n(B) = \mu(B)$.

Exercise 3.2 ($*$). *Let X be a locally compact complete separable metric space. Let $O \subset X$ be an open set and let $K \subset O$ be a non empty compact set.*
(i) Show that for every $x \in O$, there exists a compact neighborhood K_x of x such that $K_x \subset O$.
(ii) Show that there exists an open set V such that $\bar{V}$ is compact and $K \subset V \subset \bar{V} \subset O$.
(iii) (Locally compact version of Uryshon's Lemma) Show that there exists a function $g \in C(X, [0,1])$ such that $g = 1$ on K and $g = 0$ outside a compact subset of O.

Exercise 3.3 ($*$). *Let X be a locally compact complete separable metric space. Show that $C_0(X)$ is the uniform closure of $C_c(X)$.*

Hint: *Apply Exercise 3.2.*

Exercise 3.4 ($*$). *Let $L : C_0(\mathbb{R}^d) \to \mathbb{R}$ be a linear which is bounded for the uniform norm. Show that there exist bounded linear functionals $L^\pm : C_0(\mathbb{R}^d) \to \mathbb{R}$ such that $L = L^+ - L^-$ and $L^\pm(f) \geq 0$ for all non negative $f \in C_0(\mathbb{R}^d)$. Conclude that one can extend the Riesz representation theorem to L.*

Exercise 3.5 ($*$). *Show that there exists a countable set $\{\phi_j\}_{j=1}^\infty \subset C_c(\mathbb{R}^d)$, which is dense in $C_c(\mathbb{R}^d)$ for the uniform topology.*

Exercise 3.6 ($*$). *Let X be a locally compact complete separable metric space. Show that if μ is a Radon measure on X such that $\mu(X) = \infty$ then there exists $f \in C_0(X)$ such that $f \geq 0$ and $\int_X f d\mu = \infty$.*

Exercise 3.7 ($*$). *Let X be a locally compact complete separable metric space. Show that, every positive linear functional on $C_0(X)$ is bounded.*

Exercise 3.8. *Let $\mathcal{C}_0,\ \mathcal{C} \subset C_b(X)$ be non empty and such that for any $f \in \mathcal{C}$, we have*

$$\sup_h \left\{ \int_X h(x)\mu(dx) \mid h \in \mathcal{C}_0, h \leq f \right\} = \int_X f(x)\mu(dx) = \inf_h \left\{ \int_X h(x)\mu(dx) \mid h \in \mathcal{C}_0, h \geq f \right\}.$$

(i) Show that if $\{\mu\} \cup \{\mu_k\}_k \subset \mathcal{P}(X)$ is such that

$$\lim_{k\to\infty} \int_X f(x)\mu_k(dx) = \int_X f(x)\mu(dx) \tag{3.1}$$

for every $f \in \mathcal{C}_0$ then (3.1) holds for every $f \in \mathcal{C}$.
(ii) Show that if $\operatorname{Span}(\mathcal{C})$ is dense in $C_b(X)$ for the uniform norm and (3.1) holds for every $f \in \mathcal{C}$ then $(\mu_k)_k$ narrowly converges to μ.

Exercise 3.9 ($*$). *Let X be a separable metric space. Show that if $(\mu_n)_n$ narrowly converges to μ in $\mathcal{P}(X)$ then $\left(\operatorname{spt}\mu_n\right)_n$ converges to $\operatorname{spt}\mu$ in the sense of Kuratowski.*

4. Homework #4: Due on Friday 03 March

Exercise 4.1 ($*$). *Let G be a subspace of a Banach space E.*
(i) Show that every linear and continuous $g : G \to \mathbb{R}$ admits a linear and continuous extension $f : E \to \mathbb{R}$ such that $\|g\|_{G'} = \|f\|_{E'}$.
(ii) Show that for every $x_0 \in E$, there exists $f_0 \in E'$ such that

$$\|f_0\|_{E'} = \|x_0\| \quad \text{and} \quad f_0(x_0) = \|x_0\|^2.$$

(iii) Conclude that for every $x \in E$, we have

$$\|x\| = \max_f \{|f(x)| :\ f \in E', \|f\|_{E'} = 1\}.$$

Exercise 4.2 (∗)**.** *Let $X = C[0,1]$ be endowed with the norm $\|u\| = \max_{[0,1]} |u|$. Let $E := \{u \in X : u(0) = 0\}$. We consider the application $f : E \to \mathbb{R}$ defined by*

$$f(u) = \int_0^1 u(t)dt.$$

(i) Show that $f \in E'$ and compute $\|f\|_{E'}$.
(ii) Can we find $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E'}$?

Exercise 4.3 (∗)**.** *Let E be a normed vector space and let $f : E \to \mathbb{R}$ be a non–null linear functional. Show that if $\alpha \in \mathbb{R}$ then f is continuous if and only if $\{f = \alpha\}$ is closed.*

Exercise 4.4 (∗)**.** *Let C be an open convex subset of a normed space E such that $0 \in C$. Define the gauge of C (or the Minkowski functional of C) as*

$$\varrho_C(x) := \inf_{\alpha > 0} \Big\{ \alpha : \alpha^{-1} x \in C \Big\}.$$

(i) Show that there exists a constant $M > 0$ such that $\varrho_C(x) \le M\|x\|$ for all $x \in E$.
(ii) Show that $\varrho_C(x+y) \le \varrho_C(x) + \varrho_C(y)$ and $\varrho_C(\lambda x) = \lambda \varrho_C(x)$ for all $x, y \in E$ and all $\lambda > 0$.

Exercise 4.5. *Let C be a non empty open convex subset of a normed space E and let $x_0 \in E \setminus C$. Show that there exists $f \in E'$ such that $f(x) < f(x_0)$ for all $x \in C$.*

Hint. *We can assume without loss of generality that $0 \in C$. Use the gauge of C.*

Exercise 4.6 (∗)**.** *Let A and B be two non empty convex disjoint subsets of a normed space E. Show that if A is open then there exists $f \in E'$ and $\alpha \in \mathbb{E}$ such that $A \subset \{f \le \alpha\}$ and $B \subset \{f \ge \alpha\}$. We say that the closed hyperplane $\{f = \alpha\}$ separate A and B is the large sense.*

Hint. *We denote $A - b$ the translation of A which is $\{a - b : a \in A\}$. Check that $C := \cup_{b \in B}(A - b)$ is an open convex set which does not contain 0. Apply Exercise 4.5.*

Exercise 4.7 (∗)**.** *Let $(E, \|\cdot\|)$ be an infinite dimensional normed space.*
(i) Show that there exists a set $\{e_i : i \in I\}$ of unit vectors such that every $x \in E$ can be written in a unique way as $x = \sum_{i \in J} x_i e_i$ for some $J \subset I$ of finite cardinality and for a set $\{x_i : i \in J\} \subset \mathbb{R}$.
(ii) Construct a linear function $f : E \to \mathbb{R}$ which fails to be continuous.
(iii) Show that if we further assume that E is a Banach space then I cannot be countable.

Exercise 4.8 (∗)**.** *Let $(E, \|\cdot\|)$ be a vectorial normed space. Show that every hyperplane in E is either closed or dense in E.*

Hint. *Show that if $f : E \to \mathbb{R}$ is linear and discontinuous then for every $x \in E$ and $r > 0$, we have $f\big(D(x,r)\big) = \mathbb{R}$.*

Exercise 4.9 (∗)**.** *Let $(E, \|\cdot\|)$ be a Banach space and let $(x_n)_n \subset E$ be a sequence which converges weakly to x (converges for the $\sigma(E, E')$–topology). Show that if we set*

$$y_n := \frac{x_1 + \cdots + x_n}{n}$$

then $(y_n)_n \subset E$ converges weakly to x.

Exercise 4.10 (Mazur Lemma: (∗))**.** *Let $(E, \|\cdot\|)$ be a Banach space and let $(x_n)_n \subset E$ be a sequence which converges weakly to x. Show that there exists a sequence $(z_n)_n$ which converges strongly to x and such that each z_n belongs to the convex hull of $\{x_k : k \ge N_n\}$ for an increasing sequence $(N_n)_n \subset \mathbb{N}$*

5. Homework #5

Exercise 5.1. *Let E and F be Banach spaces and let $T : E \to F$ be surjective, linear and continuous.*
(i) (Open graph theorem) Show that there exists $c > 0$ such that $B_F(0, c) \subset T(B_E(0, 1))$.
(ii) Conclude that if we further assume that T is one-to-one then T^{-1} is continuous.

Hint. *Argue that E is the union of the closed set $X_n := n\overline{T(B_E(0, 1))}$ and use Baire's Lemma. This gives you only that $B_F(y_0, 4c) \subset X_1$ for some $y_0 \in F$ and $c > 0$. Argue that $-y_0 \in \overline{T(B_E(0, 1))}$ to obtain the weaker conclusion that $B_F(0, c) \subset T(B_E(0, 1))$.*

Exercise 5.2. *Let G is a Banach space for the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Show that there exist $C_1, C_2 > 0$ such that $\|\cdot\|_1 \leq C_1\|\cdot\|_2$ and $\|\cdot\|_2 \leq C_2\|\cdot\|_1$*

Hint. *Apply Exercise 5.1 (ii) to $(E, \|\cdot\|_E) = (G, \|\cdot\|_1 + \|\cdot\|_2)$ and $(F, \|\cdot\|_F) = (G, \|\cdot\|_2)$ with $T = \mathbf{id}$.*

Exercise 5.3. *Let (S, d_S) be a metric space. Show that the following are equivalent:*

(i) K is pre–compact

(ii) (Bolzano–Weierstrass) Every sequence $(x_n)_n$ in S has a cluster point.

(iii) (Sequential compactness) Every sequence in S has a convergent subsequence.

(iv) Every infinite set in S has a limit point.

Hint. *Show that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i). For the last implication, you could first show that for every $\epsilon > 0$, there exist $x_1, \cdots, x_n \in S$ such that $S = B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$. Conclude that S is separable and so, we can use Lindelöf theorem to conclude that every open cover of S has a countable sub–cover.*

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