HOMEWORK ASSIGNMENTS: MATH 131BH

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Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

2. Homework #2: Due on Wednesday 05 February

Exercise 2.1. *Show that a power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has the same radius of convergence as* $\sum_{n=0}^{\infty} c_{n+m}(x-a)^n$*, for any positive integer* m.

Exercise 2.2 $(*)$**.** *Let* $(c_n)_{n=0}^{\infty} \subset \mathbb{R}$ *with at least one non null term, let* $a \in \mathbb{R}$ *and let* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *have radius of convergence* $r > 0$*. Show that there exists* $\delta \in (0, r)$ *such that the sum of the series is nonzero for every real number* x *such that* $0 < |x-a| < \delta$.

Exercise 2.3 $(*)$**.** *Let* a, b, c, d *be real numbers such that* $a < b$ *and* $c < d$ *and let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be a continuous function.*

(i) Check that $x \to I(x) := \int_c^d f(x, y)dy$ *and* $y \to J(y) := \int_a^b f(x, y)dx$ *are continuous and so,* $\int_a^b I dx$ *and* $\int_c^d J dy$ *exist.*

(ii) Prove that $\int_a^b I dx = \int_c^d J dy$.

Hint. Compute

$$\frac{d}{dt} \int_c^d \left(\int_a^t f(x, y)dx \right) dy \quad \text{and} \quad \frac{d}{dt} \int_a^t \left(\int_c^d f(x, y)dy \right) dx.$$

Exercise 2.4 $(*)$**.** *If* $a, b, c > 0$ *compute*

$$\lim_{n\to+\infty} \left(\frac{a^{1/n} + b^{1/n} + c^{1/n}}{3} \right)^n$$

Hint. For $r \in (1, +\infty)$*, let* Δ_r *be the rectangle* $(r^{-1}, r)^3$ *and set*

$$f_n(a, b, c) := \left(\frac{a^{1/n} + b^{1/n} + c^{1/n}}{3} \right)^n.$$

Show that there exists a function $f_\infty \in C^\infty\big((0, +\infty)^3\big)$ *such that* $(f_n)_n$ *converges uniformly to* f_∞ *on* Δ_r *and express* $\frac{\partial f_\infty}{\partial a}$ *in terms of* $f_\infty(a, b, c)$.

Exercise 2.5 $(*)$**.** *For* $n = 0, 1, 2, \cdots$*, let* $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$*. Show that*

(a) $\frac{d}{dx}\big(\cos x \sin^{n-1} x\big) = (n-1)\sin^{n-2} x - n \sin^n x$

(b) $I_n = \frac{n-1}{n} I_{n-2}$ *if* $n \geq 2$

(c) For $n = 1, 2, 3, \cdots$

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}, \quad I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

Exercise 2.6 (∗)**.** *Let* $a \in (0, +\infty)$*. Show that applying the Newton's method to the function* $x^2 - a$ *gives the formula*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Prove that Newton's method works for any $x_0 > 0$ *by showing that then* $x_1 \geq \sqrt{a}$ *and the map sending* x *into* $f(x) := \frac{1}{2}\big(x + \frac{a}{x}\big)$ *is a contraction map of* $[\sqrt{a}, +\infty)$*.*

Exercise 2.7. *Prove that the equation* $\cos x - x - \frac{1}{2} = 0$ *has a unique solution. Show that the fixed point theorem is applicable to the function* $Fx) = \cos x - \frac{1}{2}$ *and the interval* $[0, \pi/4]$ *and thereby find this solution to three decimal places.*

Exercise 2.8 (∗)**.** *Find the "maximal"* U, φ *of the implicit function theorem if* $f(x, y) = x^2 + y^2 - 1$ *and* $(a, b) = (0, 1)$*.*

Exercise 2.9 (∗)**.** *Find all solutions on* $\mathbb{R}$ *of the differential equation* $y' = 3|y|^{2/3}$

Exercise 2.10 (∗)**.** *Prove that if* $u_1, \cdots, u_n, v : \mathbb{R} \to \mathbb{R}$ *are real–valued functions that are* m *times differentiable, then any solution of the differential equation*

$$y^{(n)} + u_1 y^{(n-1)} + \cdots + u_n y = v$$

is $(n + m)$ *times differentiable.*

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