Chapter 11 picks up from Chapter 3 the discussion of conservation laws, now systems of conservation laws. Unlike the general theoretical developments in Chapters 5–9, for which Sobolev spaces provide the proper abstract framework, we are forced to employ here direct linear algebra and calculus computations. We pay particular attention to the solution of Riemann’s problem and to entropy criteria.

Appendices A–E provide for the reader’s convenience some background material, with selected proofs, on inequalities, linear functional analysis, measure theory, etc.

The Bibliography primarily provides a listing of interesting PDE books to consult for further information. Since this is a textbook, and not a reference monograph, I have mostly not attempted to track down and document the original sources for the myriads of ideas and methods we will encounter. The mathematical literature for partial differential equations is truly vast, but the books cited in the Bibliography should at least provide a starting point for locating the primary sources.

1.5. PROBLEMS

1. Classify each of the partial differential equations in §1.2 as follows:
   (a) Is the PDE linear, semilinear, quasilinear or fully nonlinear?
   (b) What is the order of the PDE?

The next exercises provide some practice with the multiindex notation introduced in Appendix A.

2. Prove the Multinomial Theorem

\[ (x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \]

where \( \binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!} \), and \( x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \). The sum is taken over all multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( |\alpha| = k \).

3. Prove Leibniz' formula

\[ D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v, \]

where \( u, v : \mathbb{R}^n \to \mathbb{R} \) are smooth, \( \binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} \), and \( \beta \leq \alpha \) means \( \beta_i \leq \alpha_i \) \( (i = 1, \ldots, n) \).

4. Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is smooth. Prove

\[ f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \quad \text{as } x \to 0 \]
for each \( k = 1, 2, \ldots \). This is Taylor's formula in multiindex notation.

(Hint: Fix \( x \in \mathbb{R}^n \) and consider the function of one variable \( g(t) := f(tx) \).)
2.5. PROBLEMS

Then

\[
\dot{e}(t) = \int_{B(x_0,t_0-t)} u_t u_t + Du \cdot Du_t \, dx - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} u^2 + |Du|^2 \, dS
\]

\[
= \int_{B(x_0,t_0-t)} u_t (u_{tt} - \Delta u) \, dx
\]

\[
+ \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t \, dS - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} u^2_t + |Du|^2 \, dS
\]

\[
= \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u^2_t - \frac{1}{2} |Du|^2 \, dS.
\]

(46)

Now

\[
\left| \frac{\partial u}{\partial \nu} u_t \right| \leq |u_t| |Du| \leq \frac{1}{2} u^2_t + \frac{1}{2} |Du|^2,
\]

by the Cauchy–Schwarz and Cauchy inequalities (§B.2). Inserting (47) into (46), we find \( \dot{e}(t) \leq 0 \); and so \( e(t) \leq e(0) = 0 \) for all \( 0 \leq t \leq t_0 \). Thus \( u_t \), \( Du \equiv 0 \), and consequently \( u \equiv 0 \) within the cone \( C \).

A generalization of this proof to more complicated geometry appears later, in §7.2.4.

2.5. PROBLEMS

In the following exercises, all given functions are assumed smooth, unless otherwise stated.

1. Write down an explicit formula for a function \( u \) solving the initial-value problem

\[
\begin{cases}
  u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}.
\end{cases}
\]

Here \( c \in \mathbb{R} \) and \( b \in \mathbb{R}^n \) are constants.

2. Prove that Laplace's equation \( \Delta u = 0 \) is rotation invariant; that is, if \( O \) is an orthogonal \( n \times n \) matrix and we define

\[ v(x) := u(Ox) \quad (x \in \mathbb{R}^n), \]

then \( \Delta v = 0 \).

3. Modify the proof of the mean value formulas to show for \( n \geq 3 \) that

\[
u(0) = \int_{\partial B(0,r)} g \, dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, dx,
\]

where \( f \) is a smooth function on \( B(0,r) \).
provided
\[ \begin{align*}
-\Delta u &= f & \text{in } B^0(0, r) \\
u &= g & \text{on } \partial B(0, r).
\end{align*} \]

4. We say \( u \in C^2(\bar{U}) \) is subharmonic if
\[-\Delta u \leq 0 \quad \text{in } U.\]

(a) Prove for subharmonic \( u \) that
\[ v(x) \leq \int_{B(x,r)} v \, dy \quad \text{for all } B(x,r) \subset U. \]

(b) Prove that therefore \( \max_{\bar{U}} u = \max_{\partial U} u \).

(c) Let \( \phi : \mathbb{R} \to \mathbb{R} \) be smooth and convex. Assume \( u \) is harmonic and \( v := \phi(u) \). Prove \( v \) is subharmonic.

(d) Prove \( v := |Du|^2 \) is subharmonic, whenever \( u \) is harmonic.

5. Prove that there exists a constant \( C \), depending only on \( n \), such that
\[ \max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|) \]
whenever \( u \) is a smooth solution of
\[ \begin{align*}
-\Delta u &= f & \text{in } B^0(0,1) \\
u &= g & \text{on } \partial B(0,1).
\end{align*} \]

6. Use Poisson’s formula for the ball to prove
\[ \frac{r^n - |x|^n}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^n + |x|^n}{(r - |x|)^{n-1}} u(0) \]
whenever \( u \) is positive and harmonic in \( B^0(0,r) \). This is an explicit form of Harnack’s inequality.

7. Prove Theorem 15 in §2.2.4. (Hint: Since \( u \equiv 1 \) solves (44) for \( g \equiv 1 \), the theory automatically implies
\[ \int_{\partial B(0,1)} K(x,y) \, dS(y) = 1 \]
for each \( x \in B^0(0,1) \).

8. Let \( u \) be the solution of
\[ \begin{align*}
\Delta u &= 0 & \text{in } \mathbb{R}^n_+ \\
u &= g & \text{on } \partial \mathbb{R}^n_+.
\end{align*} \]
2.5. PROBLEMS

given by Poisson's formula for the half-space. Assume $g$ is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leq 1$. Show $Du$ is not bounded near $x = 0$. (Hint: Estimate $\frac{u(x_n) - u(0)}{x_n}$.)

9. Let $U^+$ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1, x_n > 0\}$. Assume $u \in C(\overline{U^+})$ is harmonic in $U^+$, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0, 1)$. Prove $v$ is harmonic in $U$.

10. Suppose $u$ is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(i) Show $u_\lambda(x, t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show $v(x, t) := x \cdot Du(x, t) + 2tu_x(x, t)$ solves the heat equation as well.

11. Assume $n = 1$ and $u(x, t) = v\left(\frac{x^2}{t}\right)$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$4xzv''(z) + (2 + z)v'(z) = 0 \quad (z > 0).$$

(b) Show that the general solution of (*) is

$$v(z) = c \int_0^z e^{-s/t} s^{-1/2} ds + d.$$

(c) Differentiate $v\left(\frac{x^2}{t}\right)$ with respect to $x$ and select the constant $c$ properly, so as to obtain the fundamental solution $\Phi$ for $n = 1$.

12. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

13. Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{\frac{x^2}{2(t-s)}} g(s) \, ds.$$
for a solution of the initial/boundary-value problem
\[
\begin{align*}
  u_t - u_{xx} &= 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty) \\
  u &= 0 \quad \text{on } \mathbb{R}_+ \times \{t = 0\}, \\
  u &= g \quad \text{on } \{x = 0\} \times [0, \infty).
\end{align*}
\]

(Hint: Let \( v(x, t) := u(x, t) - g(t) \) and extend \( v \) to \( \{x < 0\} \) by odd reflection.)

14. We say \( v \in C_1^2(U_T) \) is a subsolution of the heat equation if
\[
v_t - \Delta v \leq 0 \quad \text{in } U_T.
\]

(a) Prove for a subsolution \( v \) that
\[
v(x, t) \leq \frac{1}{4\pi} \int \int_{E(x, t; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} \, dy \, ds
\]
for all \( E(x, t; r) \subset U_T. \)

(b) Prove that therefore \( \max_{\partial U_T} v = \max_{\partial T} v \).

(c) Let \( \phi: \mathbb{R} \to \mathbb{R} \) be smooth and convex. Assume \( u \) solves the heat equation and \( v := \phi(u) \). Prove \( v \) is a subsolution.

(d) Prove \( v := |Du|^2 + u^2 \) is a subsolution, whenever \( u \) solves the heat equation.

15. (a) Show the general solution of the PDE \( u_{xy} = 0 \) is
\[
u(x, y) = F(x) + G(y)
\]
for arbitrary functions \( F, G \).

(b) Using the change of variables \( \xi = x + t, \quad \eta = x - t \), show \( u_{tt} - u_{xx} = 0 \) if and only if \( u_{\xi\eta} = 0 \).

(c) Use (a) and (b) to rederive d'Alembert's formula.

16. Assume \( E = (E^1, E^2, E^3) \) and \( B = (B^1, B^2, B^3) \) solve Maxwell's equations (§1.2.2). Show
\[
u_{tt} - \Delta u = 0
\]
where \( u = E^i \) or \( B^i \) (i = 1, 2, 3).

17. (Equipartition of energy). Let \( u \in C^2(\mathbb{R} \times [0, \infty)) \) solve the initial-value problem for the wave equation in one dimension:
\[
\begin{align*}
  u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
  u &= g, \ u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}.
\end{align*}
\]
2.6. REFERENCES

Suppose $g$, $h$ have compact support. The kinetic energy is $k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t) \, dx$ and the potential energy is $p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t) \, dx$.

Prove

(i) $k(t) + p(t)$ is constant in $t$,

(ii) $k(t) = p(t)$ for all large enough times $t$.

18. Let $u$ solve

$$
\begin{cases}
    u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\
    u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\},
\end{cases}
$$

where $g$, $h$ are smooth and have compact support. Show there exists a constant $C$ such that

$$
|u(x,t)| \leq C/t \quad (x \in \mathbb{R}^3, \ t > 0).
$$

2.6. REFERENCES

Section 2.2 A good source for more on Laplace’s and Poisson’s equations is Gilbarg–Trudinger [G-T, Chapters 2-4]. The proof of analyticity is from Mikhailov [M]. J. Cooper helped me with Green’s functions.

Section 2.3 See John [J, Chapter 7] or Friedman [FR2] for further information concerning the heat equation. Theorem 3 is due to N. Watson [W], as is the proof of Theorem 4. Theorem 6 is taken from John [J], and Theorem 8 follows Mikhailov [M]. Theorem 11 is from Payne [PA, §2.3].

Section 2.4 See Antman [A] for a careful derivation of the one-dimensional wave equation as a model for a vibrating string. The solution of the wave equation presented here follows Folland [F1], Strauss [ST].

Section 2.5 J. Goldstein suggested Problem 17.
for $A$ large enough. This proves (76) and a similar argument establishes that

$$y(x,t) \leq R \quad \text{if} \quad x - \sigma t = -R + (qd(1 - \varepsilon)t)^{1/2}. \quad (77)$$

5. Remember from the proof of Theorem 1 in §3.4.2 that the mapping $x \mapsto y(x,t)$ is nondecreasing. Hence (69), (76) and (77) imply for large $t$ that

$$u(x, t) - \frac{\sigma}{t} \frac{(x - \sigma t)^2}{2} \leq \frac{2}{t} \quad \text{if} \quad R - (qd(1 - \varepsilon)t)^{1/2} < x - \sigma t < -R + (qd(1 - \varepsilon)t)^{1/2}. \quad (78)$$

6. Owing to Theorem 5, we have $|u| = O(t^{-\frac{1}{2}})$ and by definition $|N| = O(t^{-\frac{1}{2}})$. In addition (71) implies $((1 \pm \varepsilon)t)^{\frac{3}{2}} - t^{\frac{3}{2}} = O(1)$. Using these bounds along with (72), (73) and (78), we estimate

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)| \, dx = O \left( t^{-1/2} \right),$$

as desired. \(\square\)

**Example 3** (continued). Observe we have $p = 0, q = 2, \sigma = 0, d = 1$ in Example 3 of §3.4.1. In this case

$$N(x,t) = \begin{cases} \frac{2}{t} & \text{if} \quad 0 < x < (2t)^{1/2} \\ 0 & \text{otherwise}, \end{cases}$$

and so in fact $u = N$ for times $t \geq 2$. \(\square\)

We will study *systems* of conservation laws in Chapter 11.

### 3.5. PROBLEMS

1. Prove

$$u(x,t,a,b) = a \cdot x - tH(a) + b \quad (a \in \mathbb{R}^n, b \in \mathbb{R})$$

is a complete integral of the Hamilton–Jacobi equation

$$u_t + H(Du) = 0.$$  

2. (a) Write down the characteristic equations for the PDE

$$u_t + b \cdot Du = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty),$$

\(\ast\).
where $b \in \mathbb{R}^n$, $f = f(x, t)$.

(b) Use the characteristic ODE to solve (*) subject to the initial condition

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$  

Make sure your answer agrees with formula (5) in §2.1.2.

3. Solve using characteristics:

(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u$,  
   $u(x_1, 1) = g(x_1)$.

(b) $uu_x + u_{x_2} = 1$,  
   $u(x_1, x_1) = \frac{1}{2} x_1$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} = 3u$,  
   $u(x_1, x_2, 0) = g(x_1, x_2)$.

4. Verify that formula (61) in §3.2.5 provides an implicit solution of the scalar conservation law.

5. Write $L = H^*$, if $H : \mathbb{R}^n \to \mathbb{R}$ is convex.

(a) Let $H(p) = \frac{1}{r} |p|^r$, for $1 < r < \infty$. Show

$$L(q) = \frac{1}{s} |q|^s, \quad \text{where } \frac{1}{r} + \frac{1}{s} = 1.$$  

(b) Let $H(p) = \frac{1}{2} \sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i=1}^n b_i p_i$, where $A = ((a_{ij}))$ is a symmetric, positive definite matrix, $b \in \mathbb{R}^n$. Compute $L(q)$.

6. Let $H : \mathbb{R}^n \to \mathbb{R}$ be convex. We say $q$ belongs to the \textit{subdifferential} of $H$ at $p$, written

$$q \in \partial H(p),$$

if

$$H(r) \geq H(p) + q \cdot (r - p) \quad \text{for all } r \in \mathbb{R}^n.$$  

Prove $q \in \partial H(p)$ if and only if $p \in \partial L(q)$ if and only if $p \cdot q = H(p) + L(q)$, where $L = H^*$.

7. Prove that the Hopf–Lax formula reads

$$u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}$$

$$= \min_{y \in B(x, R)} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}$$

for $R = \sup_{x \in \mathbb{R}^n} |DH(Dy)|$, $H = L^*$. (This proves \textit{finite propagation speed} for a Hamilton–Jacobi PDE with convex Hamiltonian and Lipschitz continuous initial function $g$. Hint: Use the previous problem.)

8. Let $E$ be a closed subset of $\mathbb{R}^n$. Show that if the Hopf–Lax formula could be applied to the initial-value problem

$$\begin{cases} 
    u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u = \begin{cases} 
    0 & x \in E \\
    +\infty & x \notin E 
\end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{cases}$$

then $u(x, t) = g(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. (Hint: Use the previous problem.)
it would give the solution
\[ u(x, t) = \frac{1}{4t} \text{dist}(x, E)^2. \]

9. Fill in all details for the proof of Lemma 4 in §3.3.3.
10. Assume \( u^1, u^2 \) are two weak solutions of the initial-value problems
\[
\begin{align*}
\begin{cases}
u^1_t + \mathcal{H}(Du^1) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u^i = g^i & \text{on } \mathbb{R}^n \times \{t = 0\} \quad (i = 1, 2),
\end{cases}
\end{align*}
\]
for \( \mathcal{H} \) as in §3.3. Prove the \( L^\infty \)-contraction inequality
\[
\sup_{\mathbb{R}} |u^1(\cdot, t) - u^2(\cdot, t)| \leq \sup_{\mathbb{R}} |g^1 - g^2| \quad (t > 0).
\]

11. Show that
\[
u(x, t) = \begin{cases} 
-\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{if } 4x + t^2 > 0 \\
0 & \text{if } 4x + t^2 < 0
\end{cases}
\]
is an (unbounded) entropy solution of \( u_t + (\frac{u^3}{2})_x = 0 \).

12. Assume \( u(x + z) - u(x) \leq Ez \) for all \( z > 0 \). Let \( u^\varepsilon = \eta_\varepsilon * u \), and show
\[
u^\varepsilon_x \leq E.
\]

13. Assume \( F(0) = 0 \), \( u \) is a continuous integral solution of the conservation law
\[
\begin{align*}
\begin{cases}
u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
u = g & \text{on } \mathbb{R} \times \{t = 0\},
\end{cases}
\end{align*}
\]
and \( u \) has compact support in \( \mathbb{R} \times [0, \infty] \). Prove
\[
\int_{-\infty}^{\infty} u(\cdot, t) \, dx = \int_{-\infty}^{\infty} g \, dx
\]
for all \( t > 0 \).

14. Compute explicitly the unique entropy solution of
\[
\begin{align*}
\begin{cases}
u_t + \left( \frac{u^3}{2} \right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
u = g & \text{on } \mathbb{R} \times \{t = 0\},
\end{cases}
\end{align*}
\]
for
\[
g(x) = \begin{cases} 
1 & \text{if } x < -1 \\
0 & \text{if } -1 < x < 0 \\
2 & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1.
\end{cases}
\]
3.6. REFERENCES

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$.

Section 3.1  A nice reference for this material is Courant–Hilbert [C-H, Chapter 2].

Section 3.2  This derivation of the characteristic differential equations is found in Carathéodory [C]. The proof of Theorem 2 follows Garabedian [G, Chapter 2], John [J, Chapter 1], etc. Chester [CH] and Sneddon [SN] are also good texts for more on first-order PDE. Example 3 in §3.2.2 is from Zwillinger [ZW].

Section 3.3  See Lions [LI], Rund [RU] and Benton [BE] for more on Hamilton–Jacobi PDE. The uniqueness proof, which is due to A. Dougis, is from [BE].

Section 3.4  See Lax [LA] and Smoller [S, Chapters 15,16] (from which I took the proof of Theorem 3, due to O. Oleinik). Theorems 5 and 6 are from DiPerna [DP] and I am indebted to M. Struwe for help with the proofs. A good overall reference on nonlinear waves is Whitham [WH].
and
\[
\| \bar{u} \|_{L^2(0,T;H^2(V))} \leq C \| u \|_{L^2(0,T;H^2(U))},
\]
for an appropriate constant \( C \). In addition, \( \bar{u}' \in L^2(0,T;L^2(V)) \), with the estimate
\[
\| \bar{u}' \|_{L^2(0,T;L^2(V))} \leq C \| u' \|_{L^2(0,T;L^2(U))}.
\]
This follows if we consider difference quotients in the \( t \)-variable, remember the methods in \S 5.8.2, and observe also that \( E \) is a bounded linear operator from \( L^2(U) \) into \( L^2(V) \).

2. Assume for the moment that \( \bar{u} \) is smooth. We then compute
\[
\frac{d}{dt} \left( \int_V |D\bar{u}|^2 \, dx \right) = 2 \int_V D\bar{u} \cdot D\bar{u}' \, dx = 2 \int_V \Delta \bar{u} \bar{u}' \, dx \leq C(\|\bar{u}\|_{H^2(V)}^2 + \|\bar{u}'\|_{L^2(V)}^2).
\]
There is no boundary term when we integrate by parts, since the extension \( \bar{u} = Eu \) has compact support within \( V \). Integrating and recalling (14), (15), it follows that
\[
\max_{0 \leq t \leq T} \| u(t) \|_{H^1(U)} \leq C \left( \| u \|_{L^2(0,T;H^2(U))} + \| u' \|_{L^2(0,T;L^2(U))} \right).
\]
We obtain the same estimate if \( u \) is not smooth, upon approximating by \( u^\varepsilon := \eta_\varepsilon \ast u \), as before. As in the previous proofs, it also follows that \( u \in C([0,T];H^1(U)) \).

3. In the general case that \( m \geq 1 \), we let \( \alpha \) be a multiindex of order \( |\alpha| \leq m \), and set \( v := D^\alpha u \). Then
\[
v \in L^2(0,T;H^2(U)), \quad v' \in L^2(0,T;L^2(U)).
\]
We apply estimate (16), with \( v \) replacing \( u \), and sum over all indices \( |\alpha| \leq m \), to derive (13). \( \square \)

5.10. PROBLEMS

In these exercises \( U \) always denotes an open subset of \( \mathbb{R}^n \).

1. Suppose \( k \in \{0,1,\ldots\} \), \( 0 < \gamma \leq 1 \). Prove \( C^{k,\gamma}(\overline{U}) \) is a Banach space.
2. Let $U, V$ be open sets, with $V \subset\subset U$. Show there exists a smooth function $\zeta$ such that $\zeta \equiv 1$ on $V$, $\zeta = 0$ near $\partial U$. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify $x_w$.)

3. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{\gamma}(U)} \leq \|u\|^{\frac{\gamma-eta}{\gamma}}_{C^{\beta}(U)} \|u\|^{\frac{\beta}{\gamma}}_{C^{\gamma}(U)}.$$ 

4. Assume $U$ is bounded and $U \subset\subset \bigcup_{i=1}^{N} V_i$. Show there exist $C^\infty$ functions $\zeta_i \ (i = 1, \ldots, N)$ such that

$$\begin{cases}
0 \leq \zeta_i \leq 1, \text{ spt } \zeta_i \subset V_i \ (i = 1, \ldots, N) \\
\sum_{i=1}^{N} \zeta_i = 1 \text{ on } U.
\end{cases}$$ 

The functions $\{\zeta_i\}_{i=1}^{N}$ form a partition of unity.

5. Prove that if $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$, then $u$ is equal a.e. to an absolutely continuous function, and $u'$ (which exists a.e.) belongs to $L^p(0,1)$.

6. Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x - y|^{1 - \frac{1}{p}} \left( \int_0^1 |u'|^p \, dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

7. Denote by $U$ the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases}
1 - x_1 & \text{if } x_1 > 0, \ |x_2| < x_1 \\
1 + x_1 & \text{if } x_1 < 0, \ |x_2| < -x_1 \\
1 - x_2 & \text{if } x_2 > 0, \ |x_1| < x_2 \\
1 + x_2 & \text{if } x_2 < 0, \ |x_1| < -x_2.
\end{cases}$$

For which $1 \leq p \leq \infty$ does $u$ belong to $W^{1,p}(U)$?

8. Integrate by parts to prove the interpolation inequality:

$$\int_U |Du|^2 \, dx \leq C \left( \int_U u^2 \, dx \right)^{1/2} \left( \int_U |D^2u|^2 \, dx \right)^{1/2}$$

for all $u \in C^\infty_c(U)$. By approximation, prove this inequality if $u \in H^2(U) \cap H^1_0(U)$.

9. Integrate by parts to prove:

$$\int_U |Du|^p \, dx \leq C \left( \int_U |u|^p \, dx \right)^{1/2} \left( \int_U |D^2u|^p \, dx \right)^{1/2}$$
5.10. PROBLEMS

for $2 \leq p < \infty$ and all $u \in W^{2,p}(U) \cap W^{1,p}_0(U)$. (Hint: $\int_U |Du|^p \, dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^p \, dx$.)

10. Suppose $U$ is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U.$$ 

Prove $u$ is constant a.e. in $U$.

11. Show by example that if we have $\|D_h u\|_{L^1(V)} \leq C$ for all $0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(V)$.

12. Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$, such that $u$ is not Lipschitz continuous on $U$. (Hint: Take $U$ to be the open unit disk in $\mathbb{R}^2$, with a slit removed.)

13. Verify that if $n > 1$, the unbounded function $u = \log \log \left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

14. Let $U$ be bounded, with a $C^1$ boundary. Show that a “typical” function $u \in L^p(U)$ ($1 \leq p < \infty$) does not have a trace on $\partial U$. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

15. Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant $C$, depending only on $n$ and $\alpha$, such that

$$\int_U u^2 \, dx \leq C \int_U |Du|^2 \, dx ,$$

provided

$$|\{x \in U \mid u(x) = 0\}| \geq \alpha , \quad u \in H^1(U).$$

16. Assume $F : \mathbb{R} \to \mathbb{R}$ is $C^1$, with $F'$ bounded. Suppose $U$ is bounded and $u \in W^{1,p}(U)$ for some $1 < p < \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u) u_{x_i} \quad (i = 1, \ldots, n).$$

17. Assume $1 < p < \infty$, and $U$ is bounded.

(i) Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$. 
(ii) Prove $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\} \end{cases}$$

(Hint: $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, for

$$F_\varepsilon(z) := \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \geq 0 \\ 0 & \text{if } z < 0. \end{cases}$$

(iii) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

18. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)},$$

the constant $C$ depending only on $s$ and $n$.

5.11. REFERENCES

Sections 5.2-8 See Gilbarg–Trudinger [G-T, Chapter 7], Lieb–Loss [L-L], Ziemer [Z] and [E-G] for more on Sobolev spaces.

Section 5.5 W. Schlag showed me the proof of Theorem 2.

Section 5.6 J. Ralston suggested an improvement in the proof of Theorem 4.

Section 5.9 See Temam [TE, pp. 248–273].