## HOMEWORK ASSIGNMENTS: MATH 131BH

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Only the Exercises marked (\*) will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

1. Homework #1: Due on Wednesday 22 January

**Exercise 1.1.** Find a sequence of functions  $f_n : \mathbb{R} \to \mathbb{R}$  such that  $\lim_{x\to 0} \lim_{n\to+\infty} f_n(x)$  and  $\lim_{n\to+\infty} \lim_{x\to 0} f_n(x)$  exist and are unequal.

**Exercise 1.2.** If  $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ , three limits we can consider are

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y), \quad \lim_{x \to 0} \lim_{y \to 0} f(x, y), \quad \lim_{(x, y) \to (0, 0)} f(x, y)$$

Compute these limits, if they exist, for

$$f(x,y) = \frac{xy}{x^2 + y^2}, \quad f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

**Exercise 1.3** (\*). Find a sequence of functions  $f_n : [0,1] \to \mathbb{R}$  that converges to the zero function and such that the sequence  $\left(\int_0^1 f_n(x) dx\right)_n$ , increases without bound.

**Exercise 1.4.** Let  $(a_n)_n \subset [0, +\infty)$ . Show that the following are equivalent

(i)  $\sum_{n=1}^{\infty} a_n$  is convergent

(ii) For any  $m \in \mathbb{N}$ ,  $\sum_{n=1}^{\infty} a_{m+n}$  is convergent

Show that if (i) holds then

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{m} a_k + \sum_{n=1}^{\infty} a_{m+n}.$$

**Exercise 1.5** (\*). Show that if  $\sum_{n=1}^{\infty} a_n$  is a convergent series of real numbers and  $(\nu_k)_k$  is a subsequence of  $(n)_{n=1}^{\infty}$  then

$$(a_1 + \dots + a_{\nu_1}) + (a_{\nu_1+1} + \dots + a_{\nu_2}) + (a_{\nu_2+1} + \dots + a_{\nu_3}) + \dots = \sum_{n=1}^{\infty} a_n.$$

**Exercise 1.6** (\*). Let  $(a_n)_n \subset [0, +\infty)$  be a sequence of positive numbers which is monotone non increasing. Show that the following hold.

(i) If  $\sum_{n=1}^{\infty} a_n$  is convergent then  $\lim_{n \to +\infty} na_n = 0$ . (ii)  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if  $\sum_{n=1}^{\infty} 2^n a_{2^n}$  is convergent **Exercise 1.7** (\*). (Integral Test) Let  $f : [1, +\infty) \to \mathbb{R}$  be a monotone non increasing function. Prove that the following are equivalent.

- (i)  $\sum_{n=1}^{\infty} f(n)$  is convergent
- (ii)  $\lim_{n \to +\infty} \int_{1}^{n} f$  exists.

**Exercise 1.8** (\*). For which p > 0 the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad \sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^p}$$

**Exercise 1.9** (\*). On the set  $\mathbb{R} \setminus \{-1, -2, \cdots\}$ , show the convergence of the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+x} \right).$$

**Exercise 1.10** (\*). (Root Test) Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers such that there exists  $r \in (0,1)$  such that  $\sqrt[n]{|a_n|} \leq r$  for all sufficiently large n. Show that  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

**Exercise 1.11** (\*). Prove that if  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are absolutely convergent series of real numbers then the series  $\sum_{m,n=1}^{\infty} a_n b_m$  is also absolutely convergent and

$$\sum_{n,n=1}^{\infty} a_n b_m = \left(\sum_{n=1}^{\infty} a_n\right) \left(\sum_{n=1}^{\infty} b_n\right)$$

**Exercise 1.12** (\*). Let  $(c_n)_{n=0}^{\infty} \subset \mathbb{R}$ . Prove that the radius of convergence of the power series  $\sum_{n=1}^{\infty} c_n x^n$  is  $1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$ .

**Exercise 1.13.** Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} n(\log n) x^n, \qquad \sum_{n=2}^{\infty} (\log n)^{\log n} x^n, \qquad \sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$$

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