Lagrangian Dynamics on an infinite-dimensional torus; a Weak KAM theorem

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Abstract

The space $L^2(0,1)$ has a natural Riemannian structure on the basis of which we introduce an $L^2(0,1)$–infinite dimensional torus $\mathbb{T}$. For a class of Hamiltonians defined on its cotangent bundle we establish existence of a viscosity solution for the *cell problem* on $\mathbb{T}$ or, equivalently, we prove a Weak KAM theorem. As an application, we obtain existence of absolute action-minimizing solutions of prescribed rotation number for the one-dimensional nonlinear Vlasov system with periodic potential.

1 Introduction

One of the fundamental problems in the theory of dynamical systems is the search for invariant sets or invariant measures. In the case of Hamiltonian flows on compact finite dimensional Riemannian manifolds there are well developed variational theories (cf. e.g. Fathi [8]) called Aubry/Mather theory, Weak KAM theory. There, the approach is based on the existence of Lipschitzian viscosity solutions for some appropriate Hamilton-Jacobi equation called *cell problem*. Our starting point in adapting this theory to partial differential equations is the following: consider a potential $W \in C^2(\mathbf{T}^1)$ ($\mathbf{T}^d$ denotes the d–dimensional torus) and a system of n particles whose initial positions and velocities are $(M_0 z, \dot{M}_0 z) \in$

*WG gratefully acknowledges the support provided by NSF grants DMS-03-54729 and DMS-06-00791.
†AT gratefully acknowledges the support provided by the School of Mathematics. Key words: mass transfer, Wasserstein metric. AMS code: 49J40, 82C40 and 47J25.

$I\!R \times I\!R$, $z \in \mathbf{Z} := \{1/n, 2/n, ..., 1\}$. Denote by $\sigma_t z$ the position of the z particle at time $t > 0$. Assume that the evolution of the system is governed by the law

$$\ddot{\sigma}_t z = -\frac{1}{n} \sum_{\bar{z} \in \mathbf{Z}} W'(\sigma_t z - \sigma_t \bar{z}), \qquad \sigma_0 z = M_0 z, \quad \dot{\sigma}_0 z = \dot{M}_0 z. \tag{1}$$

This is a Hamiltonian system for the Hamiltonian

$$h(x,p) = \frac{n}{2}|p|^2 + \frac{1}{2n^2} \sum_{i,j=1}^{n} W(x_i - x_j)$$

and $x = (x_1, ..., x_n), p = (p_1, ..., p_n) \in I\!R^n$. The Hamiltonian h is invariant under coordinate permutations and is periodic in its spatial variables. As a consequence, uniqueness in (1) ensures that, if we permute the components of the initial data, the solution for the evolutive system undergoes the same coordinate permutation. In addition, if the initial positions of two solutions differ by an integer, that property is preserved over time. Hence, (1) is an evolutive system on the n–symmetric product of the circle $\mathbf{T}^n/P_n$ where P_n is the set of permutations of n letters. The weak KAM theory has been proven to be a powerful tool for studying periodic orbits and invariant Lagrangian tori of the finite dimensional system (1). The latter are sets of the form $G_\omega := \{(x, \omega_x) |\ x \in \mathbf{T}^n\}$ where ω is a closed one–form on $\mathbf{T}^n$ and satisfies the Hamilton-Jacobi equation $h(x, \omega_x) = \lambda$ in the sense of visco sity, for some real number λ. Assume ω is a smooth closed one–form such that the function $(x, \xi) \to \omega_x(\xi)$ defined on $\mathbf{T}^n \times I\!R^n$ is invariant under the action of the group P_n. One can readily show existence of a function $u \in C^1(\mathbf{T}^n)$ that is invariant under the action of P_n such that $\omega = l_c + du$ where l_c is the linear form $\xi \to l_c(\xi) = c(\xi_1 + ... + \xi_n)$.

The goal of this paper is to extend methods of the weak KAM theory to encompass systems of infinitely many points. A more general formulation of (1) consists in substituting the set of subscript $\mathbf{Z}$ by $I := (0,1)$ so that (1) becomes

$$\ddot{\sigma}_t z = -\int_I W'(\sigma_t z - \sigma_t \bar{z}) d\bar{z}, \qquad \sigma_0 = M, \quad \dot{\sigma}_0 = N. \tag{2}$$

This is an evolutive system on the infinite dimensional manifold $L^2(I)$, a separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_{\nu_0}$ which induces the norm $\| \cdot \|_{\nu_0}$. Here, ν_0 is the one-dimensional Lebesgue measure $\mathcal{L}^1$, restricted to I. The space $L^2(I)$ has a natural differential structure and at each $M \in L^2(I)$ the tangent space at M is $\mathcal{T}_M L^2(I) = L^2(I)$. Hence, the tangent bundle is $\mathcal{T} L^2(I) := L^2(I) \times L^2(I)$ which we identify with the cotangent bundle. The system (2) is an Euler system for the Lagrangian L defined on the tangent bundle by

$$L(M,N) = \frac{1}{2}\|N\|^2_{\nu_0} - \frac{1}{2}\mathcal{W}(M), \qquad \mathcal{W}(M) := \int_{I^2} W(Mz - M\bar{z}) dz d\bar{z} \tag{3}$$

The corresponding Hamiltonian H defined on the cotangent bundle which can be identified with the tangent bundle is

$$H(M,N) = \frac{1}{2}\|N\|^2_{\nu_0} + \frac{1}{2}\mathcal{W}(M). \tag{4}$$

The standard theory of the ordinary differential equations such as the Cauchy-Lipschitz-Picard Theorem [4] provides us with a unique solution of (2). We define the Eulerian flow

$$\Psi(t, M, N) = (\Psi^1(t, M, N), \Psi^2(t, M, N)) = (\sigma_t, \dot{\sigma}_t). \tag{5}$$

The invariance property of h under the action of P_n translates into an invariance property of H under the action of a group $\mathcal{G}$. Here, $\mathcal{G}$ is the set of bijections $G : [0, 1] \to [0, 1]$ such that G, G^{-1} are Borel maps that push ν_0 forward to itself. We introduce $L^2_{\mathbb{Z}}(I)$ as the set of $M \in L^2(I)$ whose ranges are contained in $\mathbb{Z}$. The group $\mathcal{G}$ acts on $L^2(I)$ as to

$$(G, M) \in \mathcal{G} \times L^2(I) \to M \circ G.$$

It also acts on the topological subspace $L^2_{\mathbb{Z}}(I)$ and so, induces a natural action on $\mathbb{T}$ and on the tangent bundle $L^2(I) \times L^2(I)$. The latter is given by

$$(G; M, N) \in \mathcal{G} \times \mathcal{T}L^2(I) \to (M \circ G, N \circ G).$$

The periodicity property of the potential is expressed in terms of $L^2_{\mathbb{Z}}(I)$. We set

$$\mathbb{T} := L^2(I)/L^2_{\mathbb{Z}}(I)$$

and we refer to it as the $L^2(I)$–torus. We say that $\mathcal{W}$ is periodic in the sense that it is constant on the class of equivalence of any $M \in L^2(I)$.

Note that L and H are invariant under the action of $\mathcal{G}$. The curve $t \to \sigma_t \in L^2(I)$ is a solution of (2) if and only if for all $Z \in L^2_{\mathbb{Z}}(I)$ and $G \in \mathcal{G}$, $t \to \sigma_t \circ G + Z$ is also a solution of (2). In the current manuscript, we view (2) as an evolutive system on the infinite dimensional torus $\mathbb{T}$ quotiented by the group $\mathcal{G}$. In other words, we identify the paths $t \to \sigma_t \in L^2(I)$ and $t \to \sigma_t \circ G + Z$. This identification becomes even more natural as we write the kinetic system corresponding to (2). To do so, for each t define the Borel measure f_t on $I\!R \times I\!R$ as the push forward of ν_0 by $(M_t, \dot{M}_t)$:

$$f_t(B) = \nu_0\{z \in I \ : \ (M_t z, \dot{M}_t z) \in B\}$$

for $B \subset \mathcal{T}L^2(I)$ Borel. The measures f_t satisfy the nonlinear Vlasov system

$$\begin{cases} \partial_t f_t + v\partial_x f_t \ = \ \partial_v \Big(f_t \, \partial_x P_t \Big) \\ \qquad\quad P_t(x) \ = \int_{I\!R} W(x - \bar{x}) d\varrho_t(\bar{x}). \end{cases} \tag{6}$$

Here, ϱ_t is the first marginal of f_t, P and $E := \partial_x P$ represent respectively the potential and the electric field of the system. By a solution of (6) we mean $t \to f_t \in AC^2(0, \infty; \mathcal{P}_2(I\!R^2))$ such that

$$\int_{I\!R^2} F_0 df_0 + \int_0^\infty dt \int_{I\!R^2} \big(\partial_t F_t + v\partial_x F_t - \partial_x P_t \, \partial_v F_t\big) df_t = 0 \tag{7}$$

for all $F \in C^1_c([0, \infty) \times \mathbf{T}^1 \times I\!R)$. Let $G \in \mathcal{G}$, $Z \in L^2_{\mathbb{Z}}(I)$ and set

$$f^*_t := (M^*_t, \dot{M}^*_t)_{\#}\nu_0, \quad M^*_t := M_t \circ G + Z.$$

Due to the periodicity property of W it becomes apparent that

$$\int_{I\!R^2} F df_t = \int_{I\!R^2} F df_t^* \tag{8}$$

for all $t \geq 0$ and all bounded $F \in C(\mathbf{T} \times I\!R)$. This proves that $t \to f_t$ satisfies (7) if and only if $t \to f_t^*$ satisfies (7).

Let $\Omega^1(\mathbb{T})$ be the set of closed differential one-forms on $\mathbb{T}$ in the sense of definitions 5.1, 5.2 and pick $\Lambda \in \Omega^1(\mathbb{T})$. We show that there exist a continuous linear one-form C on $L^2(I)$ and $U \in C^1(\mathbb{T})$ such that $M \to \|d_M U\|$ is bounded and $\Lambda = C + dU$. Suppose further that $M \in L^2(I) \to \Lambda_M(M)$ is invariant under the action of $\mathcal{G}$. Then U, C and $(M,N) \in \mathcal{T}L^2(I) \to \Lambda_M(N)$ are invariant under the action of $\mathcal{G}$ and there exists $c \in I\!R$ such that $C(N) = c\int_I N d\nu_0$. In other words, the first equivariant de Rham cohomology group is $I\!R$. Let μ be a Borel probability measure on the tangent bundle $\mathcal{T}L^2(I)$, invariant under the flow Ψ in the sense of definition 3.13. Then

$$\int \Lambda d\mu = \int \langle C, N\rangle_{\nu_0} d\mu = \rho(\mu)\, c \quad \text{where} \quad \rho(\mu) := \int m_1 d\mu, \quad m_1(N) := \int_I N d\nu_0.$$

We refer to $\rho(\mu)$ as the *rotation number* of μ.

In the current manuscript, we are interested in several types of problems. For $c \in I\!R$ we introduce the Lagrangian and Hamiltonians

$$L_c(M,N) = L(M,N) - c\int_I N d\nu_0, \qquad H_c(M,N) = H(M, N+c).$$

The first problem is: find $\lambda \in I\!R$ and $U \in C(\mathbb{T})$ viscosity solution of

$$H_c(M, \nabla_{L^2} U) = \lambda. \tag{9}$$

We assert existence of a solution for the cell problem (9). To simplify our study, later we further assume that

$$W(-z) = W(z) \leq W(0) = 0. \tag{10}$$

We use (9) to establish a second result that is: for each $M \in L^2(I)$ which is monotone nondecreasing, there exists $N \in L^2(I)$ such that

$$\lim_{t\to\infty} \frac{\Psi^1(t,M,N)}{t} = -c, \qquad \lim_{t\to\infty} \Psi^2(t,M,N) = -c. \tag{11}$$

In fact, we have obtained an explicit estimate stronger than the first limit in (11) and which has the following consequence: given a Borel probability measure ϱ_0 on $I\!R$ of finite second moment, there exist

$$\varrho \in AC^2_{loc}(0,\infty;\mathcal{P}_2(I\!R)), \quad u : (0,\infty)\times I\!R \to I\!R \quad \text{Borel}$$

satisfying the following properties: $u_t \in L^2(\varrho_t)$ for $\mathcal{L}^1$–almost every $t > 0$, (ϱ, u) satisfy the Euler system

$$\begin{cases} \partial_t(\varrho_t u_t) + \partial_x(\varrho_t u_t^2) = -\varrho_t \partial_x P_t \\ \partial_t \varrho_t + \partial_x(\varrho_t u_t) = 0 \\ P_t(x) = \int_{I\!R} W(x - \bar{x}) d\varrho_t(\bar{x}). \end{cases} \tag{12}$$

Furthermore,

$$\sup_{t>0} \sqrt{t} \|\mathbf{id}/t + c\|_{\varrho_t} < \infty, \qquad \lim_{t\to\infty} \|u_t + c\|_{\varrho_t} = 0.$$

Note that, in particular, we obtain solutions of prescribed asymptotic velocity and kinetic energy.

Let $\mathcal{P}(\mathcal{T}L^2(I))$ be the set of Borel probabilities on $\mathcal{T}L^2(I)$. A variational problem of interest in this manuscript is: find μ^* minimizer for

$$\inf_{\mu \in \mathcal{P}(\mathcal{T}L^2(I))} \Big\{ \int L d\mu \ : \ \rho(\mu) = c, \ \mu \text{ is invariant under the flow } \Psi \Big\}. \tag{13}$$

In case (10) holds, we show that the solutions of (13) are trivial.

We have chosen the Vlasov system as a simple model to illustrate the use of the weak KAM theory in understanding qualitative behavior of PDEs appearing in kinetic theory, for several reasons. Firstly, they provide a simple link between finite and infinite dimensional systems. Secondly, they are one of the most frequently used kinetic models in statistical mechanics. Existence and uniqueness of global solutions for the initial value problem are well understood [3], [13], [7]. It has already been noticed that (6) can be regarded as an infinite-dimensional Hamiltonian ODE on the space $\mathcal{P}_2(I\!R^2)$, the set of Borel probability measures on $I\!R^2$ with finite second-order moments [1] [9] [16]. Indeed, if

$$\mathcal{H}(f) := \int_{I\!R^2} \left[\frac{v^2}{2} + \frac{1}{2} \int_{I\!R^2} W(x - \bar{x}) \mathrm{d}f(\bar{x}, \bar{v}) \right] \mathrm{d}f(x, v),$$

then in [9] they introduced a Poisson structure on $\mathcal{P}_2(I\!R^2)$, which induces a Hamiltonian vector field $X_{\mathcal{H}}$ such that (6) is equivalent to the ordinary differential equation

$$\dot{f}_t = X_{\mathcal{H}}(f_t).$$

In this paper we have searched for special solutions which allow for a connection with a more conventional way of regarding (6) as Hamiltonian. We assume the initial data to be of the form $f_0 = (M, N)_{\#}\nu_0$ where $M, \ N \in L^2(I)$ so that the unique solution of (6) retains the same structure.

For the convenience of the reader, we collect notation used throughout this manuscript.

Notation and Definitions

The euclidean norm on $I\!R^d$ and standard inner product are respectively denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$. We denote the n–dimensional torus by $\mathbf{T}^n$. If $x \in I\!R^n$, $|x|_{\mathbf{T}^n}$ is the infinum of $|x + k|$ over the set of $k \in \mathbb{Z}^n$. $\mathbf{id}$ denotes the identity map on $I\!R^d$ for $d \geq 1$.

We denote by $\mathcal{L}^d$ the Lebesgue measure on $I\!R^d$, whereas I denotes the unit interval $(0,1)$. $\mathcal{P}_2(I\!R^d)$ stands for the set of Borel probability measures μ on $I\!R^d$ with finite second moments.

If $\mu \in \mathcal{P}_2(I\!R^d)$, $L^2(\mu)$ is the set of function $\xi : I\!R^d \to I\!R^d$ which are μ measurable and such that $\int_{I\!R^d} |\xi|^2 d\mu$ is finite. This is a separable Hilbert space for the inner product $\langle \xi, \bar\xi \rangle_\mu = \int_{I\!R^d} \langle \xi, \bar\xi \rangle d\mu$. We denote the associated norm by $\|\cdot\|_\mu$.

If $(\mathbf{E}, |\cdot|)$ is a norm space, $L^2(0,T;\mathbf{E})$ is the set of Borel functions $M:(0,T) \to \mathbf{E}$ such that $\int_0^T |M_t|^2_{\mathbf{E}} dt < \infty$. Here and throughout this work, we write M_t in place of $M(t)$. When μ is a Borel probability measure on $I\!R^d$ and $\mathbf{E} = L^2(\mu)$, we identify $L^2(0,T;L^2(\mu))$ with $L^2(\mathcal{L}^1|_{(0,T)} \times \mu)$.

We also recall that if $M : I\!R^d \to I\!R^d$ is a Borel map and $\mu \in \mathcal{P}_2(I\!R^d)$ then $M_\# \mu$ is the Borel measure defined by $M_\#\mu[C] = \mu[M^{-1}(C)]$ for all Borel sets $C \subset I\!R^d$.

If μ, ν are Borel probability measures on the real line and μ is atom-free, then it is known that there exists a unique (up to a set of μ–zero measure) optimal map pushing forward μ to ν. It is called *the monotone rearrangement* and is obtained as $G^{-1} \circ F$, where F, G are the cumulative distribution functions of μ and ν. We have

$$G(y) = \nu(-\infty, y] \text{ and } G^{-1}(x) = \inf\{y \in I\!R : G(y) \geq x\}.$$

Note that G^{-1} is the left-continuous *generalized inverse* of G. In this work, optimal map on the real line always means *left continuous* optimal map.

Suppose $(\mathcal{S}, \mathrm{dist})$ is a complete metric space and $\sigma : (0,T) \to \mathcal{S}$. We write σ_t to denote the value of σ at $t : \sigma_t := \sigma(t)$. If there exists $\beta \in L^2(0,T)$ such that $\mathrm{dist}(\sigma_t, \sigma_s) \leq \int_s^t \beta(u) du$ for every $s < t$ in $(0,T)$, we say that σ is absolutely continuous. We denote by $AC^2(0,T;\mathcal{S})$ the set of $\sigma : (0,T) \to \mathcal{S}$ that are absolutely continuous.

We denote by $\mathcal{G}$, the set of bijections $G : [0,1] \to [0,1]$ such that G, G^{-1} are Borel and push ν_0 forward to itself. The operator on $\mathcal{G}$ is the composition on the set of functions.

We denote by $L^2_{\mathbb{Z}}(I)$ the set of $M \in L^2(I)$ with ranges in $\mathbb{Z}$.

Definition 1.1. *Let $U : L^2(I) \to I\!R \cup \{\infty\}$. (i) We say that U is periodic if it is constant on the class of equivalence of $M \in L^2(I)$ in $\mathbb{T}$. (ii) We say that U is invariant under the action of $\mathcal{G}$ if $U(M \circ G) = U(M)$ for all $M \in L^2(I)$ and $G \in \mathcal{G}$.*

Recall that $I := (0,1)$, ν_0 is the one-dimensional Lebesgue measure restricted to I, $\langle \cdot, \cdot \rangle_{\nu_0}$ and $\|\cdot\|_{\nu_0}$ are the inner product and norm on $L^2(I)$. We identify the one-dimensional torus $\mathbf{T}^1$ with $[0,1)$. We denote the norm on the n–dimensional torus by $\|\cdot\|_{\mathbf{T}^n}$. For $n \geq 1$ integer, P_n is the set of permutation of n letters.

2 Action of a subgroup of the set of measure preserving maps

The Aubry/Mather theory studies dynamical systems on finite dimensional manifolds without boundary. Typical examples are systems evolving on the n-dimensional torus $\mathbf{T}^n$. In this work, we are interested in systems of undistinguishable n particles of equal mass $1/n$

and the limiting systems as n tends to infinity. As it is commonly done in physics, especially in *String Theory*, we identify the set of systems of undistinguishable n particles with the so-called n^{th} symmetric product of the circle $\mathbf{T}^n/P_n$. Observe that P_n is a non commutative group which acts on $I\!R^n$ and so on $\mathbf{T}^n$: for $x = (x_1, ..., x_n) \in I\!R^n$ and $\sigma \in P_n$ the action $P_n \times \mathbf{T}^n \to \mathbf{T}^n$ associates to (σ, x), x^σ the vector obtained by permuting the components of x according to σ. A particularity of having an action is that $(x^\sigma)^\tau = x^{\sigma\circ\tau}$ for $\tau \in P_n$. The n^{th} symmetric products of the circle have been quite a bit studied in topology and its cohomology groups are well-understood (cf. the review paper [14]). When $n = 1$ this is the circle which is a smooth manifold without a boundary. When $n \geq 2$, the action is not free in the sense that we may find $x \in I\!R^n$ and $\sigma \in P_n$ such that σ is not the identity map and $x^\sigma = x$. $\mathbf{T}^n/P_n$ is then a manifold with a boundary.

For our purpose, to encompass systems of infinitely many particles, we substitute P_n by a group which has infinitely many elements, the set $\mathcal{G}$ introduced earlier. The action of $\mathcal{G}$ on $\mathbb{T}$ yields a quotient space which can be interpreted as the ∞^{th} symmetric product of the circle. The set we obtain here is different from the limit as $n \to \infty$ of the n^{th} symmetric product of the circle considered in [15].

In this section we adopt a differential structure on $L^2(I)$ and study the infinite dimensional torus on that set.

2.1 The $L^2(I)$–torus

We consider the topological group $\big(L^2(I), +\big)$ and its subset $L^2_{\mathbb{Z}}(I)$ which is a topological subgroup and is locally compact. The the greatest integer function $\hat{\ }: I\!R \to [0, 1)$ provides us with a natural map of $L^2(I)$ onto $L^2(I)/L^2_{\mathbb{Z}}(I)$ given by

$$M \to \pi(M), \quad \pi(M)(x) = \widehat{M(x)}.$$

We set

$$\mathcal{T}L^2(I) := L^2(I) \times L^2(I), \qquad \mathbb{T} := L^2(I)/L^2_{\mathbb{Z}}(I),$$

Let $|\cdot|_{\mathbf{T}^1}$ be the norm on the flat torus $\mathbf{T}^1 := I\!R/\mathbb{Z}$. To alleviate notation when there is no possible confusion we still denote the class of equivalence of $M \in L^2(I)$ by M. The norm on $L^2(I)$ induces a distance $\text{dist}_{\mathbb{Z}}$ on $\mathbb{T}$ given by

$$\text{dist}_{\mathbb{Z}}(M_1, M_2) = \inf_{Z \in L^2_{\mathbb{Z}}(I)} \|M_1 - M_2 - Z\|_{L^2(I)}. \tag{14}$$

The infimum in (14) is attained. Note that

$$\text{dist}_{\mathbb{Z}}(M_1, M_2) = \Big\| |M_1 - M_2|_{\mathbf{T}^1} \Big\|_{L^2(I)}. \tag{15}$$

Since the diameter of $\mathbf{T}^1$ is $1/2$, (15) implies that the diameter of $\mathbb{T}$ is $1/2$. Observe that $\pi : L^2(I) \to \mathbb{T}$ is 1–Lipschitz.

Proposition 2.1. $(\mathbb{T}, \text{dist}_{\mathbb{Z}})$ *is a complete, separable metric space.*

Proof: The fact that $\mathrm{dist}_{\mathbb{Z}}$ is a metric is a direct consequence of (15). The facts that π is surjective, 1–Lipschitz and $L^2(I)$ is separable imply that $\mathbb{T}$ is separable. Recall that $L^2(I)$ is complete and π is 1–Lipschitz. Furthermore, for $M_1, M_2 \in L^2(I)$ we may choose $\bar{M}_2 \in L^2(I)$ such that $\pi(\bar{M}_2) = \pi(M_2)$ and $\mathrm{dist}_{\mathbb{Z}}(M_1, M_2) = \|M_1 - \bar{M}_2\|_{L^2(I)}$. These facts imply that $(\mathbb{T}, \mathrm{dist}_{\mathbb{Z}})$ is a complete metric space. QED.

2.2 The space $\mathbb{T}/\mathcal{G}$ and the Wasserstein space $\mathcal{P}(\mathbf{T}^1)$

In this section, $W_{\mathbf{T}^1}$ denotes the Wasserstein distance on the torus $\mathbf{T}^1$. We recall that if $\mu, \nu \in \mathcal{P}(\mathbf{T})$ and $\Gamma(\mu,\nu)$ is the set of Borel measures on $\mathbf{T}^1 \times \mathbf{T}^1$ which have μ and ν as marginals, then

$$W^2_{\mathbf{T}^1}(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbf{T}^1\times\mathbf{T}^1} |x-y|^2_{\mathbf{T}^1}\, d\gamma(x,y). \tag{16}$$

We will identify $\mathcal{P}(\mathbf{T}^1)$ with $\mathcal{P}([0,1))$ using the bijection between $[0,1)$ and $\mathbf{T}^1$ given by the Borel map

$$t \in [0,1) \to e(t) = (\cos(2\pi t), \sin(2\pi t)). \tag{17}$$

The group $\mathcal{G}$ is a non commutative group which acts on $L^2(I)$: $(G, M) \to M \circ G$. This is an action which preserves the norm of M. Since $\mathcal{G}$ also acts on $L^2_{\mathbb{Z}}(I)$, it provides an action on the quotient space $\mathbb{T}$. The metric on $\mathbb{T}$ induces a function which we refer to as a weak metric on $\mathbb{T}/\mathcal{G}$: for $M, \bar{M} \in L^2(I)$ we set

$$\mathrm{dist}_{weak}(M, \bar{M}) = \inf_{G\in\mathcal{G}} \mathrm{dist}_{\mathbb{Z}}(M\circ G, \bar{M}).$$

It is symmetric and it satisfies the triangle inequality. But dist_{weak} is not a metric on $\mathbb{T}/\mathcal{G}$. Lemma 2.6 (ii) shows that dist_{weak} is not a metric on $\mathbb{T}/\mathcal{G}$ as we may have $\mathrm{dist}_{weak}(M, \bar{M}) = 0$ for M and $\bar{M}$ which do not have the same projection in $\mathbb{T}/\mathcal{G}$.

Let χ^n_i be the characteristic function of the interval $A_i := \big((i-1)/n, i/n\big)$ and set

$$\mathcal{C}_n := \Big\{ \sum_{i=1}^n x_i\, \chi^n_i \mid x \in I\!\!R^n \Big\}. \tag{18}$$

Lemma 2.2. *For any positive integer n, the restriction of dist_{weak} to the finite dimensional space $\mathcal{C}_n$ is a metric.*

Proof: It suffices to see that if $x, y \in I\!\!R^n$ and $M_x := \sum_{i=1}^n x_i\, \chi^n_i$, $M_y := \sum_{i=1}^n y_i\, \chi^n_i$ then

$$\mathrm{dist}^2_{weak}(M_x, M_y) = \frac{1}{n}\|x^\sigma - y\|^2_{\mathbf{T}^n} = W^2_{\mathbf{T}^1}\Big(\pi(M_x)_\#\nu_0, \pi(M_y)_\#\nu_0\Big), \tag{19}$$

for some $\sigma \in P_n$. QED.

For $k \geq 1$ integer, let $C^k_{per}(I\!R)$ denote the set of $f \in C^k(I\!R)$ that are 1–periodic: $f(z+1) = f(z)$ for all $z \in I\!R$. We define the relation $\sim$ on $L^2(I)$ as follows: $M \sim \bar{M}$ in $L^2(I)$ if

$$\int_I f(Mz)dz = \int_I f(\bar{M}z)dz \tag{20}$$

for all $f \in C^1_{per}(I\!R)$ (in fact, we could substitute $C^1_{per}(I\!R)$ by any $C^k_{per}(I\!R)$) or by the orthonormal basis of $L^2(I;\mathbb{C})$, $\{e^{i2\pi tk}\}_{k=0}^{\infty}$. Here $\mathbb{C}$ is the set of complex numbers and $i^2 = -1$. We denote by $[M]$ the set of $\bar{M} \in L^2(I)$ such that $M \sim \bar{M}$. If $\mu, \bar{\mu} \in \mathcal{P}_2(I\!R)$ we write $\mu \sim \bar{\mu}$ if $M_\mu \sim M_{\bar{\mu}}$.

Lemma 2.3. *If $\mu, \bar{\mu} \in \mathcal{P}([0,1))$ are such that $\mu \sim \bar{\mu}$ then $\mu = \bar{\mu}$.*

Proof: Let e be the bijection defined in (17) and set $\mu^* := e_\#\mu$, $\bar{\mu}^* := e_\#\bar{\mu}$. If $f \in C(\mathbf{T}^1)$ then $F := f \circ e \in C_{per}(I\!R)$ and so,

$$\int_{\mathbf{T}^1} f d\mu^* = \int_I F d\mu = \int_I F d\bar{\mu} = \int_{\mathbf{T}^1} f d\bar{\mu}^*.$$

Thus, $\mu^* = \bar{\mu}^*$ and so, $\mu = (e^{-1})_\#\mu^* = (e^{-1})_\#\bar{\mu}^* = \bar{\mu}$. QED.

Corollary 2.4. *Let $M, M^* \in L^2(0,1)$. Then $M \sim M^*$ if and only if $(M - \hat{M})_\#\nu_0 = (M^* - \hat{M}^*)_\#\nu_0$.*

Proof: Let $M, M^* \in L^2(0,1)$, set $\mu := (M - \hat{M})_\#\nu_0$ and $\mu^* := (M^* - \hat{M}^*)_\#\nu_0$. Observe that $\mu, \mu^* \in \mathcal{P}([0,1))$ and if $F \in C^1_{per}(I\!R)$ then

$$\int_I F d\mu = \int_{I\!R} F(M) d\nu_0, \qquad \int_I F d\mu^* = \int_{I\!R} F(M^*) d\nu_0. \tag{21}$$

Therefore, $M \sim M^*$ if and only if $\int_{I\!R} F(M)d\nu_0 = \int_{I\!R} F(M^*)d\nu_0$, which, by (21), is equivalent to $\int_I F d\mu = \int_I F d\mu^*$. By lemma 2.3 this is equivalent to $\mu = \mu^*$. QED.

Remark 2.5. *Let $\mu, \nu \in \mathcal{P}(\mathbf{T}^1)$. The Monge-Kantorovich duality gives existence of two periodic functions $u, v : I\!R \to I\!R$ that are 1–Lipschitz such that $u(x) + v(y) \leq |x-y|^2_{\mathbf{T}^1}$ for all $x, y \in I\!R$ and*

$$W^2_{\mathbf{T}^1}(\mu, \nu) = \int_{\mathbf{T}^1} u d\mu + \int_{\mathbf{T}^1} v d\nu.$$

Lemma 2.6. *Let $M, \bar{M} \in L^2(0,1)$. (i) If M and $\bar{M}$ have the same projection in the quotient space $\mathbb{T}/\mathcal{G}$ then $M \sim \bar{M}$. (ii) $M \sim \bar{M}$ if and only if $\mathrm{dist}_{weak}(M, \bar{M}) = 0$.*

Proof: Part (i) is trivial and so, we shall only prove (ii). It is also straightfoward to obtain that if $\mathrm{dist}_{weak}(M,\bar{M})=0$ then $M\sim\bar{M}$. To prove the converse statement, we assume in the sequel that $M\sim\bar{M}$. We may assume without loss of generality that $M,\bar{M}$ have their ranges in $[0,1)$. Set $M^n=\Pi^n(M)$ and $\bar{M}^n=\Pi^n(\bar{M})$ where Π^n is the orthogonal projection on $\mathcal{C}_n$. Thanks to remark 2.5 there exist two periodic functions $u_n,v_n:I\!\!R\to I\!\!R$ that are 1–Lipschitz such that $u_n(x)+v_n(y)\le|x-y|^2_{\mathbf{T}^1}$ for all $x,y\in I\!\!R$ and

$$
\begin{aligned}
W^2_{\mathbf{T}^1}(M^n_{\#}\nu_0,\bar{M}^n_{\#}\nu_0) &= \int_{\mathbf{T}^1}\big[u_n(M^n)+v_n(\bar{M}^n)\big]d\nu_0\\
&= \int_{\mathbf{T}^1}\big[u_n(M^n)-u_n(M)+v_n(\bar{M}^n)-v_n(\bar{M})\big]d\nu_0\\
&+ \int_{\mathbf{T}^1}\big[u_n(M)+v_n(M)\big]d\nu_0 \qquad (22)\\
&\le \int_{\mathbf{T}^1}(|M-M^n|+|\bar{M}-\bar{M}^n|)d\nu_0. \qquad (23)
\end{aligned}
$$

To obtain (22) we have used that $\int_{\mathbf{T}^1}v_n(M)d\nu_0=\int_{\mathbf{T}^1}v_n(\bar{M})d\nu_0$, whereas the last expression in it has disappeared because $u_n(x)+v_n(y)\le|x-y|^2_{\mathbf{T}^1}$. We exploit (19), (23) and Young's inequality to conclude that

$$
\mathrm{dist}_{weak}(M^n,\bar{M}^n)\le\|M-M^n\|_{\nu_0}+\|\bar{M}-\bar{M}^n\|_{\nu_0}. \qquad (24)
$$

The fact that dist_{weak} satisfies the triangle inequality and is bounded above by $\|\cdot\|_{\nu_0}$ yields

$$
\mathrm{dist}_{weak}(M,\bar{M})\le\|M-M^n\|_{\nu_0}+\mathrm{dist}_{weak}(M^n,\bar{M}^n)+\|\bar{M}-\bar{M}^n\|_{\nu_0}.
$$

This, together with (23), (24) and the fact that Π^n converges pointwise to the identity map in $L^2(I)$, yields the desired result. QED.

Definition 2.7. *We say that $U:L^2(I)\to I\!\!R$ is rearrangement invariant if it is invariant under the action of $\mathcal{G}$: $U(M)=U(M\circ G)$ for every $M\in L^2(I)$ and $G\in\mathcal{G}$.*

Proposition 2.8. *Let $U:L^2(I)\to I\!\!R$ be continuous and periodic. Then the following assertions are equivalent: (i) U is rearrangement invariant*
(ii) $U(M)=U(\bar{M})$ for all $M,\bar{M}\in L^2(I)$ such that $[M]=[\bar{M}]$.

Proof: Clearly, (ii) implies (i). Next suppose (i) and let $M,\bar{M}\in L^2(I)$ be such that $[M]=[\bar{M}]$. By lemma 2.6, $\mathrm{dist}_{weak}(M,\bar{M})=0$ and so, we may find $Z_n\in L^2_{\mathbb{Z}}(I)$, $G_n\in\mathcal{G}$ such that

$$
\lim_{n\to\infty}\|M-\bar{M}\circ G_n-Z_n\|_{\nu_0}=0.
$$

This, together with the fact that U is continuous implies

$$
U(M)=\lim_{n\to\infty}U(\bar{M}\circ G_n+Z_n)=U(\bar{M}).
$$

QED.

2.3 The first equivariant de Rham cohomology group on $\mathbb{T}$ under the $\mathcal{G}$ action

Recall that $U : L^2(I) \to I\!R$ is Fréchet differentiable if for each $M \in L^2(I)$ there exists $\xi \in \mathcal{T}_M L^2(I) := L^2(I)$ such that $U(M+X) = U(M) + \langle X, \xi\rangle_{\nu_0} = o(\|X\|_{\nu_0})$. In that case, ξ is uniquely determined and we call it the gradient of U at M. We write $\nabla_{L^2} U(M) = \xi$. The differential of U at M is the one-form

$$dU : \mathcal{T} L^2(I) \to I\!R, \qquad dU_M(X) = \langle X, \xi\rangle_{\nu_0}$$

We next compute the first cohomology group of $\mathbb{T}$, and then the first equivariant cohomology group of $\mathbb{T}$ under the action of $\mathcal{G}$. According to subsection 5.1 every differentiable closed one-form Λ on $L^2(I)$ is an exact form in the sense that $\Lambda = dU$ for some real valued differentiable function U defined on $L^2(I)$. The smoothness properties imposed on Λ in that subsection imply that U is twice differentiable. Here are going to state a result on a larger class of one–forms in the sense that we do not require them to be differentiable.

Proposition 2.9. *Assume $S : L^2(I) \to I\!R$ is Fréchet differentiable and Lipschitz. (i) If dS is periodic in the sense that $d_{M+Z}S = d_M S$ for all $M \in L^2(I)$ and $Z \in L^2_{\mathbb{Z}}(I)$, then there exist a unique $C \in L^2(I)$ and $U : L^2(I) \to I\!R$ periodic such that $S(M) = U(M) + \langle C, M\rangle_{\nu_0}$. (ii) If, in addition, $M \to d_M S(M)$ is rearrangement invariant then C is a constant function and U is rearrangement invariant.*

Proof: 1. We assume without loss of generality that $S(0) = 0$ and let κ be the Lipschitz constant of S. Note that if $Z \in L^2_{\mathbb{Z}}(I)$ then because dS is periodic, the gradient of $M \to S(M+Z) - S(M)$ vanishes and so, $M \to S(M+Z) - S(M)$ depends only on Z. We write

$$\mathcal{C}(Z) = S(M+Z) - S(M). \tag{25}$$

Clearly, $\mathcal{C}$ is additive and Lipschitz and so, the function η defined on the Borel subsets of $[0,1]$ by

$$\eta(E) = \mathcal{C}(\chi_E) = S(\chi_E)$$

is countably additive. Since $|\eta(E)| = |S(\chi_E) - S(0)| \le \kappa \|\chi_E\|_{\nu_0}$, η is absolutely continuous. By the Radon-Nikodym theorem, there exists $C \in L^1(I)$ such that $\eta(E) = \int_E C d\nu_0$ for all $E \subset [0,1]$ Borel. Define

$$\tilde{\mathcal{C}}(X) = \int_I C X d\nu_0 \qquad X \in L^\infty(I).$$

2. Claim. $C \in L^2(I)$ and $\tilde{\mathcal{C}} = \mathcal{C}$.
Proof: It suffices to show that $\tilde{\mathcal{C}}(X) \le \kappa \|X\|_{\nu_0}$ and $\tilde{\mathcal{C}}(X) = \mathcal{C}(X)$ for all

$$X = \sum_{i=1}^n \frac{p_i}{q} \chi_{A_i}, \qquad p_i, q \in \mathbb{Z}, q > 0.$$

In case X is of the above form, the second assertion is easy to check. Hence,

$$|\tilde{\mathcal{C}}(X)| = \frac{1}{q}\Big|\mathcal{C}(\sum_{i=1}^n p_i\chi_{A_i})\Big| = \frac{1}{q}\Big|S(\sum_{i=1}^n p_i\chi_{A_i})\Big| \leq \frac{\kappa}{q}\Big\|\sum_{i=1}^n p_i\chi_{A_i})\Big\|_{\nu_0} = \kappa\|X\|_{\nu_0},$$

which proves the claim.

3. Define $s^n : I\!\!R^n \to I\!\!R$ by

$$s^n(x) = S(M_x), \quad x = (x_1, ..., x_n) \in I\!\!R^n, \quad M_x := \sum_{i=1}^n x_i\chi_{\left(\frac{i-1}{n}, \frac{i}{n}\right)}.$$

Then s^n is $\kappa/\sqrt{n}$–Lipschitz, differentiable everywhere and ∇s^n is periodic:

$$\nabla s^n(x+k) = \nabla s^n(x) \quad \forall x \in I\!\!R^n, k \in \mathbb{Z}^n.$$

Because the de Rham cohomology group of $\mathbf{T}^n$ is $I\!\!R^n$, we obtain existence of a $c^n = (c_1^n, ..., c_n^n) \in I\!\!R^n$ and $u^n : I\!\!R^n \to I\!\!R$ periodic, Lipschitz, differentiable such that

$$s^n(x) = u^n(x) + \langle M_x, M_{c^n}\rangle_{\nu_0}.$$

Let e_i^n be the ith standard unit vector of $I\!\!R^n$. We have by claim 2,

$$\frac{1}{n}c_i^n = \mathcal{C}(M_{e_i^n}) = \int_{\frac{i-1}{n}}^{\frac{i}{n}} C d\nu_0. \tag{26}$$

For $x \in I\!\!R^n$ we set $\tilde{x} = (x_1, x_1, ..., x_n, x_n)$. This provides an embedding of $I\!\!R^n$ into $I\!\!R^{2n}$. Note that $M_{\tilde{x}} = M_x$ and thanks to (26) $\langle M_{\tilde{x}}, M_{c^{2n}}\rangle_{\nu_0} = \langle M_x, M_{c^n}\rangle_{\nu_0}$. Hence,

$$u^{2n}(\tilde{x}) = S(M_{\tilde{x}}) - \langle M_{\tilde{x}}, M_{c^{2n}}\rangle_{\nu_0} = S(M_x) - \langle M_x, M_{c^n}\rangle_{\nu_0} = u^n(x).$$

Let $\mathcal{H}$ be the union of all the $\{M_x |\ x \in I\!\!R^{2^k}\}$. We have proven existence of a function U on $\mathcal{H}$ such that $U(M_x) = u^n(x)$ for $x \in I\!\!R^n$ and

$$S(M) = U(M) + \langle M, C\rangle_{\nu_0} \tag{27}$$

for $M \in \mathcal{H}$. Since S and $\mathcal{C}$ are Lipschitz, so is U. Consequently, U admits a unique existension on $L^2(I)$ (the closure of $\mathcal{H}$) still denoted by U. It is obvious that (27) still holds on $L^2(I)$ and so, U is differentiable and Lipschitz on $L^2(I)$ as the difference of two functions satisfying these properties. Because the restriction of U to each $\{M_x |\ x \in I\!\!R^{2^k}\}$ is periodic, U is periodic on $L^2(I)$.

4. Suppose, in addition, that $M \to d_M S(M)$ is rearrangement invariant. Since

$$S(M) = S(0) + \int_0^1 \langle \nabla_{L^2} S(tM), M\rangle_{\nu_0} dt,$$

we obtain that S is invariant under the action of $\mathcal{G}$ and so, setting $M \equiv 0$ in (25) we conclude that $\mathcal{C}$ is also invariant under the action of $\mathcal{G}$. Using that $\mathcal{C}(Z) = \int_I CZ d\nu_0$ we conclude that C is constant. QED.

Remark 2.10. *There is an obvious one-to-one correspondence between the set of rearrangement invariant maps $U : L^2(I) \to I\!R$ and the set of maps $\bar{U} : \mathcal{P}_2(I\!R) \to I\!R$. Indeed, if $\bar{U} : \mathcal{P}_2(I\!R) \to I\!R$ we can define $U(M) = \bar{U}(\mu)$ where $M_{\#}\nu_0 = \mu$. However, $\bar{U}$ maybe differentiable in the sense of [2] whereas U may not be. For instance $\bar{U}(\mu) = \int_{I\!R\times I\!R} |z - w| d\mu(z) d\mu(w)$ is differentiable in the sense of [2] whereas $U(M) = \int_{I\times I} |Mz - Mw| dz dw$ is not.*

2.4 Topological properties of $\mathbb{S}$

We shall show that the norm $\|\cdot\|_{L^2(I)}$ induces a metric $dist_{\mathbb{S}}$ on $\mathbb{S} := L^2(0,1)/\sim$ defined by

$$dist_{\mathbb{S}}([M],[\bar{M}]) = \inf_{(M^*,\bar{M}^*)} \big\{ \|M^* - \bar{M}^*\|_{L^2(I)} : \; M \sim M^*, \bar{M} \sim \bar{M}^* \big\}. \tag{28}$$

Remark 2.11. *Recall that the map $L^2(I) \ni M \to M^{mon} \in L^2(I)$ such that M^{mon} is monotone nondecreasing and $M^{mon}_{\#}\nu_0 = M_{\#}\nu_0$ is 1–Lipschitz (cf. e.g. [10]).*

Lemma 2.12. *Let $M, \bar{M} \in L^2(I)$. Then the minimum in (28) is attained for a pair $(M^*, \bar{M}^*)$ such that $M^* : [0,1] \to [0,1)$ is monotone nondecreasing, $\bar{M}^* : [0,1] \to [-3/2, 3/2]$ is monotone nondecreasing and $|M^* - \bar{M}^*|_{\mathbf{T}^1} = |M^* - \bar{M}^*|$.*

Proof: Let $M_k \sim M$, $\bar{M}_k \sim \bar{M}$ be such that

$$dist_{\mathbb{S}}([M],[\bar{M}]) = \lim_{n\to\infty} \|M_k - \bar{M}_k\|_{L^2(I)}.$$

We may assume without loss of generality that $|M_k - \bar{M}_k|_{\mathbf{T}^1} = |M_k - \bar{M}_k|$ and by remark 2.11 suppose that $M_k, \bar{M}_k$ are monotone nondecreasing. We may also assume that the range of M_k is contained in $[0,1)$, that of $\bar{M}_k$ is contained in $[-3/2, 3/2]$. By corollary 2.4 $M_{k\#}\nu_0 = M_{1\#}\nu_0$. Because both M_k and M_1 are monotone nondecreasing, we have $M_k = M_1 =: M^*$. Passing to a subsequence if necessary and using Helly's theorem, we may assume that $\{\bar{M}_k\}_{k=1}^\infty$ converges pointwise and in $L^2(I)$ to a monotone nondecreasing function $\bar{M}^* \sim \bar{M}$. Observe that this convergence ensures that $M^* \sim M$. Hence,

$$dist_{\mathbb{S}}([M],[\bar{M}]) = \lim_{k\to\infty} \|M_k - \bar{M}_k\|_{L^2(I)} = \|M^* - \bar{M}^*\|_{L^2(I)}$$

and so, $dist_{\mathbb{S}}([M],[\bar{M}]) = \|M^* - \bar{M}^*\|_{L^2(I)}$. QED.

Theorem 2.13. $(\mathbb{S}, dist_{\mathbb{S}})$ *is a metric space.*

Proof: The fact that $dist_{\mathbb{S}}$ is symmetric is obvious. Let $M, \bar{M}, \tilde{M} \in L^2(I)$. Using the fact that there exists a minimizer in (28), one readily obtain that $dist_{\mathbb{S}}([M],[\bar{M}]) = 0$ if and only if $[M] = [\bar{M}]$. By lemma 2.12, we may find $M^* \sim M$, $\bar{M}^* \sim \bar{M} \sim \bar{M}^*_1$ and $\tilde{M}^* \sim \tilde{M}$ satisfying the following properties:

$$dist_{\mathbb{S}}([M],[\bar{M}]) = \|M^* - \bar{M}^*\|_{L^2(I)}, \quad dist_{\mathbb{S}}([\bar{M}],[\tilde{M}]) = \|\bar{M}^*_1 - \tilde{M}^*\|_{L^2(I)}, \tag{29}$$

$\bar{M}^*$ and $\bar{M}_1^*$ are monotone nondecreasing and have their ranges in $[0,1)$. Since $\bar{M}^* \sim \bar{M}_1^*$, corollary 2.4 gives that $\bar{M}^*_{\#}\nu_0 = (\bar{M}_1^*)_{\#}\nu_0 =: \bar{\mu}$ and so, $\bar{M}^* = \bar{M}_1^*$. This, together with (29) implies

$$dist_{\mathbb{S}}([M],[\bar{M}]) + dist_{\mathbb{S}}([\bar{M}],[\tilde{M}]) = \|M^* - \bar{M}^*\|_{L^2(I)} + \|\bar{M}^* - \tilde{M}^*\|_{L^2(I)} \geq \|M^* - \tilde{M}^*\|_{L^2(I)}.$$

This proves that

$$dist_{\mathbb{S}}([M],[\bar{M}]) + dist_{\mathbb{S}}([\bar{M}],[\tilde{M}]) \geq dist_{\mathbb{S}}([M],[\tilde{M}]).$$

QED.

Lemma 2.14. *The map $\Phi : \mathbb{S} \to \mathcal{P}(\mathbf{T}^1)$ defined by $\Phi([M]) = (M - \hat{M})_{\#}\nu_0$ is an isometry.*

Proof: By corollary 2.4, Φ is well-defined. It is clear that Φ is surjective. Let $M, \bar{M} \in L^2(I)$. It remains to prove that $W_{\mathbf{T}^1}(\mu, \bar{\mu}) = dist_{\mathbb{S}}([M],[\bar{M}])$ where $\mu := \Phi([M])$ and $\bar{\mu} := \Phi([\bar{M}])$. By lemma 2.12, we may assume without loss of generality that the following hold: $M : (0,1) \to [0,1)$ is monotone nondecreasing, $\bar{M} : (0,1) \to [-3/2, 3/2]$ and both functions satisfy $|M - \bar{M}|_{\mathbf{T}^1} = |M - \bar{M}|$,

$$dist_{\mathbb{S}}([M],[\bar{M}]) = \|M - \bar{M}\|_{L^2(I)}. \tag{30}$$

1. Observe that $\gamma := \Big(M \times (\bar{M} - \hat{\bar{M}})\Big)_{\#}\nu_0$ satisfies

$$\int_I F d\mu = \int_{I\times I} F(z) d\gamma(z,w), \qquad \int_I F d\bar{\mu} = \int_{I\times I} F(w) d\gamma(z,w) \tag{31}$$

for all $F \in C_{per}(I\!R)$. By (30) and (31)

$$dist^2_{\mathbb{S}}([M],[\bar{M}]) = \int_{I\times I} |M - \bar{M}|^2_{\mathbf{T}^1} d\nu_0 = \int_{I\times I} |z - w|^2_{\mathbf{T}^1} d\gamma(z,w) \geq W^2_{\mathbf{T}^1}(\mu, \bar{\mu}). \tag{32}$$

2. Suppose first that $M_{\#}\nu_0 << \mathcal{L}^1$. Let $\psi \in C(I\!R)$ be a convex function such that $\psi' - \mathbf{id}$ is periodic,

$$W^2_{\mathbf{T}^1}(\mu, \bar{\mu}) = \int_I |z - \psi'(z)|^2_{\mathbf{T}^1} d\mu(z) = \int_I |z - \psi'(z)|^2 d\mu(z) \tag{33}$$

and

$$\int_I F \circ \psi' d\mu = \int_I F d\bar{\mu} \tag{34}$$

for all $F \in C_{per}(I\!R)$. Recall that $M : [0,1) \to [0,1)$ pushes ν_0 forward to μ. Setting $\bar{T} := \psi' \circ M$ and using (34), we have $\bar{T} \sim \bar{M}$. This, together with (33) implies

$$W^2_{\mathbf{T}^1}(\mu, \bar{\mu}) = \int_I |z - \psi'(z)|^2 d\mu(z) = \int_I |M_\mu - \bar{T}|^2 d\nu_0 \geq dist^2_{\mathbb{S}}([M],[\bar{M}]). \tag{35}$$

Suppose next that $\mu << \mathcal{L}^1$ fails. Let $\mu_k \in \mathcal{P}([0,1))$ be such that spt$\mu_k \subset [0, 1-1/k]$ and $\{\mu_k\}_{k=1}^{\infty}$ converges to μ in $\mathcal{P}([0,1)$. Let $M_k : [0,1] \to [0, 1-1/k]$ be monotone nondecreasing such that $(M_k)_{\#}\nu_0 = \mu_k$. Recall that $M : (0,1) \to [0,1)$ is monotone nondecreasing and so, $\{M_k\}_{k=1}^{\infty}$ converges to M pointwise and in $L^2(I)$. Observe

$$W_{\mathbf{T}^1}(\mu, \bar\mu) = \lim_{k\to\infty} W_{\mathbf{T}^1}(\mu_n, \bar\mu) \geq \liminf_{k\to\infty} dist_{\mathbb{S}}([M_k], [\bar M]) = dist_{\mathbb{S}}([M], [\bar M]). \tag{36}$$

We have used (35) to obtain the inequality in (36). (32) and (36) yield the proof of the lemma. QED.

Corollary 2.15. $(\mathbb{S}, dist_{\mathbb{S}})$ *is a compact, complete, separable metric space.*

Proof: Since by lemma 2.14 $(\mathbb{S}, dist_{\mathbb{S}})$ is isometric to $(\mathcal{P}(\mathbf{T}^1), W_{\mathbf{T}^1})$ which is a compact, complete, separable space, we conclude the proof. QED.

3 Mather Theory and Weak KAM theory on $L^2(I)$

The action of $\mathcal{G}$ on $L^2(I)$ induces an action on its tangent bundle $\mathcal{T}L^2(I)$:

$$(G; M, N) \to (M \circ G, N \circ G),$$

if $G \in \mathcal{G}$ and $M, N \in L^2(I)$. Throughout this section $c \in I\!\!R$ and

$$2L(M,N) = \|N\|^2_{\nu_0} - \mathcal{W}(M), \;\; L_c(M,N) = L(M,N) - c\int_I N d\nu_0, \;\; \tilde L(M,N) = L_c(M, -N). \tag{37}$$

Here, $\mathcal{W} : L^2(I) \to I\!\!R$ is periodic, κ_w–Lipschitz and differentiable invariant under the action of $\mathcal{G}$. We define the Legendre transforms of $L(M,\cdot)$ and $L_c(M,\cdot)$:

$$H(M,N) = \frac{1}{2}\|N\|^2_{\nu_0} + \frac{1}{2}\mathcal{W}(M), \quad H_c(M,N) = H(M, N+c), \quad \tilde H(M,N) = H_c(M, -N).$$

Since these Lagrangians and Hamiltonians are invariant under $\mathcal{G}$ and periodic in the position variables, they are well-defined on $\mathbb{S} \times L^2(I)$. Recall that by proposition 2.8 every periodic continuous function U invariant under the action of $\mathcal{G}$ can be identified with a continuous function U^* on $\mathbb{S}$. Note that the extrema of U and U^* are the same. If U is κ–Lipschitz then U^* is also κ–Lipschitz. We write $U : \mathbb{S} \to I\!\!R$ is κ–Lipschitz. In this section, we will make no distinction between U and U^*. Let $M \in L^2(I)$ so that $[M] \in \mathbb{S}$. We use corollary 2.4 to find a unique $M^* \in [M]$ such that M^* is monotone nondecreasing, $M^*_{\#}\nu_0 = (M - \hat M)_{\#}\nu_0$ and M^* has its range in $[0,1)$. We shall use the convention $M^* \in \mathbb{S}$. Similarly, we define $\phi \in C^k(\mathbb{S})$ and $\phi \in C^k(\mathbb{S} times L^2(I))$. From the above comments we obtain that $\mathcal{W}$ achieves its maximum w^+ at a point $M^+ \in \mathbb{S}$ and its minimum w^- at a point $M^- \in \mathbb{S}$. We assume, without loss of generality, that $w^+ = 0$.

3.1 Viscosity sub and super solutions

Definition 3.1. *Let V be a real valued proper functional defined on $L^2(I)$ with values in $I\!R \cup \{\pm\infty\}$. Let $M \in L^2(I)$ and $\xi \in L^2(I)$. (i) We say that ξ belongs to the subdifferential of V at M and we write $\xi \in \partial_\cdot V(M)$ if $V(M+X) - V(M) \geq \langle \xi, X\rangle + o\big(\|X\|_{\nu_0}\big)$ for all $M \in L^2(I)$. (ii) We say that ξ belongs to the superdifferential of V at M and we write $\xi \in \partial^\cdot V(M)$ if $-\xi \in \partial_\cdot(-V)(M)$.*

Remark 3.2. *As expected, when the sets $\partial_\cdot V(M)$ and $\partial^\cdot V(M)$ are both nonempty, then they coincide and consist of a single element. That element is $\nabla_{L^2} V(M)$, the gradient of V at M_0.*

We can now define [5] the notion of viscosity solution for a general Hamilton-Jacobi equation of the type

$$F(M, \nabla_{L^2} U(M)) = 0. \tag{HJ}$$

Definition 3.3. *Let $V : L^2(I) \to I\!R$ be continuous. (i) We say that V is a* viscosity subsolution *for (HJ) if $F(M,\zeta) \leq 0$ for all $M \in L^2(I)$ and all $\zeta \in \partial^\cdot V(M)$. (ii) We say that V is a* viscosity supersolution *for (HJ) if $F(M,\zeta) \geq 0$ for all $M \in L^2(I)$ and all $\zeta \in \partial_\cdot V(M)$. (iii) We say that V is a* viscosity solution *for (HJ) if V is both a subsolution and a supersolution for (HJ).*

3.2 A preliminary stationary Hamilton-Jacobi equation

Throughout this subsection, $\varepsilon \in (0,1)$ is fixed, $\mathcal{C}_n$ is the finite dimensional subspace of $L^2(I)$ defined in (18) and Π^n is the orthogonal projection onto it. We define the action

$$\mathcal{A}_\varepsilon(\sigma) := \int_0^\infty e^{-\varepsilon t} \tilde{L}(\sigma, \dot\sigma) dt,$$

which is well-defined for $\sigma \in AC^2_{loc}(0,\infty; L^2(I))$ since $\tilde{L}$ is bounded below by $c^2/2$. We do not display its dependence on c to keep the notation simpler. We set

$$V_\varepsilon(M) := \inf_\sigma \{\mathcal{A}_\varepsilon(\sigma) \ : \ \sigma \in AC^2_{loc}(0,\infty; L^2(I)),\ \sigma(0) = M\}. \tag{38}$$

Since $\tilde{L}$ is invariant under the action of $\mathcal{G}$, so is V_ε. The fact that $L(\cdot, N)$ is periodic ensures that V_ε is periodic.

Remark 3.4. *We do not know that (38) admits a minimizer unless M is monotone nondecreasing. Indeed, in the latter case, let $\{\sigma^k\}_{k=1}^\infty$ be a minimizing sequence. By remark 5 [10], we may assume without loss of generality that σ_t^k is monotone nondecreasing for each k and t and $(t,z) \to \sigma_t^k(z)$ is bounded in $L^2([0,T]\times I)$ for each $T > 0$. This, together with remark 6 [10] yields that $(t,z) \to \sigma_t^k(z)$ is bounded in $BV([0,T]\times[r,1-r])$ for all $r \in (0,1)$. These facts are used to conclude existence of a minimizer σ in (38). Furthermore, $\sigma \in H^2_{loc}(0,\infty; L^2(I))$ and satisfies the Euler-Lagrange equation*

$$\ddot\sigma = \varepsilon\dot\sigma - 1/2 \nabla_{L^2} \mathcal{W}(\sigma). \tag{39}$$

One of the main properties of σ is that $\int_0^1 \|\dot\sigma_t\|^2_{\nu_0} dt$ is bounded on $[0,1]$ by a constant independent of M. Exploiting (39), one concludes that the supremum of $\|\dot\sigma_t\|_{\nu_0}$ on $[0,1]$ is bounded by a constant independent of M.

The first task we accomplish in this section is to show existence of a constant N_∞ independent of M and a minimizing sequence $\{\sigma^\delta\}_{\delta\in D}$ in (38) such that

$$\sup_{\delta\in D}\sup_{t\in[0,1]} \|\dot\sigma^\delta_t\|_{\nu_0} \le N_\infty. \tag{40}$$

We next prove a sequence of lemmata which provide properties of minimizing sequences of (38).

Lemma 3.5. *There exists an increasing real valued function $R \to N_R$ (depending on ε and c) satisfying the following properties: if $\sigma \in AC^2_{loc}(0,\infty; L^2(I)$ and $\mathcal{A}_\varepsilon(\sigma) \le R$ then*

$$\int_0^1 \|\dot\sigma - c\|^2_{\nu_0} dt, \int_0^1 \|\dot\sigma\|^2_{\nu_0} dt \le N_R^2, \quad \textit{and} \quad \|\sigma_1 - \sigma_0\|^2_{\nu_0} \le N_R^2.$$

Proof: Suppose σ satisfies the assumption of the lemma and set $a = -(c^2/2 + w^+)$ so that $\tilde L \ge a$. We have

$$R \ge \int_0^\infty e^{-\varepsilon t}\tilde L(\sigma,\dot\sigma)dt = \int_0^\infty e^{-\varepsilon t}(\tilde L(\sigma,\dot\sigma) - a)dt + \frac{a}{\varepsilon} \ge e^{-\varepsilon}\int_0^1 (\tilde L(\sigma,\dot\sigma) - a)dt + \frac{a}{\varepsilon}.$$

We use this, together with the fact that $\mathcal{W}$ is bounded to obtain the first two inequalities in the lemma. The third one is a straightforward consequence of the second one. QED.

Lemma 3.6. *There exists a constant N_∞ independent of M and a minimizing sequence $\{\sigma^\delta\}_{\delta\in D}$ in (38) such that (40) holds.*

Proof: Let $M \in L^2(I)$. *Part 1. The discrete problem.* Standard methods of the calculus of variations ensure existence of a minimizer σ in

$$\inf_\sigma\{\mathcal{A}_\varepsilon(\sigma) \;:\; \sigma \in AC^2_{loc}(0,\infty;\mathcal{C}_n), \quad \sigma_0 = \Pi^n(M)\}. \tag{41}$$

We have $\sigma \in H^2_{loc}(0,\infty;\mathcal{C}_n)$ and its Euler-Lagrange equation is (39). Set $\sigma^*_t = \Pi^n(M)$ for all $t \ge 0$. Since $2\varepsilon\mathcal{A}_\varepsilon(\sigma) \le 2\varepsilon\mathcal{A}_\varepsilon(\sigma^*) \le -w^-$, we use lemma 3.5 to obtain a constant N (independent of n and M) such that $\int_0^1 \|\dot\sigma\|^2_{\nu_0}dt \le N$. As a consequence, $\|\dot\sigma_{\bar t}\|^2_{\nu_0} \le N^2$ for some $\bar t \in [0,1]$. These facts, together with the fact that σ satisfies (39) ensure that the suppremum of $\|\dot\sigma_t\|_{\nu_0}$ over $[0,1]$ is bounded by a finite constant $\bar N$ which depends only on ε and c (is independent of n and M). We assume without loss of generality that $\bar N \ge 1$.

Part 2. An appropriate minimizing sequence. For each $\delta \in (0,1)$ we may find $\sigma^\delta \in AC^2_{loc}(0,\infty; L^2(I))$ such that $\sigma^\delta_0 = M$ and

$$V_\varepsilon(M) \ge -\delta^2 + \mathcal{A}_\varepsilon(\sigma^\delta). \tag{42}$$

Set $\sigma^n_t := \Pi^n(\sigma^\delta_t)$. Because Π^n is a linear projection we have that $\sigma^n \in AC^2_{loc}(0,\infty;\mathcal{C}_n)$ and $\dot\sigma^n = \Pi^n(\dot\sigma^\delta_t)$. We choose n large enough so that $\|M-\Pi^n(M)\|_{\nu_0} \le \delta^2$ and

$$\mathcal{A}_\varepsilon(\sigma^\delta) \ge -\delta^2 + \mathcal{A}_\varepsilon(\sigma^n). \tag{43}$$

Let $\bar\sigma^n$ be a minimizer of (41). By (42), (43) and the minimality property of $\bar\sigma^n$, we have

$$V_\varepsilon(M) \ge -2\delta^2 + \mathcal{A}_\varepsilon(\bar\sigma^n) \tag{44}$$

and by the first part of the proof, $\|\dot\sigma^{n,\delta}_s\|_{\nu_0} \le \bar N$ for all $s \ge \delta^2$, where we have set

$$\sigma^{n,\delta}_s := \begin{cases} (1-\frac{s}{\delta^2})M + \frac{s}{\delta^2}\Pi^n(M) & \text{if} \quad 0 \le s \le \delta^2 \\ \\ \bar\sigma^n_{s-\delta^2} & \text{if} \quad s \ge \delta^2. \end{cases} \tag{45}$$

For $s \le \delta^2$ we have

$$\|\dot\sigma^{n,\delta}_s\|_{\nu_0} = \frac{\|M-\Pi^n(M)\|_{\nu_0}}{\delta^2} \le 1 \le N_\infty.$$

Hence,

$$\|\dot\sigma^{n,\delta}_s\|_{\nu_0} \le N_\infty \qquad \forall s \ge 0. \tag{46}$$

As a consequence of (44) we have

$$\begin{aligned} V_\varepsilon(M) \quad &\ge \quad -2\delta^2 + \mathcal{A}_\varepsilon(\sigma^{n,\delta}) + (e^{\varepsilon\delta^2}-1)\int_{\delta^2}^\infty e^{-\varepsilon t}\tilde L(\bar\sigma^n,\dot{\bar\sigma}^n)dt \\ &+ \quad \frac{1}{2}\int_0^{\delta^2}\mathcal{W}(\sigma)dt - \frac{1-e^{-\varepsilon\delta^2}}{\varepsilon}\Big(\frac{\|M-\Pi^n(M)\|^2_{\nu_0}}{2\delta^4} + c\int_I \frac{\Pi^n(M)-M}{\delta^2}\Big). \end{aligned} \tag{47}$$

Let a be the minimum of $\tilde L$. We use that

$$\int_{\delta^2}^\infty e^{-\varepsilon t}\tilde L(\bar\sigma^n,\dot{\bar\sigma}^n)dt = \int_{\delta^2}^\infty e^{-\varepsilon t}(\tilde L(\bar\sigma^n,\dot{\bar\sigma}^n)-a)dt + a\frac{1-e^{-\varepsilon\delta^2}}{\varepsilon} \ge a\frac{1-e^{-\varepsilon\delta^2}}{\varepsilon}$$

and (47) to conclude that

$$V_\varepsilon(M) \ge -2\delta^2 + \mathcal{A}_\varepsilon(\sigma^{n,\delta}) + a\frac{1-e^{-\varepsilon\delta^2}}{\varepsilon} - \delta^2\Big(\frac{1-w^-}{2} + |c|\Big). \tag{48}$$

Hence,

$$V_\varepsilon(M) \ge \delta^2\left(|a| + |c| + \frac{5-w^-}{2}\right) + \mathcal{A}_\varepsilon(\sigma^{n,\delta}).$$

This proves that $\{\sigma^{n,\delta}\}_{n,\delta}$ is a minimizing sequence satisfying the desired property. QED.

We define the cost between $M, M^* \in L^2(I)$ to be

$$W_T(M,M^*) := \inf_\sigma\Big\{\int_0^T \tilde L(\sigma,\dot\sigma)dt \;:\; \sigma_0 = M, \sigma_T = M^*, \sigma \in AC^2(0,T;L^2(I))\Big\}. \tag{49}$$

We do not display its dependence on c to alleviate notation. Note that

$$W_T(M,M^*) \le \frac{1}{2T}\|M-M^*\|^2_{\nu_0} + |c|\|M-M^*\|_{\nu_0} + \frac{T}{2}|w^-|. \tag{50}$$

Remark 3.7. *Standard computations give that for each $T > 0$*

$$V_\varepsilon(M) = \inf_{M^*} \big\{ e^{-\varepsilon T} V_\varepsilon(M^*) + W_T(M, M^*) \ : \ M \in L^2(I) \big\}.$$

Proposition 3.8. *The function V_ε defined in (38) is a κ_c–Lipschitz function and is a viscosity solution of $\varepsilon V_\varepsilon + H_c(M, \nabla_{L^2} V_\varepsilon) = 0$. Here,*

$$\kappa_c := w + \frac{1}{2} + |c| + \frac{|w^-|}{2}, \quad 2\varepsilon w := c^2 - w^-.$$

Proof: We use that $-c^2 \le 2\tilde{L}(M, N)$ and $2\tilde{L}(M, 0) \le -w^-$ to conclude that

$$-c^2 \le 2\epsilon V_\varepsilon(M) \le -w^-. \tag{51}$$

1. *Claim: V_ε is Lipschitz.*
Proof. Let $M, M^* \in L^2(I)$ and set $T := \|M - M^*\|_{\nu_0}$. By remark 3.7,

$$\begin{aligned} V_\varepsilon(M) - V_\varepsilon(M^*) &\le (e^{-\varepsilon T} - 1) V_\varepsilon(M^*) + W_T^c(M, M^*) \\ &\le \varepsilon T V_\varepsilon(M^*) + W_T^c(M, M^*) \le \kappa_c \|M - M^*\|_{\nu_0}. \end{aligned}$$

To obtain the previous inequality, we have used (51) and (50). This proves that V_ε is κ_c–Lipschitz. Proving that V_ε is a viscosity supersolution of $\varepsilon V_\varepsilon + H_c(M, \nabla_{L^2} V_\varepsilon) = 0$ is harder compared to proving it is a viscosity subsolution. We only prove the hardest part while we refer the reader to [11] theorem 3.9, where one can easily adapt the method there to establish that V_ε is a viscosity subsolution.

2. *Claim: V_ε is a viscosity supersolution.*
Proof. Let $M \in L^2(I)$ and $P \in \partial_\cdot V_\varepsilon(M)$. By lemma 3.6, for each $\delta \in (0, 1)$ choose $\sigma^\delta \in AC^2_{loc}(0, \infty; L^2(I))$ such that $\sigma_0^\delta = M$,

$$V_\varepsilon(M) \ge -\delta^2 + \int_0^\infty e^{-\varepsilon t} \tilde{L}(\sigma^\delta, \dot{\sigma}^\delta) dt \ge -\delta^2 + \int_0^\delta e^{-\varepsilon t} \tilde{L}(\sigma^\delta, \dot{\sigma}^\delta) dt + e^{-\varepsilon\delta} V_\varepsilon(\sigma_\delta^\delta) \tag{52}$$

and

$$\sup_{\delta \in (0,1)} \sup_{t \in [0,1]} \|\dot{\sigma}_t^\delta\|_{\nu_0} \le N_\infty. \tag{53}$$

for a constant $N_\infty < \infty$. By the fact that $P \in \partial_\cdot V_\varepsilon(M)$, we may choose a nonnegative real valued function $\bar{o}$ such that $\bar{o}(t)/t$ tends to 0 as t tends to 0 and

$$\begin{aligned} V_\varepsilon(\sigma_\delta^\delta) &\ge V_\varepsilon(M) + \langle P, \sigma_\delta^\delta - M \rangle_{\nu_0} + \bar{o}(\|\sigma_\delta^\delta - M\|_{\nu_0}) \\ &= V_\varepsilon(M) + \int_0^\delta \langle P, \dot{\sigma}^\delta \rangle_{\nu_0} dt + \bar{o}\big(\|\sigma_\delta^\delta - M\|_{\nu_0}\big). \end{aligned}$$

This, together with (52), yields

$$\begin{aligned} V_\varepsilon(M)(1 - e^{-\varepsilon\delta}) &\ge -\delta^2 + \int_0^\delta e^{-\varepsilon t} \Big(\langle P, \dot{\sigma}^\delta \rangle_{\nu_0} + \tilde{L}(\sigma^\delta, \dot{\sigma}^\delta) \Big) dt \\ &+ \int_0^\delta (e^{-\varepsilon\delta} - e^{-\varepsilon t}) \langle P, \dot{\sigma}^\delta \rangle_{\nu_0} dt + \bar{o}\big(\|\sigma_\delta^\delta - M\|_{\nu_0}\big) \end{aligned}$$

and so,

$$V_\varepsilon(M)(1-e^{-\varepsilon\delta}) + \int_0^\delta e^{-\varepsilon t} H_c(\sigma^\delta, -P)dt \;\; \geq \;\; -\delta^2 + \int_0^\delta (e^{-\varepsilon\delta} - e^{-\varepsilon t})\langle P, \dot\sigma^\delta\rangle_{\nu_0} dt + \bar{o}\big(\|\sigma^\delta_\delta - M\|_{\nu_0}\big). \tag{54}$$

By (53), $\|\sigma^\delta_\delta - M\|_{\nu_0} = 0(\delta)$. Dividing both sides of (54) by δ and letting δ tend to 0 in the subsequent inequality, we obtain $\varepsilon V_\varepsilon(M) + H_c(M, -P) \geq 0$. This proves that V_ε is a viscosity supersolution. QED.

3.3 The cell problem

Throughout this section $\kappa := |c| + \sqrt{c^2 - w^-}$, V_ε is the value function defined in section 3.2 and $U_\varepsilon := V_\varepsilon - \inf V_\varepsilon$.

Proposition 3.9. *(i) The function U_ε is κ–Lipschitz and $V_\varepsilon(M) = V_\varepsilon(\bar M)$ whenever $[M] = [\bar M]$.*

(ii) Every subfamily of $\{U_\varepsilon\}_{\varepsilon\in(0,1)}$ admits a subsequence converging to some U which is κ–Lipschitz. Every subfamily of $\{\varepsilon V_\varepsilon\}_{\varepsilon\in(0,1)}$ admits a subsequence converging to a constant depending on c which we denote $-\bar H(c)$.

Proof: By proposition 3.8 V_ε is Lipchitz and is a viscosity solution for $\varepsilon V_\varepsilon + \tilde H(M, \nabla_{L^2} V_\varepsilon) = 0$. In particular, V_ε is a viscosity subsolution for $\|\nabla_{L^2} V_\varepsilon(M)\|^2_{\nu_0} \leq \kappa^2$. Because $L(\cdot, N)$ is periodic, so is V_ε. Since L is invariant under the action of $\mathcal{G}$, so is V_ε. In light of lemma 2.8, we conclude the proof of (i).

By (i), we may identify V_ε with a function on $\mathbb{S}$ which is κ–Lipschitz and so, U_ε is a function on $\mathbb{S}$ which is κ–Lipschitz. Since by corollary 2.15 $\mathbb{S}$ is compact, the minumum of U_ε are achieved and is null. Thus, $\{U_\varepsilon\}_{\varepsilon\in(0,1)}$ is equicontinuous and bounded on the compact set $\mathbb{S}$. The Ascoli-Arzela lemma yields the first part of (ii). We apply arguments similar to the previous ones to $\{\varepsilon V_\varepsilon\}_{\varepsilon\in(0,1)}$ to conclude that any of its subfamilies admits a subsequence converging to a function F whose Lipschitz constant is null. Thus, F is the constant function and so, (ii) is established. QED.

Remark 3.10. *In a forthcoming paper [12], we show that the constant $\bar H(c)$ found above coincides with the effective Hamiltonian of H at $M \equiv c$.*

We set

$$\tilde{\mathcal{A}}_T(\sigma) := \int_0^T \tilde L(\sigma, \dot\sigma)dt, \qquad \sigma \in AC^2(0,T; L^2(I)).$$

Theorem 3.11. *Let U be the function obtained in proposition 3.9. Then, for every $T > 0$ and $\sigma \in AC^2(0,T; L^2(I))$, we have $U(\sigma_0) - U(\sigma_T) \leq T\bar H(c) + \tilde{\mathcal{A}}_T(\sigma)$. If $M \in L^2(I)$ is monotone nondecreasing, then there exists $N \in L^2(I)$ independent of T such that*

$$U(M) = U(\sigma^*_T) + T\bar H(c) + \tilde{\mathcal{A}}_T(\sigma^*), \qquad \sigma^*_T := \Psi^1(T, M, N), \tag{55}$$

where Ψ is the flow (5) defined on the tangent bundle $\mathcal{T}L^2(I)$. We have that σ_t^* is monotone nondecreasing for each $t > 0$. Futhermore, σ^* minimizes $\tilde{\mathcal{A}}_T$ over the set of $\sigma \in AC^2(0,T;L^2(I))$ such that $\sigma_0 = M$ and $\sigma_T = \sigma_T^*$.

Proof: Let $\sigma \in AC^2(0,T;L^2(I))$ be such that $\sigma_0 = M$. By remark 3.7

$$U_\varepsilon(M) \le e^{-\varepsilon T} U_\varepsilon(\sigma_T) + (e^{-\varepsilon T} - 1) \inf V_\varepsilon + \int_0^T e^{-\varepsilon t} \tilde{L}(\sigma^*, \dot{\sigma}^*) dt.$$

Letting ε tend to 0 in the previous inequality and using proposition 3.9 we have

$$U(M) \le U(\sigma_T) + T\bar{H}(c) + \tilde{\mathcal{A}}_T(\sigma). \tag{56}$$

This establishes the first assertion of the theorem.

Next, suppose $M \in L^2(I)$ is monotone nondecreasing. By remark 3.4 there exists $\sigma^\varepsilon \in AC^2_{loc}(0,\infty;L^2(I))$ such that σ_t^ε is monotone nondecreasing for each t, $\sigma_0^\varepsilon = M$, and

$$U_\varepsilon(\sigma_0^\varepsilon) = e^{-\varepsilon T} U_\varepsilon(\sigma_T^\varepsilon) + (e^{-\varepsilon T} - 1) \inf V_\varepsilon + \int_0^T e^{-\varepsilon t} \tilde{L}(\sigma^\varepsilon, \dot{\sigma}^\varepsilon) dt. \tag{57}$$

Remark 6 [10] ensures that $\{\sigma^\varepsilon\}_{\varepsilon \in (0,1)}$ is bounded in $BV([0,T] \times [r, 1-r])$ for all $r \in (0,1)$. Hence, it admits a point of accumulation σ^*. We have that σ_t^* is monotone nondecreasing for each $t > 0$ and

$$\liminf_{\varepsilon \to 0^+} \int_0^T e^{-\varepsilon t} \tilde{L}(\sigma^\varepsilon, \dot{\sigma}^\varepsilon) dt \ge \tilde{\mathcal{A}}_T(\sigma^*).$$

Letting ε tend to 0 in (57), using proposition 3.9, and the previous inequality we obtain

$$U(\sigma_0) \ge U(\sigma_T^*) + T\bar{H}(c) + \tilde{\mathcal{A}}_T(\sigma^*). \tag{58}$$

Since (56) holds for arbitrary paths, the previous inequality is, in fact, an equality:

$$U(M) = U(\sigma_T^*) + T\bar{H}(c) + \tilde{\mathcal{A}}_T(\sigma^*). \tag{59}$$

We have constructed a path σ^* which a priori depends on T. In fact, one can readily adapt the previous arguments to show existence of a $\sigma^* \in AC^2_{loc}(0,\infty;L^2(I))$ independent of T such that (59) holds. Indeed, that for any $h > 0$ we can apply the above construction to obtain a path $\bar{\sigma} \in AC^2(T, T+h; L^2(I))$ such that $\bar{\sigma}_T = \sigma_T^*$ and

$$U(\sigma_T^*) = U(\bar{\sigma}_{T+h}) + h\bar{H}(c) + \int_T^{T+h} \tilde{L}(\bar{\sigma}(s), \dot{\bar{\sigma}}(s)) ds.$$

Summing up this equation and (59), we notice that the path obtained from σ^* and $\bar{\sigma}$ by concatenation does the job on $[0, T+h]$. Thus, the existence of a path independent of $t > 0$ for which (55) holds is proved. Let now $\sigma \in AC^2_{loc}(0,\infty;L^2(I))$ be an arbitrary path satisfying $\sigma_0 = M$. By (56) and (59) we have

$$\tilde{\mathcal{A}}_T(\sigma) \ge U(M) - U(\sigma_T) - T\bar{H}(c) = \tilde{\mathcal{A}}_T(\sigma^*).$$

Hence, σ^* minimizes $\tilde{\mathcal{A}}_T$ over the set of paths whose endpoints are σ_0^* and σ_T^*. Its Euler-Lagrange equation is

$$\ddot{\sigma}^* = -\frac{1}{2}\nabla_{L^2}\mathcal{W}(\sigma^*), \qquad -(\sigma_0^* + c) \in \partial^{\cdot} U(\sigma_0^*). \tag{60}$$

Hence, $(\sigma_t^*, \dot{\sigma}_t^*) = \Psi(t, M, N)$ where $N = \dot{\sigma}_0$. QED.

Proposition 3.12. *The function U is a viscosity solution of $H_c(M, \nabla_{L^2}U) = \bar{H}(c)$.*

Proof: Set

$$F_\varepsilon(M,N) = H_c(M,N) + \varepsilon V_\varepsilon(M), \quad F(M,N) = H_c(M,N) - \bar{H}(c).$$

By proposition 3.9, $\{F_\varepsilon\}_{\varepsilon\in(0,1)}$ converges uniformly to F on $L^2(I) \times L^2(I)$ and $\{U_\varepsilon\}_{\varepsilon\in(0,1)}$ converges uniformly to U on $L^2(I)$. According to proposition 3.8, U_ε is a viscosity solution of $F_\varepsilon(M, \nabla_{L^2}U_\varepsilon) = 0$. We use the stability property of viscosity solutions to conclude that U is a viscosity solution of $F(M, \nabla_{L^2}U) = 0$ (cf. [6]). QED.

3.4 Rotation number of invariant measures

Throughout this section Ψ is the flow defined in (5) and we write Ψ_t in place of $\Psi(t,\cdot,\cdot)$. If $U : L^2(I) \to I\!R$ is of class C^k, periodic and rearrangement invariant, we write $U \in C^k(\mathbb{S})$. Similarly, if $\phi : \mathcal{T}L^2(I) \to I\!R$ is continuous, invariant under the action of $\mathcal{G}$ and periodic in the position variables, we write $\phi \in C(\mathcal{T}\mathbb{S})$. The following continuous functions will play a special role:

$$m_1(N) := \int_I N d\nu_0, \quad m_p^*(N) := \|N\|_{\nu_0}^p, \qquad N \in L^2(I), \quad p \geq 1.$$

Note that these two functions belong to $C(\mathcal{T}\mathbb{S})$ and if $F \in C(\mathcal{T}\mathbb{S})$ then $F \circ \Psi_t \in C(\mathcal{T}\mathbb{S})$ for $t \geq 0$.

Definition 3.13. *(i) We say that a Borel probability measure on $\mathcal{T}L^2(I)$ is invariant under the flow Ψ if $\int \phi \circ \Psi(t,\cdot,\cdot)d\mu = \int \phi d\mu$ for all $\phi \in C(\mathcal{T}\mathbb{S})$.*

(ii) If μ is a measure on $\mathcal{T}L^2(I)$ such that m_p^* is μ–measurable, we say that the p–moment of μ (in the velocity variable) is finite if $\int m_p^* d\mu$ is finite.

Remark 3.14. *Suppose μ is a Borel probability measure on $\mathcal{T}L^2(I)$ such that its 1–moment in the velocity variable is finite and set $\mu_t := \Psi_{t\,\#}\mu$. (i) If $U \in C^1(\mathbb{S})$ then*

$$\int U d\mu_t = \int U d\mu + t \int dU d\mu + o(t).$$

This proves that $\int dU d\mu = 0$ if μ is invariant under the flow Ψ.

(ii) Suppose that $S : L^2(I) \to I\!R$ is Lipschitz of class C^1 such that $L^2(I) \ni M \to d_M S$ is $L^2_{\mathbb{Z}}$-periodic and $L^2(I) \ni M \to d_M S(M)$ is rearrangement invariant. In light of section 2.3, there exists $c \in I\!R$ such that the equivariant de Rham cohomology class of dS is the set of $c + dU$ where $U \in C^1(\mathbb{S})$ is Lipschitz. By (i)

$$\int dS d\mu = c\rho(\mu), \quad \textit{where} \quad \rho(\mu) := \int m_1 d\mu.$$

Definition 3.15. *Let μ be a Borel probability measure on $\mathcal{T}L^2(I)$ such that its 1–moment in the velocity variable is finite. We say that μ is weakly invariant if $\int dU d\mu = 0$ for all $U \in C^1(\mathbb{S})$. In that case, we define its rotation number to be $\rho(\mu) := \int m_1 d\mu$.*

Example 3.16. *Set $\sigma_t = ct - \widehat{ct}$ and define the measure μ_T^* on $\mathcal{T}L^2(I)$ by*

$$\int \phi d\mu_T := \frac{1}{T}\int_0^T \phi(ct - \widehat{ct}, c)dt.$$

for $\phi : \mathcal{T}L^2(I) \to I\!R$ continuous. The set $\{\mu_T\}_{T>0}$ admits a point of accumulation μ^* for the narrow convergence, which is weakly invariant and of rotation number c. If $\nabla_{L^2}\mathcal{W}$ vanishes on the set of constant functions, then μ^* is invariant under Ψ.

Proof: We consider the closed subset

$$\mathcal{C}_* := \{(M, N) \in \mathcal{T}L^2 \ : \ \exists b \in [0, 1] \text{ such that } (M, N) \equiv (b, c)\}$$

and the function ϕ_0 defined by $\phi_0(M, N) = \|M\|_{\nu_0} + \|N\|_{\nu_0}$ on $\mathcal{C}_*$. We set $\phi_0 \equiv \infty$ on the complement of $\mathcal{C}_*$. Observe that ϕ_0 is lower semicontinuous, its sublevel sets are compact and $T \to \int \phi_0 d\mu_T$ is bounded on $(0, \infty)$. Hence, there exists an increasing unbounded sequence $\{T_n\}_{n=1}^\infty \subset (0, \infty)$ such that $\{\mu_{T_n}\}_{n=1}^\infty$ converges narrowly to some Borel probability measure μ^* (cf. e.g. remark 5.1.5 [2]). If $U \in C^1(\mathbb{S})$ then U is bounded and using the periodicity of dU in the position variable we have

$$\begin{aligned}\int dU d\mu^* &= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} d_{ct-\hat{ct}}U(c)dt \\ &= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} d_{\sigma_t}U(\dot{\sigma}_t)dt = \lim_{n\to\infty}\frac{U(\sigma_{T_n}) - U(\sigma_0)}{T_n} = 0,\end{aligned}$$

where $\sigma_t = ct$. Hence μ^* is weakly invariant. Its rotation number is

$$\int m_1 d\mu^* = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} m_1(c)dt = c.$$

We further assume that $\nabla_{L^2}\mathcal{W}$ vanishes on the set of constant functions. Let $t_0 > 0$ and let $\phi \in C(\mathcal{T}\mathbb{S})$. Note that if $(M, N) = (b, c)$ are constant and we set $\sigma_s^* = (b, c) + (sc, 0)$

then σ^* satisfies (60) and $(\sigma_0^*, \dot\sigma_0^*) = (b, c)$. Hence, $\Psi_s(M,N) = (\sigma_s^*, \dot\sigma_s^*)$. In particular, $\Psi_{t_0}(ct - \hat{ct}, c) = (ct - \hat{ct}, c) + (ct_0, 0)$ and so,

$$\int \phi \circ \Psi_{t_0} d\mu = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \phi(c(t+t_0) - \hat{ct}, c)dt = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \phi(c(t+t_0), c)dt, \qquad (61)$$

where we have used that ϕ is periodic in the position variable. Since ϕ is bounded, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \phi(c(t+t_0), c)dt &= \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \phi(cs, c)ds \\ &= \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \phi(cs - \hat{cs}, c)ds = \int \phi d\mu. \end{aligned}$$

This, together with (61) and the fact that $t_0 > 0$ is arbitrary proves invariance of μ^* under Ψ. QED.

4 Application: The Vlasov System

Throughout this section Ψ is the flow defined in (5). We write Ψ_t in place of $\Psi(t, \cdot, \cdot)$. We assume that $W \in C^2(I\!\!R)$ is 1–periodic, even, $W(z) \le W(0) = 0$, and

$$\mathcal{W}(M) := \int_{I\times I} W(M(z) - M(\bar z))dzd\bar z, \qquad M \in L^2(I).$$

Given $c \in I\!\!R$, we define Γ_c to be the set of μ Borel probability measures on $\mathcal{T}L^2(I)$ invariant under the flow Ψ and whose 1–moment in the velocity variable is finite. A problem of great interest is:

$$\inf_{\mu\in\Gamma_c} \int L d\mu. \qquad (62)$$

We do not establish existence of minimizers for general potentials $\mathcal{W}$. In the next section, we keep our focus on potentials including those obtained by regularizing the classical Vlasov potential $W^*(z) := |z|_{\mathbf{T}^1}$. We shall see that for those potentials the minimizers of (62) are trivial.

We recall that if $\phi : \mathcal{T}L^2(I) \to I\!\!R$ is continuous, by abuse of notation we say that $\phi \in C(\mathcal{T}\mathbb{S})$ if ϕ is invariant under the action of $\mathcal{G}$ and $\phi(\cdot, N)$ is periodic: for $M, N \in L^2(I)$, $Z \in L^2_{\mathbb{Z}}(I)$ and $G \in \mathcal{G}$,

$$\phi(M,N) = \phi(M\circ G, N\circ G) = \phi(M+Z, N).$$

Similarly, if $U : L^2(I) \to I\!\!R$ is continuous and rearrangement invariant, we write $U \in C(\mathbb{S})$. If U is of class C^k, we write $U \in C^k(\mathbb{S})$ and denote by $d_M U$ its differential. The Lagrangian L_c defined in section 3 satisfies

$$L_c(M,N) = \frac{1}{2}\|N - c\|^2_{\nu_0} - \frac{1}{2}\int_{I\times I} W(M(z) - M(\bar z))dzd\bar z - \frac{c^2}{2} \ge -\frac{c^2}{2}. \qquad (63)$$

It attains its minimum at (M_0, N_0) such that M_0 is constant and $N_0 \equiv c$.

Remark 4.1. *Let $c \in I\!R$. (i) If $\mu \in \Gamma_c$, (63) implies $\int L d\mu = \int (L_c + c^2) d\mu \geq c^2/2$.*

(ii) Let μ^* be the measure provided by example 3.16. Then μ^* minimizes $\int L d\mu$ over Γ_c and $0 = \int (L_c + c^2/2) d\mu^*$. If $\bar{\mu}$ is another measure minimizing $\int L d\mu$ over Γ_c then it is supported by the set of (M, N) such that $N \equiv c$ and $W(M(z) - M(\bar{z})) = 0$ for almost every $(z, \bar{z}) \in I\!R^2$.

(iii) We have $\bar{H}(c) = c^2/2$ where $\bar{H}(c)$ is the constant defined in proposition 3.9.

Proof: We only prove (ii) and (iii). Let μ^* be as above. We use that $L_c(ct - \hat{ct}, c) = 0$ to conclude that $\int L_c d\mu^* = 0$. This, together with (i) proves that μ^* is a measure minimizing $\int L d\mu$ over Γ_c. Suppose $\bar{\mu}$ is another measure minimizing $\int L d\mu$ over Γ_c. Then $0 = \int (L_c + c^2/2) d\bar{\mu}$. This, together with (63) yields that $L_c(M, N) = -c^2/2$ for μ–almost every (M, N). We conclude the proof of (ii).

Let V_ε be as in proposition 3.9. By the fact that $\tilde{L} \geq -c^2/2$, we have that $V_\varepsilon(M) \geq -c^2/(2\varepsilon)$. We set $\sigma_t \equiv ct$ and use that $V_\varepsilon(0) \leq \mathcal{A}^c(\sigma) = -c^2/(2\varepsilon)$ to conclude that the minimum value of V_ε is $c^2/(2\varepsilon)$ and is attained at $M \equiv 0$. Thus $\varepsilon V_\varepsilon = -c^2/2$. The definition of $\bar{H}(c)$ as provided by proposition 3.9 yields that $\bar{H}(c) = c^2/2$. QED.

Theorem 4.2. *For each $c \in I\!R$ and $M_0 \in L^2(I)$ monotone nondecreasing there exists $N_0 \in L^2(I)$ such that*

$$\sup_{t>0} \sqrt{t} \Big\| \frac{\Psi^1(t, M_0, N_0) - M_0}{t} + c \Big\|_{\nu_0} \leq 2\sqrt{\kappa}, \qquad \lim_{t \to \infty} \Psi^2(t, M_0, N_0) = -c. \tag{64}$$

Here, $\kappa := |c| + \sqrt{c^2 - w^-}$ where w^- is the minimum of $\mathcal{W}$.

Proof: Theorem 3.11 provides us with a periodic rearrangement invariant κ–Lipschitz function such that

$$\begin{aligned} U(M_0) &= U(\Psi^1(t, M_0, N_0)) + \int_0^T \Big(\tilde{L}(\Psi(t, M_0, N_0)) + \frac{c^2}{2} \Big) dt \\ &\geq U(\Psi^1(t, M_0, N_0)) + \frac{1}{2} \int_0^T \|\Psi^2(t, M_0, N_0) + c\|_{\nu_0}^2 dt \end{aligned} \tag{65}$$

for some $N_0 \in L^2(I)$. We have used that by remark 4.1, $\bar{H}(c) = c^2/2$. Here, κ is explicitly given in section 3.3 as a function of c and the minimum value of $\mathcal{W}$. Since U is periodic, the supremum of $|U(M) - U(\bar{M})|$ over $L^2(I) \times L^2(I)$ coincides with its supremum over the set of $(M, \bar{M})$ such that $0 \leq M, \bar{M} \leq 1$. Thus, it satisfies $|U(M) - U(\bar{M})| \leq \kappa \|M - \bar{M}\|_{\nu_0} \leq 2\kappa$ and so, by (65)

$$\int_0^\infty \|\Psi^2(t, M_0, N_0) + c\|_{\nu_0}^2 dt \leq 4\kappa. \tag{66}$$

The derivative of $t \to \|\Psi^2(t, M_0, N_0) + c\|_{\nu_0}^2$ is $2\langle \Psi^2(t, M_0, N_0) + c, \dot{\Psi}^2(t, M_0, N_0) \rangle_{\nu_0}$, which we claim is bounded. Indeed, the bound on $\|\dot{\Psi}^2(t, M_0, N_0)\|_{\nu_0}$ is an immediate consequence of the Euler-Lagrange equation satisfied by the flow Ψ (in fact, the properties of W ensure that

$\dot{\Psi}^2(t, M_0, N_0)$ is essentially bounded, uniformly with respect to t). Also as a consequence of the Euler-Lagrange equation, the Hamiltonian $\tilde{H}$ is conserved along the flow, i.e.

$$\|\Psi_t^2(M_0, N_0) - c\|_{\nu_0}^2 + \mathcal{W}(\Psi_t^1(M_0, N_0)) = \|N_0 - c\|_{\nu_0}^2 + \mathcal{W}(M_0),$$

which implies the uniform bound on $\|\Psi^2(t, M_0, N_0) + c\|_{\nu_0}$. Thus, (66) gives the second limit in (64). We have

$$\begin{aligned} \|\Psi^1(t, M_0, N_0) - M_0 + ct\|_{\nu_0} &= \Big\| \int_0^t (\Psi^2(s, M_0, N_0) + c) ds \Big\|_{\nu_0} \\ &\leq \int_0^t \big\|\Psi^2(s, M_0, N_0) + c)\big\|_{\nu_0} ds. \end{aligned}$$

By Hölder's inequality, this, together with (66), yields the first inequality in (64). QED.

Corollary 4.3. *For each $c \in I\!R$ and $\varrho_0 \in \mathcal{P}_2(I\!R)$ there exists a solution (ϱ^*, u^*) for the Euler system (12), satisfying the following properties: (i) $\varrho^* \in AC^2_{loc}(0, \infty; \mathcal{P}_2(I\!R))$.*

(ii) $u_t^* \in L^2(\varrho_t^*)$ for $\mathcal{L}^1$–almost every $t > 0$.

(iii)

$$\|\mathbf{id}/t + c\|_{\varrho_t^*} \leq \frac{2\sqrt{\kappa}}{\sqrt{t}} + \frac{1}{t}\sqrt{\int_{I\!R} x^2 \varrho_0(x) dx}, \qquad \lim_{t \to \infty} \|u_t^* + c\|_{\varrho_t^*} = 0. \tag{67}$$

Proof: Since $\varrho_0 \in \mathcal{P}_2(I\!R)$, the Monge-Kantorovich theory ensures existence of a monotone nondecreasing map $M_0 \in L^2(I)$ that pushes ν_0 forward to ϱ_0. Let N_0 be as in theorem 4.2. Recall that in light of theorem 3.11, for $T > 0$ $\sigma_t^* := \Psi^1(t, M_0, N_0)$ minimizes $\sigma \to \int_0^T \tilde{L}(\sigma, \dot{\sigma}) dt$ over the set of $\sigma \in AC^2(0, T; L^2(I))$ such that $\sigma_0 = M_0$ and $\sigma_T = \sigma_T^*$. In addition, σ_t^* is monotone nondecreasing. Let ϱ_t^* be the push forward of ν_0 by σ_t^*. By remark 1 [10], $\varrho^* \in AC^2(0, T; \mathcal{P}_2(I\!R))$ and by proposition 6 [10] there exists a Borel map $u^* : (0, T) \times I\!R \to I\!R$ such that $u_t^* \in L^2(\varrho_t^*)$ for $\mathcal{L}^1$–almost every $t \in (0, T)$ and $\partial_t \varrho_t^* + \partial_z(\varrho_t^* u_t^*) = 0$ in the sense of distributions. Furthermore, $\dot{M}_t z = u_t^*(M_t z)$ for almost every $(t, z) \in (0, T) \times I$ and so,

$$\int_0^T dt \int_I |\dot{M}|^2 d\nu_0 = \int_0^T \|u_t^*\|_{\varrho_t^*}^2 dt.$$

One uses this to check that (ϱ^*, u^*) minimizes

$$\frac{1}{2} \int_0^T \Big(\|u_t\|_{\varrho_t}^2 - \int_{I\!R \times I\!R} W(x - \bar{x}) dx d\bar{x} \Big) dt$$

over the set of (ϱ, u) such that $\varrho_0 = \varrho_0^*$, $\varrho_T = \varrho_T^*$ and $\partial_t \varrho_t + \partial_z(\varrho_t u_t) = 0$ in the sense of distributions. One writes the Euler-Lagrange equations satisfied by (ϱ^*, u^*) to discover as in [11], that it is nothing but (12). Using that ϱ_t^* is the push forward of ν_0 by σ_t^* in (64), we obtain the inequality in (67). Using that $\dot{M}_t z = u_t^*(M_t z)$ for almost every $(t, z) \in (0, T) \times I$ in (64), we obtain that the limit in (67) holds. QED.

5 Appendix

5.1 A differential structure on $L^2(I)$

Let $U : L^2(I) \to I\!R$ be Fréchet differentiable (cf. subsection 2.3). We say that U is twice differentiable if for each $M \in L^2(I)$ there exists a self-adjoint continuous operator $\bar{B}_M : \mathcal{T}_M L^2(I) \to \mathcal{T}_M L^2(I)$ satisfying the following: for each $r > 0$

$$\sup_{\|M\| \le r, \|H\| \le \epsilon} \frac{|U(M+H) - U(M) - \langle H, \xi \rangle_{\nu_0} - \frac{1}{2}\langle \bar{B}_M H, H \rangle_{\nu_0}|}{\|H\|^2} = 0(\epsilon).$$

We next give a general definition of a differential form on $\mathcal{T}L^2(I)$.

Definition 5.1. *We say that $\Lambda : L^2(I) \times (\mathcal{T}L^2(I))^k \to I\!R$ is a differential k-form on $L^2(I)$ if for each $M \in L^2(I)$, $\Lambda_M := \Lambda(M, \cdot, ..., \cdot)$ is multilinear and continuous. If in addition Λ is periodic in the sense that $\Lambda_{M+Z}(N_1, ..., N_k) = \Lambda_M(N_1, ..., N_k)$ for all $M, N_1, ..., N_k \in L^2(I)$ and all $Z \in L^2_{\mathbb{Z}}(I)$, we say that Λ is a k-form on $\mathbb{T}$.*

We next consider only differential forms which satisfy some strong uniform differentiability conditions.

Definition 5.2. *Let $\mathcal{L}(\mathcal{T}_M L^2(I), \mathcal{T}_M L^2(I))$ be the set of linear continuous functionals from $\mathcal{T}_M L^2(I)$ into itself. Let Λ be a one-form on $\mathcal{T}L^2(I)$ so that by the Riesz representation theorem $\Lambda_M(X) = \langle X, A_M \rangle_{\nu_0}$ for some $A_M \in L^2(I)$. We say that Λ is differentiable if for each $M \in L^2(I)$ there exists a linear continuous map $B_M : L^2(I) \to \mathcal{L}(\mathcal{T}_M L^2(I), \mathcal{T}_M L^2(I))$ such that*

$$\|A_{M+H} - A_M - B_M(H)\| \le \|H\|_{L^2(I)} \min\{c(A), 0(\|H\|_{L^2(I)})\} \tag{68}$$

for all $M, H \in L^2(I)$. Here, $c(A)$ is independent of M and depends only on Λ. We further impose that for each compact set $\mathcal{K} \subset L^2(I)$

$$b(\mathcal{K}) := \sup_{M \in \mathcal{K}} \|\Lambda_M\| + \|B_M\| < \infty. \tag{69}$$

Remark 5.3. *Let Λ be a differentiable one-form on $\mathcal{T}L^2(I)$ and let B be defined as above. If $X, Y \in \mathcal{T}_M L^2(I)$ then $t \to \Lambda_{M+tX}(Y)$ is differentiable at 0 and*

$$\frac{d}{dt}\Lambda_{M+tX}(Y)\Big|_{t=0} = \langle B_M(X), Y \rangle_{\nu_0}.$$

This motivates the following definition.

Definition 5.4. *Let Λ be a differentiable one-form on $\mathcal{T}L^2(I)$ and let B be defined as above. If $X, Y \in \mathcal{T}_M L^2(I)$ we define the differential of Λ to be $d\Lambda : L^2(I) \times (\mathcal{T}L^2(I))^2 \to I\!R$ defined by*

$$d\Lambda_M(X, Y) := \langle B_M(X), Y \rangle_{\nu_0} - \langle B_M(Y), X \rangle_{\nu_0}.$$

Remark 5.5. *Note that if Λ be a differentiable one-form on $\mathcal{T}L^2(I)$ then $d\Lambda$ is a differential 2–form on $\mathcal{T}L^2(I)$.*

We use the notation $AC^2(a,b;L^2(I))$ to denote the set of paths $t \to \sigma_t \in L^2(I)$ which are 2–absolutely continuous. We refer the reader to [2] for its definition and properties (cf. also [10]). We denote by $\dot\sigma$ its functional time-derivative and by $|\sigma'|$ its metric derivative. We recall that elements of $AC^2(a,b;L^2(I))$ are 1/2–Hölder continuous.

Lemma 5.6. *Suppose Λ is a differentiable one-form on $\mathcal{T}L^2(I)$ and $\sigma \in AC^2(a,b;L^2(I))$. Then (i) $t \to \|\Lambda_{\sigma_t}\|_{\nu_0}$ is uniformly bounded.*

(ii) $(t,z) \to \Lambda_{\sigma_t}(z)$ is in $L^2((a,b)\times I)$ and $(t,z) \to \Lambda_{\sigma_t}(\dot\sigma_t)$ is in $L^2((a,b))$.

(iii) If $r:[c,d] \to [a,b]$ is a Lipschitz map and $\sigma^*_s = \sigma_{r(s)}$ then $\sigma^* \in AC^2(c,d;L^2(I))$ and

$$\int_a^b \Lambda_{\sigma_t}(\dot\sigma_t)dt = \int_c^d \Lambda_{\sigma^*_t}(\dot\sigma^*_t)dt.$$

Proof: Since σ is continuous $\sigma[a,b]$ is a compact subset of $L^2(I)$ and so by (69) (i) holds. For $n \geq 1$ integer, set

$$A^n_t(z) = A_{\sigma_{a_i}}(z), \quad t \in [a_i, a_{i+1}), \quad \text{where} \quad a_i := a + i\frac{b-a}{n}.$$

Then $A^n \in L^2((a,b)\times I)$ and

$$\|A_{\sigma_t} - A^n_t\|_{\nu_0} \leq b(\mathcal{K})\|\sigma_t - \sigma_{a_i}\|_{\nu_0} \leq b(\mathcal{K})\sqrt{t-a_i}\|\dot\sigma\|_{L^2((a,b)\times I)}$$

if $t \in [a_i, a_{i+1})$. This proves that $\{A^n\}_{n=1}^\infty$ converges to $(t,z) \to A_{\sigma_t}(z)$ and so, the latter map belongs to $L^2((a,b)\times I)$. The map $(t,z) \to \langle A_{\sigma_t}(z), \dot\sigma_t(z)\rangle = \Lambda_{\sigma_t}(\dot\sigma_t)$ is measurable as the inner product of two measurable functions. Its $L^2(a,b)$ norm is bounded by the product of $b(\mathcal{K})$ and $\|\dot\sigma\|_{L^2((a,b)\times I)}$. This proves (ii). We obtain (iii) by using that $\dot\sigma^*_s = \dot r(s)\dot\sigma_{r(s)}$ and the change of variables formula. QED.

We next prove an infinite dimensional analogue of Green's formula on the manifold $L^2(I)$. Suppose $\sigma \in AC^2(a,b;L^2(I))$ and consider the "two-dimensional annulus" $\sigma^s_t = s\sigma_t$ where $s \in [\varepsilon, 1]$ and $t \in [a,b]$.

Lemma 5.7. *Suppose Λ is a differentiable one-form on $\mathcal{T}L^2(I)$, $\sigma \in AC^2(a,b;L^2(I))$ and $\sigma_a = \sigma_b$. Then*

$$\int_a^b \Lambda_{\sigma_t}(\partial_t\sigma^1_t)dt - \int_a^b \Lambda_{\sigma^\varepsilon_t}(\partial_t\sigma^\varepsilon_t)dt = -\int_a^b dt \int_\varepsilon^1 d\Lambda\Big(\partial_t\sigma^s_t, \partial_s\sigma^s_t\Big)\, ds. \tag{70}$$

In particular if Λ is closed then $\int_a^b \Lambda_{\sigma_t}(\partial_t\sigma^1_t)dt = 0$.

Proof: Reparametrizing σ if necessary (cf. [2]), we may assume without loss of generality that $t \to \|\dot\sigma_t\|_{\nu_0} \in L^\infty(a,b)$. Lemma 5.6 ensures that the left handside of (70) is invariant under reparametrization. We use (69) and the fact that σ is Lipschitz to conclude that $t \to \Lambda_{\sigma_t^s}(\partial_s \sigma_t^s)$ and $s \to \Lambda_{\sigma_t^s}(\partial_t \sigma_t^s)$ are Lipschitz,

$$\partial_t \Big(\Lambda_{\sigma_t^s}(\partial_s \sigma_t^s) \Big) - \partial_s \Big(\Lambda_{\sigma_t^s}(\partial_t \sigma_t^s) \Big) = d\Lambda_{\sigma_t^s} \Big(\partial_t \sigma_t^s, \partial_s \sigma_t^s \Big). \tag{71}$$

We integrate both sides of (71), use that $\sigma_b^s = \sigma_a^s$ and $\partial_s \sigma_t^s = \sigma_t$ to obtain (70). If, in addition, Λ is closed, letting ε tend to 0 in (70) concludes the proof. QED.

Corollary 5.8. *Let Λ be a differentiable closed one-form on $\mathcal{T}L^2(I)$ and denote by 0 the null function. For $\sigma \in AC^2(0,b;L^2(I))$ such that $\sigma_0 \equiv 0$, $U(\sigma_b) := \int_a^b \Lambda_{\sigma_t}(\dot\sigma_t)$ depends only on σ_b and is, in particular, independent of b. (i) We have $dU = \Lambda$. (ii) If $M \to \Lambda_M(M)$ is rearrangement invariant then U is also rearrangement invariant.*

Proof: The fact that $U(\sigma_b)$ depends only on σ_b is a direct consequence of lemma 5.7. In particular,

$$U(M) = \int_0^1 \Lambda_{tM}(M) dt = \int_0^1 \frac{1}{t} \Lambda_{tM}(tM) dt. \tag{72}$$

If $M, H \in L^2(I)$ by (68) and (72)

$$|U(M+H) - U(M) - \langle A_M, H \rangle_{\nu_0} - \frac{1}{2} \langle B_M(H), H \rangle_{\nu_0}| \leq \frac{1}{2} c(A) \|H\|_{\nu_0}^2.$$

This proves that U is differentiable and $d\Lambda = U$. Suppose now that $M \to \Lambda_M(M)$ is rearrangement invariant and let $G \in \mathcal{G}$. Then, by (72),

$$U(M \circ G) = \int_0^1 \frac{1}{t} \Lambda_{(tM \circ G)}(tM \circ G) dt = \int_0^1 \frac{1}{t} \Lambda_{(tM)}(tM) dt = U(M).$$

Thus U is rearrangement invariant. QED.

Acknowledgements

A special thank to A. Swiech with whom the first author had numerous discussions. Fruitful discussions were also provided by S. Garoufalidis. We would like to thank J. Bellisard, J. Etnyre, V. Kaloshin, T. Nguyen and T. Pacini for their valuable comments and suggestions. This project started while A.T. was holding a visiting position at Georgia Tech whose financial support and hospitality are gratefully acknowledged. W.G. gratefully acknowledges the support provided by NSF grants DMS-03-54729 and DMS-06-00791.

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