SUMS OVER PRIMES OF THE FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT CUSP FORMS

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The following is a brief summary of work presented in detail in [**1**]. Our intention is to expose main ideas and to compare and contrast our methods with more traditional ones used in similar contexts.

We are concerned with certain properties of the Fourier coefficients of half-integral weight cusp forms. More specifically we give non-trivial estimates for various bilinear forms in these coefficients. These are then applied in Vinogradov's method to give an estimate for their sum over primes.

In general, given an arithmetic function f_n one is often interested in the sum over primes

$$S(X) = \sum_{p \le X} f_p \quad .$$

If f_n is normalized so that

$$\sum_{n \le X} |f_n|^2 \ll X$$

then Cauchy's inequality gives the trivial bound

$$S(X) \ll \frac{X}{\log^{\frac{1}{2}} X} \quad .$$

For most randomly oscillating f_n essentially the best bound one could hope for would be

$$S(X) \ll_\epsilon X^{\frac{1}{2}+\epsilon} \quad .$$

For multiplicative functions f_n one traditionally considers the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n n^{-s} = \prod_{p \text{ prime}} \left(\sum_{m=0}^{\infty} f_{p^m} p^{-ms} \right)$$

for $\mathrm{Re}(s) > 1$. If $F(s)$ may be continued to an entire function, satisfies a standard type of functional equation relating $F(s)$ to $F(1-s)$, and if $F(s) \neq 0$ for $\mathrm{Re}(s) > \theta \ge \frac{1}{2}$ then we may deduce classically that

$$S(X) \ll_\epsilon X^{\theta+\epsilon} \quad .$$

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Of course, this last condition is not known for $\theta < 1$ for any interesting $F(s)$. If, for example, $f_n = \chi(n)$ is a non-trivial Dirichlet character or if $n^{\frac{k-1}{2}} f_n$ is the n^{th} Fourier coefficient of a cuspidal eigenform of weight k then the method of Hadamard and de la Vallée Poussin gives only

$$S(X) \ll X \exp(-c \log^{\frac{1}{2}} X)$$

for some $c > 0$ depending on the "conductor" of f_n.

For non-multiplicative f_n Vinogradov's method may be applicable. This was first applied by him in 1937 to $f_n = e(n\alpha)$. We apply it to the Fourier coefficients of certain half-integral weight cusp forms. To define these, let χ be a Dirichlet character modulo N, where $N \equiv 0 \pmod 4$, and $k = \frac{1}{2} + l$ where $l \in \mathbf{Z}$, $l \geq 2$. H is the upper half-plane acted on by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R}) \text{ where } \gamma z = \frac{az+b}{cz+d} \quad .$$

$S_k(N, \chi)$ is the finite-dimensional Hilbert space of holomorphic functions on H which satisfy

$$f(\gamma z) = \nu(\gamma)(cz+d)^k f(z) \quad \text{for } \gamma \in \Gamma = \Gamma_0(N)$$

and are such that $y^{\frac{k}{2}} |f(z)|$ is uniformly bounded on H. Here $\nu(\gamma) = \chi(d)\left(\frac{c}{d}\right)\bar{\epsilon}_d$ where $\left(\frac{c}{d}\right)$ is the extended quadratic residue symbol (see [6]) and

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ i^{2k}, & \text{if } d \equiv -1 \pmod 4. \end{cases}$$

The inner product on $S_k(N, \chi)$ is defined by

$$\langle f, g \rangle = \int_{\Gamma \backslash H} f(z) \overline{g}(z) y^k \frac{dx\,dy}{y^2} \quad .$$

Any $f \in S_k(N, \chi)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \hat{f}_n e(nz) \quad .$$

Define $f_n = n^{-\frac{k-1}{2}} \hat{f}_n$. It is expected that for all $\epsilon > 0$, $f_n \ll_\epsilon n^\epsilon$, the best known estimate being $f_n \ll_\epsilon n^{\frac{3}{14}+\epsilon}$ which follows from [3]. By the Rankin-Selberg method

$$\sum_{n \leq X} |f_n|^2 \ll_f X$$

so the trivial bound for $S(X)$ is, as before,

$$S(X) \ll_f \frac{X}{\log^{\frac{1}{2}} X} \quad .$$

We prove in [**1**] the following

THEOREM : *For all* $\epsilon > 0$

$$S(X) = \sum_{p \leq X} f_p \ll X^{\frac{155}{156}+\epsilon} \quad as \quad X \to \infty \quad ,$$

where the summation is over p prime and the implied constant depends only on ϵ and f.

Vinogradov's method reduces the estimation of $S(X)$ to that of certain bilinear forms in f_{mn} for various ranges of m and n. We actually use a variant of this method given by Vaughan in [**7**] which, among other things, simplifies the arguments. We shall consider here only one of the required estimates, namely

$$\sum_{n \leq X} \sum_{m \leq Y}{}' a_m b_n f_{mn} \ll_{\epsilon, f} (X^{\frac{3}{4}} + X^{\frac{1}{4}} Y)(XY)^{\epsilon} \|a\| \, \|b\| \tag{1}$$

where the prime restricts the variable of summation to squarefree values, a_m and b_n are any complex numbers, and $\|a\|^2 = \sum_{m \leq Y} |a_m|^2$.

The trivial bound for the left-hand side of (1) is $(XY)^{\frac{1}{2}+\epsilon}\|a\|\|b\|$ and this would generally be best possible if f_n were multiplicative. In this case we see that f_n is not multiplicative since (1) is non-trivial if $X \gg Y^{2+\epsilon} \gg 1$.

By Cauchy's inequality the proof of (1) is reduced to

$$\sum_{n \leq X} \left| \sum_{m \leq Y}{}' a_m f_{mn} \right|^2 \ll_{\epsilon, f} (X + X^{\frac{1}{2}} Y^2)(XY)^{\epsilon} \|a\|^2 \quad . \tag{2}$$

We conjecture that the following self-dual form in fact holds :

$$\sum_{n \leq X}{}' \left| \sum_{m \leq Y}{}' a_m f_{mn} \right|^2 \ll_{\epsilon, f} (X + Y)(XY)^{\epsilon} \|a\|^2 \quad . \tag{3}$$

This would be somewhat analogous to the self-dual inequality

$$\sum_{n \leq X}{}'' \left| \sum_{m \leq Y}{}'' a_m \left(\frac{m}{n} \right) \right|^2 \ll_{\epsilon} (X + Y)(XY)^{\epsilon} \|a\|^2 \tag{4}$$

where the double prime restricts the summation to squarefree integers congruent to 1 modulo 4. The possibility that an inequality of this type might hold was suggested orally by Montgomery to the second author in 1984. Jutila has given results related to (4) (see [**4**] and [**5**]) but (4) itself is unknown.

For the proof of (2) we require the following estimate :

$$\sum_{n\geq 1} f_{nr}\overline{f}_{ns}\exp\left(-\frac{n}{X}\right) \ll_{\epsilon,f} \delta_{rs}X + (rsX)^{\frac{1}{2}+\epsilon} \tag{5}$$

where r, s are squarefree integers congruent modulo 4 and prime to a number depending only on f. Here δ_{rs} denotes the Kronecker delta. It is important to realize that the uniformity of this estimate in r and s is its key feature. For its proof we may assume that $f(z)$ is a Poincaré series and then f_{ns} may be evaluated as a sum of Kloosterman sums. We substitute this evaluation of f_{ns} into (5). Then we apply Poisson's summation for f_{nr} twisted by an additive character (resulting from the Kloosterman sum) to obtain Gauss-Ramanujan sums. These are estimated to give (5).

Our approach shares some features with the Rankin-Selberg method. However, we avoided this because we had difficulty in establishing the required uniformity in r and s through the functional equation of the Eisenstein series.

The estimate in our theorem should be compared with that of Heath-Brown and Patterson for sums of cubic Gauss sums over prime moduli [**2**]. Their technique is, however, rather different from ours since they use a kind of twisted multiplicativity of cubic Gauss sums to estimate the relevant bilinear forms. Such multiplicativity does not exist in the case of half-integral weight cusp forms.

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