# Table of contents of Volume I



# When is the product of two Hecke eigenforms an eigenform?


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*Dedicated to A. Schinzel on the occasion of his sixtieth birthday*

## Introduction and statement of result

From an elementary point of view the classical identity between divisor functions

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m), \tag{1}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, is rather surprising. Although it and other related identities may be proved using elementary means (see [Ra]), it becomes transparent in the context of modular forms. It is equivalent to the identity $E_8 = E_4^2$ where, for even $k \geq 4$, $E_k$ is the Eisenstein series

$$E_k(z) = 1 + \gamma_k \sum_{n\geq 1} \sigma_{k-1}(n) q^n \tag{2}$$

with certain rational $\gamma_k$, where $q = e(z) = e^{2\pi i z}$. This identity is forced by the fact that the space of modular forms of weight 8 is 1-dimensional.

Call a modular form $f$ for the full modular group an *eigenform* if it is either $E_k$ or a cusp form which is an eigenfunction of the Hecke operators, normalized to have Fourier expansion

$$f(z) = q + \sum_{n\geq 2} a_f(n) q^n.$$

The Fourier coefficients of an eigenform $f$ are a multiplicative arithmetic function:

$$a_f(n)a_f(m) = \sum_{d|(m,n)} d^{k-1} a_f(mn/d^2), \qquad m,n \in \mathbb{Z}^+.$$

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Research supported in part by NSF grant DMS-9500797.

For $k = 12, 16, 18, 20, 22, 26$ there is a unique cuspidal eigenform of weight $k$ which we will denote by $F_k$. In particular

$$F_{12} = q \prod_{n \geq 1} (1 - q^n)^{24},$$

the discriminant function, whose Fourier coefficients were made famous by Ramanujan.

We have the factorizations

$$\begin{gathered} E_8 = E_4^2 \qquad E_{10} = E_4 E_6 \qquad E_{14} = E_4 E_{10} = E_6 E_8 \\ F_{16} = E_4 F_{12} \qquad F_{18} = E_6 F_{12} \qquad F_{20} = E_8 F_{12} = E_4 F_{16} \\ F_{22} = E_{10} F_{12} = E_6 F_{16} = E_4 F_{18} \\ F_{26} = E_{14} F_{12} = E_{10} F_{16} = E_8 F_{18} = E_6 F_{20} = E_4 F_{22}. \end{gathered} \tag{3}$$

Each of these is equivalent to an additive identity of the type (1) between multiplicative arithmetic functions. As in the case of (1), each is an immediate consequence of the dimension formulae for the spaces of modular forms $M_k$ and cusp forms $S_k$ of weight $k$, valid for even $k > 2$:

$$\dim M_k = [k/12] + \delta_k, \qquad \dim S_k = \dim M_k - 1$$

where $\delta_k = 0$ or 1 according to whether $k \equiv 2 \pmod{12}$ or not (see p. 88 of [Se]).

One may ask whether other such factorizations are possible. My aim in this note is show that this list is complete.

**Theorem.** *Every factorization of an eigenform into a product of two eigenforms occurs in the list* (3).

In fact the proof of this Theorem shows that in the unique decomposition of a product of two eigenforms into a linear combination of eigenforms none of the coefficient may vanish. Hence, in general, a product of two eigenforms is as far from being an eigenform as the dimension allows. The main idea used is simply that an Euler product, in this case a Rankin-Selberg convolution of eigenforms, cannot vanish at a point in the region of absolute convergence.

## Proof

First observe that any factorization may involve at most one cusp form so we need only examine the possibility that

$$F = f E_\ell, \qquad \ell \geq 4$$

where $f$ is an eigenform of weight $k$ and $F$ an eigenform of weight $k + \ell$. Also, the cases where $f = E_k$ may be found by identifying the first Fourier coefficients of



both sides of $E_k E_\ell = E_{k+l}$ from (2), namely solving the equation

$$\gamma_{k+\ell} = \gamma_k + \gamma_\ell. \tag{4}$$

Using the standard fact (see p. 93 of [Se]) that

$$\gamma_k = \frac{(2\pi i)^k}{(k-1)!\zeta(k)},$$

a straightforward analysis shows that the equation (4) admits only the solutions corresponding to the factorizations on line 1 of (3). Thus we are reduced to considering the case that $f$ is a cuspidal eigenform of weight $k$.

Suppose for the moment that $f$ and $g$ are any modular forms for $\Gamma = SL(2, \mathbb{Z})$ of weights $k, k + \ell$ respectively with Fourier expansions

$$f(z) = \sum_{n \geq 0} a_f(n) q^n, \qquad g(z) = \sum_{n \geq 0} a_g(n) q^n.$$

Define the Rankin-Selberg convolution $\Lambda(f \otimes g, s) = (4\pi)^{-s}\Gamma(s) L(f \otimes g, s)$ where

$$L(f \otimes g, s) = \sum_{n \geq 1} a_f(n) \bar{a}_g(n) n^{-s}.$$

**Lemma.** *Suppose at least one of $f$ or $g$ is a cusp form. Then*

$$\langle f E_\ell, g \rangle = \Lambda(f \otimes g, k + \ell - 1)$$

*where $\langle \cdot , \cdot \rangle$ is the Petersson inner product.*

*Proof.* This follows from the integral representation

$$\Lambda(f \otimes g, s + k + \ell/2 - 1) = \int_{\Gamma \backslash \mathcal{H}} f(z) \bar{g}(z) E_\ell(z, s) y^{k + \ell/2} \frac{dxdy}{y^2}. \tag{5}$$

Here

$$E_\ell(z, s) = y^s/2 \sum_{(c,d)=1} ((cz + d)/|cz + d|)^{-\ell} |cz + d|^{-2s}$$

is a non-holomorphic Eisenstein series which transforms like

$$E_\ell(\gamma z, s) = ((cz + d)/|cz + d|)^{\ell} E_\ell(z, s)$$

for $\gamma \in \Gamma$ where $\gamma z = (az + b)/(cz + d)$ and which has an analytic continuation to the entire $s$-plane (see [Si]). Since $h(z) = f(z)\bar{g}(z) y^{k+\ell/2}$ transforms like

$$h(\gamma z) = ((cz + d)/|cz + d|)^{-\ell} h(z)$$

the integral

$$\int_{\Gamma \backslash \mathcal{H}} h(z) E_\ell(z, s) \frac{dxdy}{y^2}$$

is well defined and convergent for any $s$. By the "unfolding trick" it is

$$\int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}(\gamma z)^s)\frac{dxdy}{y^2} = \int_0^\infty \int_0^1 h(x+iy)dx\, y^{s-1}\frac{dy}{y}$$

where $\Gamma_\infty \subset \Gamma$ is the stabilizer of the cusp at infinity. Since

$$\int_0^1 h(x+iy)dx = y^{k+\ell/2}\sum_{n\geq 1} a_f(n)\bar{a}_g(n)e^{-4\pi ny}$$

and

$$\int_0^\infty y^{k+\ell/2+s-1}\sum_{n\geq 1} a_f(n)\bar{a}_g(n)e^{-4\pi ny}\frac{dy}{y}$$
$$= (4\pi)^{-(s+k+\ell/2-1)}\Gamma(s+k+\ell/2-1)L(f\otimes g, s+k+\ell/2-1),$$

we get (5). The Lemma follows from the fact that

$$E_\ell(z,\ell/2) = y^{\ell/2}E_\ell(z).$$

□

We may now prove the Theorem. Suppose that $f$ is a cuspidal eigenform and $F = fE_\ell$ for $\ell \geq 4$. Provided $k+\ell \geq 24$ and $k+\ell \neq 26$ there will be a cuspidal eigenform $g$ of weight $k+\ell$ which is different from $F$, hence orthogonal to $F$: $\langle fE_\ell, g\rangle = 0$. On the other hand in this case

$$L(f\otimes g, s) = \prod_p \sum_{m\geq 0} a_f(p^m)a_g(p^m)p^{-ms}$$

an Euler product which converges absolutely for $\mathrm{Re}(s) > k+\ell/2$. Thus

$$\Lambda(f\otimes g, k+\ell-1) \neq 0$$

since $\ell > 2$. By the Lemma this cannot happen and so this completes the proof of the Theorem. □

**Remark.** Taking $f = E_k$ and $g$ to be any cuspidal eigenform of weight $2k$, the Rankin-Selberg $L$-function factors giving rise to the formula

$$-\frac{B_k}{2k}L(f\otimes g, 2k-1) = \frac{L(g,k)L(g,2k-1)}{\zeta(k)}$$

where $L(g,s) = \sum_{n\geq 1} a_g(n)n^{-s}$ is the $L$-function associated to $g$. The fact that $E_k^2 \neq E_{2k}$ for $k > 4$ then actually implies that the central critical value $L(g,k) \neq 0$ for some $g$. This was observed in [Z] (see also [CF]).

**Acknowledgements.** I thank J. Tunnell for posing the question answered here and P. Čížek for independently verifying that the list of factorizations involving only Eisenstein series is complete. I also thank the referee for pointing out an error in an earlier version of this note. In proof: I learned that M. Larsen has obtained the Theorem of this note independently by different methods.



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