HOMEWORK ASSIGNMENTS: MATH 131BH

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Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

4. Homework #4: Due on Friday 07 March

Exercise 4.1 $(*)$**.** *Let $I_1, \cdots, I_N$ be disjoint open intervals in $\mathbb{R}^n$. Show that if $J_1, \cdots, J_M$ are open intervals in $\mathbb{R}^n$ such that $I_1 \cup \cdots \cup I_N \subset J_1 \cup \cdots \cup J_M$ then*

$$\mathrm{vol}(I_1) + \cdots + \mathrm{vol}(I_N) \leq \mathrm{vol}(J_1) + \cdots + \mathrm{vol}(J_M).$$

Exercise 4.2 $(*)$**.** *Let f be a real–valued function on a subset A of $\mathbb{R}^n$. Show that if $\int_A f dx$ exists, then so does $\int_A |f| dx$, and $\left| \int_A f dx \right| \leq \int_A |f| dx$.*

Exercise 4.3 $(*)$**.** *Let $I \subset \mathbb{R}^n$ and let f and g be real–valued functions on I.*

(a) Show that if I is a closed interval and f, g are integrable then so are f^2 and fg.

(b) Show that if $\int_I f dx$ and $\int_I g dx$ exist, then $\int_I fg dx$ exists (here, I is not necessarily a closed interval).

(c) Show that if $\int_I f dx$ exists and $B \subset I$ has volume, then $\int_B f dx$ exists.

(d) Show that if the subsets A and B of $\mathbb{R}^n$ have volume, then so do the sets $A \cap B$, $A \cup B$ and $A \setminus B$.

Exercise 4.4 $(*)$**.** *Show that if $A \subset \mathbb{R}^n$ has volume, then the interior of A has the same volume.*

Exercise 4.5 $(*)$**.** *Show that a bounded subset $A \subset \mathbb{R}^n$ has volume if and only if the boundary of A has volume zero.*

Exercise 4.6 $(*)$**.** *Let f be a bounded real–valued function on a closed interval I of $\mathbb{R}^n$. Prove that f is integrable on I if and only if, for any $\epsilon, \delta > 0$, I is the union of a finite set of closed subintervals such that the sum of the volumes of those subintervals on which f varies by at least ϵ is less than δ.*

Exercise 4.7 $(*)$**.** *Prove that if $A \subset \mathbb{R}^n$ has positive volume and f is a positive–valued function on A such that $\int_A f dx$ exists, then $\int_A f dx > 0$.*

Hint. Reduce to the case where A is a closed interval and for any positive $r > 0$, we have $\mathrm{vol}\big(\{x \in A : \; f(x) \geq r\big) = 0$*, then try to use compactness.*

Exercise 4.8. *Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, let f and g be integrable real–valued functions on A and B respectively, and let π_A and π_B be the projections of $A \times B$ onto its factors, that is*

$\pi_A(x,y) = x$ and $\pi_B(x,y) = y$ if $x \in A$ and $y \in B$. Show that

$$\int_{A\times B} (f\circ\pi_A)(g\circ\pi_B) = \Big(\int_A f\Big)\Big(\int_B g\Big).$$

Hint. Exercise 4.3 can simplify the proof

Exercise 4.9. *Prove that if f is a real–valued function on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} f$ exists, then*

$$\int_{\mathbb{R}^2} f(x,y)dxdy = \int_D f(r\cos\theta, r\sin\theta) r dr d\theta,$$

where $D = [0,+\infty)\times[0,2\pi]$.

Hint. Prove this if f is zero on some open set containing the positive x–axis. Exercise 4.3 (c) can help in passing to the general case.

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