HOMEWORK ASSIGNMENTS: MATH 245 A

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Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

1. Homework #1: Due on Friday 08 October

Exercise 1.1. *Let μ be an outer measure on a set X and let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subsets of X.*

(i) Prove that if $\mu[A_1] = 0$, then A_1 is μ-measurable.

(ii) Prove that A_1 is μ-measurable if and only if its complement A_1^c is μ-measurable.

(iii) Assume in the sequel that $\{A_k\}_{k=1}^{\infty}$ is a sequence of μ–measurable sets. Prove that if $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k$, then $\mu[A_1] + \mu[A_2 \setminus A_1] + \cdots + \mu[A_k \setminus A_{k-1}] = \mu[A_k]$.

(iv) Prove that if $A_1 \supset A_2 \supset \cdots \supset A_k \supset A_k \cdots$ and $\mu[A_1] < +\infty$ then $\lim_{k \to +\infty} \mu[A_k] = \mu[\cap_{k=1}^{\infty} A_k]$. Given a counter example to this statement when $\mu[A_1] = +\infty$

(v) Assume that $B \subset X$. Show that if A_1 is μ–measurable then A_1 is $\mu_{|B}$–measurable.

Exercise 1.2 $(*)$**.** *Suppose μ is a measure on $\mathbb{Z}^+ := \mathbb{Z} \cap [0, +\infty)$ such that every subset of $\mathbb{Z}^+$ is μ–measurable. Show that there exists a sequence $(\omega_k)_{k=0}^{\infty} \subset [0, +\infty]$ such that*

$$\mu(E) = \sum_{k \in E} \omega_k$$

Exercise 1.3 $(*)$**.** *Suppose μ is an outer measure on X such that $\mu(X) < \infty$ and suppose $\mathcal{S}$ is the set of μ–measurable subsets. Suppose $\mathcal{A} \subset \mathcal{S}$ is such that for every $A \in \mathcal{A}$, $\mu(A) > 0$ and for every $A, B \in \mathcal{A}$ such that $A \neq B$ then $A \cap B = \emptyset$. Show that $\mathcal{A}$ is countable.*

Exercise 1.4 $(*)$**.** *Let μ be an outer measure on a set X and let $\mathcal{S}$ be the collection of μ–measurable subsets on X. Show that we cannot have*

$$\{\mu(E) \,:\, E \in \mathcal{S}\} = [0, 1).$$

Exercise 1.5 $(*)$**.** *Let μ be a measure on a set X such that $\mu[X] = 1$ and let $\mathcal{M}$ be the set of μ–measurable sets. Suppose that $\mu[M] > 0$ for each $\emptyset \neq M \in \mathcal{M}$ and let*

$$\alpha(x) = \inf_{M \in \mathcal{M}} \{\mu[M] :\ x \in M\} \qquad\qquad (x \in X).$$

(a) Show that there exists a set $A_x \in \mathcal{M}$ such that $x \in A_x$ and $\mu[A_x] = \alpha(x)$.
(b) Show that the sets $\{A_x\}$ are either disjoint or identical.

Exercise 1.6. *Let μ be an measure on X such that $\mu[X] < \infty$. Show that if $\{x\}$ is $\mu-$measurable for every $x \in X$ then there are at most countably many $x \in X$ such that $\mu[\{x\}] > 0$.*

Exercise 1.7 $(*)$**.** *Show that $\mathcal{S} = \{\cup_{n \in K}(n, n+1] : K \subset \mathbb{Z}\} \cup \{\emptyset\}$ is a σ–algebra on $\mathbb{R}$.*

Exercise 1.8 (*). *Prove that* $\sigma(C_1) = \sigma(C_2) = \sigma(C_3) = \mathcal{B}$, *where* $\mathcal{B}$ *is the Borel* σ*–algebra on* $\mathbb{R}$. *Here,*

$$C_1 = \{(r,s] : r, s \in \mathbb{Q}\}, \quad C_2 = \{(r,n] : r \in \mathbb{Q}, n \in \mathbb{Z}\}, \quad C_3 = \{[r,+\infty) : r \in \mathbb{Q}\}.$$

2. Homework #2: Due on Friday 22 October

Exercise 2.1 (*). *Prove that if* $B \subset \mathbb{R}^d$ *is a Borel set,* $t \in \mathbb{R}$ *and* $c \in \mathbb{R}^d$ *then* $tB = \{tx : x \in B\}$ *and* $c + B = \{c + x : x \in B$ *are Borel sets, where* $\mathcal{B}$ *is the Borel* σ*–algebra on* $\mathbb{R}$.

Exercise 2.2. *Let* (X, μ) *be an outer measure space and* $(A_n)_n$ *be a sequence of* μ*–measurable subsets of* X. *Show that if* $\sum_{n=1}^{\infty} \mu[A_n] < \infty$ *then* $\mu[A] = 0$ *if we set*

$$A = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$$

Exercise 2.3. *Let* $B \subset \mathbb{R}^d$.

(i) Show that if μ *is a Borel measure on* $\mathbb{R}^d$ *then* $\mu_{|B}$ *is also a Borel measure on* $\mathbb{R}^d$.

(ii) Show that if μ *is a Borel regular measure on* $\mathbb{R}^d$, *if* B *is* μ*–measurable and of finite measure then* $\nu := \mu_{|B}$ *is a Radon measure on* $\mathbb{R}^d$.

Exercise 2.4 (*). *Suppose* μ *is an outer measure on* X, $E_1, \cdots, E_n$ *are disjoint subsets of* X *and* $c_1, \cdots, c_n$ *are distinct nonzero real numbers. Prove that* $c_1\chi_{E_1} + \cdots + c_n\chi_{E_n}$ *is a* μ*–measurable function if and only if* $E_1, \cdots, E_n$ *are* μ*–measurable sets.*

Exercise 2.5 (*). *Suppose* $X \subset \mathbb{R}$ *is a Borel set and* $f : X \to \mathbb{R}$ *is a function such that* $\{x \in X : f$ *is not continuous at* $x\}$ *is a countable set. Prove that* f *is a Borel measurable function.*

Exercise 2.6 (*). *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable at every element of* $\mathbb{R}$. *Prove that* f' *is a Borel measurable function from* $\mathbb{R}$ *to* $\mathbb{R}$.

Exercise 2.7 (*). *Let* h *be a continuous and strictly increasing real valued function on* $\mathbb{R}$. *Prove that if* $B \subset \mathbb{R}$, *then* $h(B)$ *is a Borel set if and only if* B *is a Borel set.*

Exercise 2.8 (*). *Suppose* $f : B \to \mathbb{R}$ *is a Borel measurable function. Define* $g : \mathbb{R} \to \mathbb{R}$ *to be the function which coincides with* f *on* B *and is identically equal to* 0 *on* $\mathbb{R} \setminus B$. *Prove that* g *is a Borel measurable function.*

Exercise 2.9 (*). *Say whether or not every open subset of* $\mathbb{R}^d$ *is a countable union of closed subsets of* $\mathbb{R}^d$. *Justify your answer (Your argument should be short, say, half a page).*

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