# HOMEWORK ASSIGNMENTS: MATH 245B

WILFRID GANGBO

Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

You are allowed to use Exercise $N$ to solve Exercise $N+1$ even if you failed to answer correctly Exercise $N$; but not allowed to go the other way around. In Exercise $N$, you are allowed to use question $k$ to solve question $k+1$ even if you failed to answer correctly question $k$.

If $x \in \mathbb{R}^d$ and $r > 0$, we denote by $B(x,r)$ the closed ball in $\mathbb{R}^d$, centered at $x$ and of radius $r$.

**Definition 1.** *Let $\Sigma$ be a $\sigma$–algebra on a set $X$. We call $\mu : \Sigma \to [-\infty, +\infty]$ a signed measure if $\mu(\emptyset) = 0$ and whenever $(A_k)_k \subset \Sigma$ is a sequence of pairwise disjoint sets, then $\sum_{k=1}^\infty \mu(A_k)$ is defined as an extended value and*

$$\sum_{k=1}^\infty \mu(A_k) = \mu\big( \cup_{k=1}^\infty A_k \big).$$

**Theorem A** (Jordan decomposition)**.** *Let $\Sigma$ be a $\sigma$–algebra on a set $X$ and let $\mu : \Sigma \to [-\infty, +\infty]$ be a signed measure. Then there exists two measures $\mu_1$ and $\mu_2$, at least one of which must be finite such that $\mu = \mu_2 - \mu_1$, and $\mu_1 \perp \mu_2$. In fact*

$$\mu_2(E) = \sup_A \Big\{ \mu(A) :\ A \subset E, A \in \Sigma \Big\}, \quad \mu_1(E) = \sup_A \Big\{ -\mu(A) :\ A \subset E, A \in \Sigma \Big\}$$

## 2. Homework #2: Due on Friday 28 January

**Exercise 2.1** $(*)$**.** *Let $\mu$ be a Borel (outer) measure on a separable metric space $X$. Let $V$ be the union of all open sets $O \subset X$ such that $\mu(U) = 0$ ($\operatorname{spt} \mu := X \setminus V$ is called the support of $\mu$).*

*(i) Show that $X$ is second countable (has a countable basis of open sets).*

*(ii) Show that $\mu(X \setminus \operatorname{spt} \mu) = 0$.*

*(iii) Further assume that $\mu(X) < \infty$. Show that $\operatorname{spt} \mu$ is the smallest closed subset $K$ such that $\mu(K) = \mu(X)$.*

**Exercise 2.2.** *Let $E \subset \mathbb{R}^d$ be a $\mathcal{L}^d$–measurable set of finite measure and let $\mathcal{V}$ be a family of non–degenerate closed balls, which is a fine cover for $E$. Show that there exists a finite or countably infinite disjoint subcollection $\{U_j\}_{j \in I} \subset \mathcal{V}$ such that*

$$\mathcal{L}^d \Big( E \setminus \cup_{j \in I} U_j \Big) = 0.$$

Hint. Use the Vitali's covering Lemma or its Corollaries.

**Exercise 2.3.** *Let $\mu$ and $\nu$ be two Radon (outer) measures on $\mathbb{R}^d$ and fix $\alpha \in (0, \infty)$. For $A \subset \mathbb{R}^d$ (not necessarily measurable), show that the following hold:*

*(i) If $A \subset \{x \in \mathbb{R}^d :\ \underline{D}_\mu \nu(x) \le \alpha\}$ then $\nu(A) \le \alpha \mu(A)$.*

*(ii) If* $A \subset \{x \in \mathbb{R}^d : \overline{D}_\mu \nu(x) \geq \alpha\}$ *then* $\nu(A) \geq \alpha\mu(A)$.

**Exercise 2.4** (*)**.** *Let* $\mathcal{L}$ *be the class of Lebesgue measurable sets in* $\mathbb{R}^d$ *and let* $\mu$ *be the Lebesgue measure on* $\mathbb{R}^d$. *Let* $(\nu_j)_j$ *be a monotone sequence of Radon measures on* $\mathbb{R}^d$ *such that* $\nu(E) := \lim_{j\to\infty} \nu_j(E)$ *is a Radon measure on* $\mathbb{R}^d$. *Show that*

$$D_\mu \nu = \lim_{j\to\infty} D_\mu \nu_j \quad a.e.$$

**Exercise 2.5.** *Let* $\mu$ *be the Lebesgue measure on* $\mathbb{R}^d$ *and let* $f \in L^p(\mathbb{R}^d, \mu)$ *for some* $p \in [1, \infty)$. *Show that*

$$\lim_{B\to x} \frac{1}{\mu(B)} \int_B |f - f(x)|^p \mu(dy) = 0, \text{for } \mu \text{ a.e. } x \in \mathbb{R}^d.$$

*Here,* $B \to x$ *means that* $x \in B$ *and the radius of* $B$ *tends to* $0$.

**Exercise 2.6** (*)**.** *Let* $a < b$ *be real numbers and let* $F : [a, b] \to \mathbb{R}$ *be a continuous function.*

*(i) Show that the set* $A$ *of points of differentiability of* $F$ *is a Borel set.*

*(ii) Suppose that* $F$ *is monotone nondecreasing and* $\mu_F$ *is the associated Lebesgue–Stieltjes measure. Let* $B_\infty := \{F' = +\infty\}$. *Show that for any Borel set* $E$

$$\mu_F(E) = \int_E F'(t)dt + \mu_F(E \cap B_\infty).$$

Similar results can be obtained when $F$ is of bounded variations and $\mu_F$ is a signed measure.

**Exercise 2.7.** *Let* $(\nu_j)_j$ *be a sequence of Borel measures on a metric space* $\Omega$ *and let* $\mu$ *be a Borel measure on* $\Omega$. *Show that if* $\nu_j \perp \mu$ *for all* $j$ *then* $\sum_{j=1}^\infty \nu_j \perp \mu$.

**Exercise 2.8** (*)**.** *Let* $X = [0, 1]$, $\Sigma$ *is the* $\sigma$ *algebra of Borel subset of* $X$, $m := \mathcal{L}^1|_X$ *is the Lebesgue measure on* $X$ *and* $\mu$ *is the counting measure.*

*(i) Show that* $m << \mu$ *but there is no* $f$ *such that* $f = dm/d\mu$.

*(ii) Show that* $\mu$ *has not Lebesgue decomposition with respect to* $m$.

**Exercise 2.9.** *Let* $\mu$ *be a Radon measure on* $\mathbb{R}^d$ *and let* $r > 0$. *Show that* $x \to \mu\big(B(x, r)\big)$ *is upper semicontinuous in the sense that* $\limsup_{y\to x} \mu\big(B(y, r)\big) \leq \mu\big(B(x, r)\big)$.

**Exercise 2.10** (*)**.** *Let* $\mu$ *and* $\nu$ *be Borel measures on* $\mathbb{R}^d$, *which are regular and such that* $\nu << \mu$. *Show that if* $\lambda := \mu + \nu$ *and* $f = d\nu/d\lambda$, *then* $0 \leq f < 1$ $\mu$ *a.e. and* $d\nu/d\mu = f/(1 - f)$.

**Exercise 2.11** (*)**.** *Let* $(\mathbb{R}^d, \Sigma, \mu)$ *be a finite Borel regular measure space. Let* $\mathcal{N}$ *be a sub-algebra of* $\Sigma$ *and let* $\nu := \mu|_\mathcal{N}$. *Show that if* $f \in L^1(\mu)$, *then there exists a unique* $g \in L^1(\mathcal{N})$ *(which implies* $g$ *is* $\nu$*–measurable) such that* $\int_E f d\mu = \int_E g d\nu$ *for all* $E \in \mathcal{N}$. *In probability theory,* $g$ *is called the conditional expectation of* $f$ *on* $\mathcal{N}$ *and is denoted by* $\mathbb{E}(f|\mathcal{N})$.

The results remain true if $(\mathbb{R}^d, \Sigma, \mu)$ is replaced by any $\sigma$–finite measure space $(X, \Sigma, \mu)$.

Department of Mathematics, UCLA, Los Angeles, CA 90095

*Email address*: wgangbo@math.ucla.edu