ON THE ZEROS AND COEFFICIENTS OF CERTAIN WEAKLY HOLOMORPHIC MODULAR FORMS

W. DUKE AND PAUL JENKINS

To J-P. Serre on the occasion of his eightieth birthday.

1. Introduction

For this paper we assume familiarity with the basics of the theory of modular forms as may be found, for instance, in Serre's classic introduction [12]. A weakly holomorphic modular form of weight $k \in 2\mathbb{Z}$ for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is a holomorphic function f on the upper half-plane that satisfies

$$f(\tfrac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \quad \text{for all} \quad (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$$

and that has a q-expansion of the form $f(\tau) = \sum_{n \geq n_0} a(n) q^n$, where $q = e^{2\pi i \tau}$ and $n_0 = \mathrm{ord}_\infty(f)$. Such an f is holomorphic if $n_0 \geq 0$ and a cusp form if $n_0 \geq 1$. Let $\mathcal{M}_k$ denote the vector space of all weakly holomorphic modular forms of weight k. Any nonzero $f \in \mathcal{M}_k$ satisfies the valence formula

$$\tfrac{1}{12} k = \mathrm{ord}_\infty(f) + \tfrac{1}{2}\mathrm{ord}_i(f) + \tfrac{1}{3}\mathrm{ord}_\rho(f) + \sum_{\tau \in \mathcal{F} \setminus \{i,\rho\}} \mathrm{ord}_\tau(f) \quad (\rho = -\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}), \tag{1}$$

where $\mathcal{F}$ is the usual fundamental domain for Γ. Write $k = 12\ell + k'$ with uniquely determined $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. An important consequence of (1) is that

$$\mathrm{ord}_\infty(f) \leq \ell \tag{2}$$

for a nonzero $f \in \mathcal{M}_k$.

For each $k \geq 4$ we have a holomorphic form in $\mathcal{M}_k$ given by the Eisenstein series

$$E_k(\tau) = 1 + A_k \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \quad \text{where} \quad A_k = -\tfrac{2k}{B_k}, \tag{3}$$

with B_k the Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. These give rise to the weight 12 cusp form

$$\Delta(\tau) = \tfrac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = q \prod_{n \geq 1}(1-q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

and the weight 0 modular function

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + \sum_{n \geq 1} c(n) q^n, \tag{4}$$

known simply as the j-function.

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In this paper we are interested in various properties of a certain natural basis for $\mathcal{M}_k$ defined as follows. For each integer $m \geq -\ell$, there exists a unique $f_{k,m} \in \mathcal{M}_k$ with q-expansion of the form

$$f_{k,m}(\tau) = q^{-m} + O(q^{\ell+1}). \tag{5}$$

It can be constructed explicitly in terms of Δ, j and $E_{k'}$, where we set $E_0 = 1$. In fact,

$$f_{k,m} = \Delta^{\ell} E_{k'} F_{k,D}(j), \tag{6}$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$ with integer coefficients. The uniqueness of $f_{k,m}$ is a consequence of (2). These $f_{k,m}$ with $m \geq -\ell$ form a basis for $\mathcal{M}_k$; any modular form $f \in \mathcal{M}_k$ with Fourier coefficients $a(m)$ can be written

$$f = \sum_{n_0 \leq n \leq \ell} a(n) f_{k,-n}, \tag{7}$$

again by (2). When $\ell > 0$, the set $\{f_{k,-\ell}, f_{k,-\ell+1}, \dots, f_{k,-1}\}$ is a basis for the subspace of cusp forms, which thus has dimension ℓ. For

$$k = 4, 6, 8, 10, 14 \quad \text{we have} \quad A_k = 240, -504, 480, -264, -24, \tag{8}$$

respectively. Therefore $E_{k'}, \Delta, j$ and $F_{k,m}$ all have integer coefficients and it follows that the coefficients $a_k(m,n)$ defined by

$$f_{k,m}(\tau) = q^{-m} + \sum_n a_k(m,n)\, q^n$$

are integral.

The functions $f_{k,-\ell} = \Delta^{\ell} E_{k'}$ play a special role, and we will denote them by f_k and their Fourier coefficients by $a_k(n)$. The $f_{k,m}$ are also familiar when $k = 0$, where they are central in the theory of singular moduli (see [14]); the first few are given by

$$\begin{aligned}
f_{0,0}(\tau) &= 1 \\
f_{0,1}(\tau) &= j(\tau) - 744 &&= q^{-1} + 196884\, q + 21493760\, q^2 + \cdots, \\
f_{0,2}(\tau) &= j(\tau)^2 - 1488 j(\tau) + 159768 &&= q^{-2} + 42987520\, q + 40491909396\, q^2 + \cdots.
\end{aligned}$$

More generally, the $f_{k,m}$ have been studied extensively when $k \in \{0, 4, 6, 8, 10, 14\}$ and when $m = 0$, where $f_{k,0}(\tau) = 1 + O(q^{\ell+1})$. For example, in all of these cases, the zeros of $f_{k,m}$ in $\mathcal{F}$ are known to lie on the unit circle; the proofs vary depending on the case. One aim of this paper is to provide a general result on the location of the zeros that holds for all k and to give a unified method of proof. This is given as Theorem 1 below. Its proof is based on the following generating function for the $f_{k,m}$ (Theorem 2), to which a simple type of circle method is applied:

$$\sum_{m \geq -\ell} f_{k,m}(z) q^m = \frac{f_k(z) f_{2-k}(\tau)}{j(\tau) - j(z)}.$$

Another consequence of the generating function is the following duality between the coefficients in weights k and $2 - k$:

$$a_k(m,n) = -a_{2-k}(n,m).$$

This duality, well known when $\ell = 0$, is illustrated by the weights $k = 12$

$$\begin{aligned}
f_{12,-1}(\tau) &= q \quad - 24q^2 \quad + 252q^3 \quad - 1472q^4 \quad + \cdots, \\
f_{12,0}(\tau) &= 1 \quad + 196560q^2 \quad + 16773120q^3 \quad + 398034000q^4 \quad + \cdots, \\
f_{12,1}(\tau) &= q^{-1} \quad + 47709536q^2 \quad + 39862705122q^3 \quad + 7552626810624q^4 \quad + \cdots,
\end{aligned}$$

and $2-k=-10$

$$\begin{array}{rcccccc} f_{-10,2}(\tau) & = & q^{-2} & +24q^{-1} & -196560 & -47709536q & +\cdots, \\ f_{-10,3}(\tau) & = & q^{-3} & -252q^{-1} & -16773120 & -39862705122q & +\cdots, \\ f_{-10,4}(\tau) & = & q^{-4} & +1472q^{-1} & -398034000 & -7552626810624q & +\cdots. \end{array}$$

Note that $f_{12,-1}=f_{12}=\Delta$.

It follows from a paper of Siegel [13] that if $k>0$, then the coefficient $a_k(0,\ell+1)$ is divisible by every prime p with $(p-1)|k$. Thus, for example, when $k=12$ we have

$$a_{12}(0,2)=196560=2^4\cdot 3^3\cdot 5\cdot 7\cdot 13.$$

To see this, by (7) the Fourier coefficients $a(n)$ of any $f\in\mathcal{M}_k$ must satisfy

$$a(\ell+1)=\sum_{n\le\ell} a_k(-n,\ell+1)a(n). \tag{9}$$

Applying this to E_k from (3) for $k\ge 4$ gives the formula

$$a_k(0,\ell+1)\frac{B_k}{2k}=\sum_{0<n\le\ell} a_k(-n,\ell+1)\sigma_{k-1}(n)-\sigma_{k-1}(\ell+1).$$

It follows that $a_k(0,\ell+1)$ is divisible by the denominator of $\frac{B_k}{2k}$, hence the result is a consequence of the Staudt-Clausen theorem. Siegel argued using the dual form of (9), namely

$$\sum_{n\le\ell+1} a_{2-k}(-n)a(n)=0. \tag{10}$$

Siegel's observation suggests that it might be interesting to examine the divisors of $a_k(m,n)$ in other cases. Consider, for example, the following factorizations when $k=14$ and $n=1$:

$$\begin{aligned} a_{14}(1,3)&=-2\cdot 3^{16}\cdot 5^2\cdot 19,\\ a_{14}(1,7)&=-3^4\cdot 5^2\cdot 7^{14}\cdot 2129,\\ a_{14}(1,15)&=-3^{17}\cdot 5^{14}\cdot 7\cdot 25679\cdot 26879,\\ a_{14}(1,32)&=-2^{72}\cdot 5^2\cdot 34610493144432841. \end{aligned}$$

In each case, the coefficient of q^n is divisible by high powers of the prime factors of n. As a special case of Theorem 3, we will show that $n^{13}|\, a_{14}(1,n)$ holds for all $n\ge 1$. Since

$$f_{14,1}=E_{14}(j-720),$$

this implies the following recursive congruence for the coefficients $c(n)$ of the j-function:

$$c(n)\equiv 24^2\sigma_{13}(n)+24\,\sigma_{13}(n+1)+24\sum_{i=1}^{n-1}\sigma_{13}(n-i)c(i)\pmod{n^{13}},$$

which holds for all $n\ge 1$.

Finally, we mention that Lehmer's famous conjecture that $\tau(n)\neq 0$ for $n\ge 1$ is equivalent to the non-vanishing of the "leading" term in the n^{th} basis function in weight -10 since by duality

$$f_{-10,n}(\tau)=q^{-n}-\tau(n)q^{-1}+\cdots.$$

More generally, we can write

$$f_{k,n}(\tau)=q^{-n}-a_{2-k}(n)q^{\ell+1}+\cdots,$$

where $a_{2-k}(n)$ is the n^{th} coefficient of $f_{2-k}=\Delta^{-\ell-1}E_{14-k'}$. It is easily checked that $a_{2-k}(n)\neq 0$ for $n\ge -\ell-1$ when $k\in\{-12,-8,-6,-4,-2\}$ or when $k\ge 4$ and $k\equiv 2\pmod 4$. Siegel

[13, Satz 2] showed that $a_{2-k}(0) \neq 0$ when $k > 0$. It seems to be an interesting problem to find other such non-vanishing results.

2. Statement of results

The following result concerning the location of the zeros of the $f_{k,m}$ is proved in Section 5.

Theorem 1. *If $m \geq |\ell| - \ell$, then all of the zeros of $f_{k,m}$ in $\mathcal{F}$ lie on the unit circle.*

The condition $m \geq 0$ of Theorem 1 excludes cusp forms; in fact, the conclusion of Theorem 1 does not hold in general without some restriction on m. The form $f_{132,-9}$ of weight 132 is the first positive weight example where it fails, and the form $f_{-256,23}$ of weight -256 is the first example of negative weight where it fails. Of course, it always holds for $f_k = f_{k,-\ell}$. A list of weights where each basis function has all of its zeros in $\mathcal{F}$ on the unit circle is given at the end of Section 6.

Theorem 1 is related in various ways to previously known results. When $k \in \{4, 6, 8, 10, 14\}$, a comparison of q-expansions shows that for $m \geq 0$

$$f_{k,m} = P_{k,-m}, \tag{11}$$

where $P_{k,m}$ is the convergent Poincaré series

$$P_{k,m}(\tau) = \tfrac{1}{2} \sum_{(c,d)=1} e(m\, \tfrac{a\tau+b}{c\tau+d})(c\tau + d)^{-k}, \tag{12}$$

defined for any $k \geq 4$ and $m \in \mathbb{Z}$. Here the sum is over all coprime pairs (c, d), where for each pair $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is arbitrarily chosen (see [10]). As a special case of a more general result, R. Rankin [9] showed in 1982 that for $m \geq 0$ and even $k \geq 4$, all of the zeros of $P_{k,-m}$ in $\mathcal{F}$ lie on the unit circle. When $m = 0$, so that $P_{k,0} = E_k$, this result had been obtained already in 1970 by F. Rankin and Swinnerton-Dyer [8]. They introduced the idea of approximating (a multiple of) the modular form by an elementary function having the required number of zeros on the arc $\{e^{i\theta} : \theta \in (\frac{\pi}{2}, \frac{2\pi}{3})\}$. Some variation on this idea appears in the known proofs of almost all such results. For Poincaré series, this approximation makes use of the definition (12). Asai, Kaneko, and Ninomiya [1] extended Rankin's result by proving Theorem 1 for the case $k = 0$. As they mention, their proof can be modified to cover all cases when $\ell = 0$. In place of Poincaré series for the approximation, they use the fact that when $\ell = 0$

$$f_{k,1} \,|\, T_m = m^{k-1} f_{k,m}, \tag{13}$$

where T_m is the Hecke operator and $m \geq 1$. Finally, when $m = 0$, Theorem 1 was proved by Getz [5], using a generalization of the method of [8]. As can be seen from the proof, this is the most delicate case of Theorem 1.

In order to prove Theorem 1 in general, we will avoid the use of Poincaré series and Hecke operators, since the relations (11) and (13) need not hold when $\ell \neq 0$. Instead, we derive an integral formula for $f_{k,m}$, for which approximation by residues leads to Theorem 1. Computing the first few terms of the approximation via a circle method-type argument is enough to prove the theorem. The integral formula, given in Lemma 2, is equivalent to the following generating function for $f_{k,m}$.

Theorem 2. *For any even integer k we have*

$$\sum_{m \geq -\ell} f_{k,m}(z) q^m = \frac{f_k(z) f_{2-k}(\tau)}{j(\tau) - j(z)},$$

where $f_k = \Delta^\ell E_{k'}$ with $k = 12\ell + k'$.

For the case $\ell = 0$, this was given in [1]. In fact, such formulas were first discovered by Faber [3, 4] as early as 1903 for quite general conformal maps, and $F_{k,D}(x)$ from (6) is a generalized Faber polynomial. For completeness, we will give the short proof of Theorem 2 in Section 4. A readily proved corollary is the following duality between coefficients for weights k and $2-k$.

Corollary 1. *Let k be an even integer. For all integers m, n the equality*

$$a_k(m,n) = -a_{2-k}(n,m)$$

holds for the Fourier coefficients of the modular forms $f_{k,m}$ and $f_{2-k,n}$.

This also follows from the fact that $f_{k,m}f_{2-k,n}$ is the derivative of a polynomial in j, hence has vanishing zeroth Fourier coefficient. A variant of this idea was used in [13] to obtain (10). Similar duality theorems hold for modular forms of half integral weight (see [14] and [2]).

The divisibility result mentioned at the end of the Introduction is a special case of the following.

Theorem 3. *Let $k \in \{4, 6, 8, 10, 14\}$. If $(m, n) = 1$, then $n^{k-1} \mid a_k(m, n)$.*

This is proved next and follows from basic properties of the Hecke operators. In the case $m = 1$ this easily implies the following congruences for the coefficients $c(n)$ of the j-function.

Corollary 2. *For each $k \in \{4, 6, 8, 10, 14\}$ and for all $n \geq 1$, we have the congruence*

$$c(n) \equiv A_k^2\sigma_{k-1}(n) - A_k\sigma_{k-1}(n+1) - \sum_{i=1}^{n-1} A_k\sigma_{k-1}(n-i)c(i) \pmod{n^{k-1}},$$

where the value of A_k is given in (8).

3. Proof of Theorem 3

Theorem 3 is an immediate consequence of the following result (c.f. [7]). (Note that $a_k(m, n) = 0$ if m or n is not an integer.)

Lemma 1. *Let p be a prime and $k \in \{4, 6, 8, 10, 14\}$. Then*

$$a_k(m, np^r) = p^{r(k-1)}\Big(a_k(mp^r, n) - a_k\big(mp^{r-1}, \tfrac{n}{p}\big)\Big) + a_k\big(\tfrac{m}{p}, np^{r-1}\big).$$

For positive integers N, the Hecke operator T_N of weight k sends modular forms in $\mathcal{M}_k$ to modular forms in $\mathcal{M}_k$. For $k \geq 2$, we denote the coefficient of q^n in $f_{k,m}(\tau)|T_N$ by $a_k(m, n, N)$, so that

$$f_{k,m}(\tau)|T_N = \sum a_k(m, n, N)q^n.$$

Standard formulas for the action of the Hecke operator (for example, in VII.5.3 of [12]) give that for a prime p,

$$a_k(m, n, p) = a_k(m, np) + p^{k-1}a_k(m, \tfrac{n}{p}) \quad \text{if } k \geq 2. \tag{14}$$

Suppose now that $k \in \{4, 6, 8, 10, 14\}$ and that $m \geq 1$, so that $f_{k,m}(\tau) = q^{-m} + O(q)$. Since an equation similar to (14) is valid for $n < 0$, we calculate that the q-expansion of $f_{k,m}(\tau)|T_p$ begins

$$f_{k,m}(\tau)|T_p = p^{k-1}q^{-mp} + q^{-m/p} + O(q),$$

where the second term is omitted if $p \nmid m$. Because there are no cusp forms in $\mathcal{M}_k$, the non-positive powers of q completely determine the decomposition of $f_{k,m}(\tau)|T_p$ into basis elements $f_{k,m}(\tau)$, and we obtain the formula

$$f_{k,m}(\tau)|T_p = p^{k-1} f_{k,mp} + f_{k,m/p},$$

where $f_{k,\alpha} = 0$ if α is not an integer. The coefficients of q^n on each side give

$$a_k(m,n,p) = p^{k-1} a_k(mp,n) + a_k\left(\tfrac{m}{p}, n\right). \tag{15}$$

Combining equations (14) and (15), then, we obtain

$$a_k(m,np) = p^{k-1}\left(a_k(mp,n) - a_k\left(m, \tfrac{n}{p}\right)\right) + a_k\left(\tfrac{m}{p}, n\right). \tag{16}$$

These observations are enough to prove the lemma.

To see this, let r be a positive integer. Note that for $1 \le i \le r-1$, replacing m with $p^i m$ and n with $p^{r-i-1} n$ in (16) gives

$$\begin{aligned} &p^{i(k-1)}\left(a_k(mp^i, np^{r-i}) - a_k(mp^{i-1}, np^{r-i-1})\right) \\ &\quad = p^{(i+1)(k-1)}(a_k(mp^{i+1}, np^{r-i-1}) - a_k(mp^i, np^{r-i-2})). \end{aligned} \tag{17}$$

We now replace n with np^{r-1} in equation (16) to obtain

$$a_k(m, np^r) = p^{k-1}(a_k(mp, np^{r-1}) - a_k(m, np^{r-2})) + a_k\left(\tfrac{m}{p}, np^{r-1}\right),$$

and use (17) a total of $(r-1)$ times to obtain

$$a_k(m, np^r) = p^{r(k-1)}\left(a_k(mp^r, n) - a_k\left(mp^{r-1}, \tfrac{n}{p}\right)\right) + a_k\left(\tfrac{m}{p}, np^{r-1}\right),$$

thus proving Lemma 1.

We remark that Lemma 1 may be generalized to weights with $\ell > 0$ without much difficulty, although the presence of cusp forms in these spaces adds additional terms.

4. Proof of Theorem 2

By Cauchy's integral formula it suffices to prove the following.

Lemma 2. *We have*

$$f_{k,m}(z) = \tfrac{1}{2\pi i} \oint_C \frac{\Delta^\ell(z) E_{k'}(z) E_{14-k'}(\tau)}{\Delta^{1+\ell}(\tau)(j(\tau) - j(z))}\, q^{-m-1}\, dq,$$

for C a (counterclockwise) circle centered at 0 in the q-plane with a sufficiently small radius.

First observe that by (5) and (6)

$$\Delta^\ell E_{k'} F_{k,D}(j) = q^{-m} + O(q^{\ell+1}).$$

Thus by Cauchy's integral formula we have, for C' a (counterclockwise) circle centered at 0 in the j-plane with a sufficiently large radius, that

$$F_{k,D}(\zeta) = \tfrac{1}{2\pi i} \oint_{C'} \frac{F_{k,D}(j)}{j - \zeta} dj = \tfrac{1}{2\pi i} \oint_{C'} \frac{q^{-m}}{\Delta(j)^\ell E_{k'}(j)(j-\zeta)}\, dj.$$

Changing variables $j \mapsto q$ and using the well-known identity

$$q\frac{dj}{dq} = \frac{-E_{14}}{\Delta},$$

we see that

$$F_{k,D}(\zeta) = \tfrac{1}{2\pi i} \oint_C \frac{E_{14-k'}(\tau)\, q^{-m-1}}{\Delta(\tau)^{1+\ell}(j(\tau) - \zeta)} dq.$$

Replacing ζ with $j(z)$, multiplying by $\Delta(z)^\ell E_{k'}(z)$ and applying (6), we finish the proof of Lemma 2 and hence Theorem 2.

5. Proof of Theorem 1

The zeros of $E_{k'}$ in $\mathcal{F}$ occur in $\{i, \rho\}$ with easily determined multiplicities, and Δ has no zeros in $\mathcal{F}$. Thus, by (6) and the valence formula (1), to prove Theorem 1 it is enough to show that when $D = \ell + m \geq |\,\ell|$, the function $f_{k,m}$ has D zeros on the arc $\{e^{i\theta} : \theta \in (\frac{\pi}{2}, \frac{2\pi}{3})\}$. In fact, we will see that these zeros are simple. An easy argument [5, Prop. 2.1] shows that for any weakly holomorphic modular form f of weight k with real coefficients, the quantity $e^{ik\theta/2} f(e^{i\theta})$ is real for $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$. We will show that for these θ, the following lemma holds.

Lemma 3. *For all* $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$,

$$\left|e^{ik\theta/2} e^{-2\pi m \sin\theta} f_{k,m}(e^{i\theta}) - 2\cos\left(\tfrac{k\theta}{2} - 2\pi m\cos\theta\right)\right| < 1.985.$$

This inequality is enough to prove the theorem. To see this, note that as θ increases from $\pi/2$ to $2\pi/3$, the quantity

$$\tfrac{k\theta}{2} - 2\pi m \cos\theta \tag{18}$$

increases from $\pi(3\ell + k'/4)$ to $\pi(3\ell + k'/3 + D)$, hitting $D + 1$ distinct consecutive integer multiples of π (this is independent of the choice of k'). A short computation shows that if $D \geq |\ell|$, then the quantity given in (18) is strictly increasing on this interval. Thus, there are exactly $D + 1$ values of θ in the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ where the function

$$2\cos\left(\tfrac{k\theta}{2} - 2\pi m \cos\theta\right)$$

has absolute value 2, alternating between $+2$ and -2 as θ increases. In view of Lemma 3 and the intermediate value theorem, then, the real-valued function $e^{ik\theta/2} e^{-2\pi m\sin\theta} f_{k,m}(e^{i\theta})$ must have at least D distinct zeros as θ moves through the interval $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$. This accounts for all D nontrivial zeros of $f_{k,m}$.

It remains to prove Lemma 3. Changing variables $q \mapsto \tau$ in the formula of Lemma 2 and deforming the resulting contour by Cauchy's theorem gives that for $A > 1$,

$$f_{k,m}(z) = \int_{-\frac{1}{2}+iA}^{\frac{1}{2}+iA} \frac{\Delta(z)^\ell}{\Delta(\tau)^{1+\ell}} \frac{E_{k'}(z) E_{14-k'}(\tau)}{j(\tau) - j(z)}\, e^{-2\pi i m\tau} d\tau.$$

For brevity, we write

$$G(\tau, z) = \frac{\Delta(z)^\ell}{\Delta(\tau)^{1+\ell}} \frac{E_{k'}(z) E_{14-k'}(\tau)}{j(\tau) - j(z)}\, e^{-2\pi i m\tau},$$

so that

$$f_{k,m}(z) = \int_{-\frac{1}{2}+iA}^{\frac{1}{2}+iA} G(\tau, z) d\tau.$$

We now assume that $z = e^{i\theta}$ for some $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3})$, and move the contour of integration downward to a height A'. As we do so, each pole τ_0 of $G(\tau, z)$ in the region defined by

$$-\tfrac{1}{2} \leq \mathrm{Re}(\tau) < \tfrac{1}{2} \text{ and } A' < \mathrm{Im}(\tau) < A$$

will contribute a term $2\pi i \cdot \mathrm{Res}_{\tau=\tau_0} G(\tau, z)$ to the equation. The poles of $G(\tau, z)$ occur only when $\tau = z$ or when τ is equivalent to z under the action of Γ. In moving the contour, then, the first nonzero contributions occur at $\tau = z = e^{i\theta}$ and $\tau = -1/z = e^{i(\pi-\theta)}$, and these are the only poles for $\sqrt{3}/2 < A' < A$. The residues can be easily calculated using the alternative formula

$$G(\tau, z) = \frac{e^{-2\pi i m\tau}}{-2\pi i} \frac{\Delta^\ell(z)E_{k'}(z)}{\Delta^\ell(\tau)E_{k'}(\tau)} \frac{\frac{d}{d\tau}(j(\tau) - j(z))}{j(\tau) - j(z)}.$$

If $\sqrt{3}/2 < A' < \sin\theta$, the result is the equation

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} G(\tau, z)d\tau = f_{k,m}(z) - e^{-2\pi i m z} - z^{-k}e^{-2\pi i m(-1/z)}.$$

We replace z with $e^{i\theta}$ and multiply by $e^{ik\theta/2}e^{-2\pi m\sin\theta}$; simplifying, we find that

$$e^{ik\theta/2}e^{-2\pi m\sin\theta}f_{k,m}(e^{i\theta}) - 2\cos\left(\tfrac{k\theta}{2} - 2\pi m\cos\theta\right),$$

which is the quantity we are trying to bound, is equal to

$$e^{ik\theta/2}e^{-2\pi m\sin(\theta)} \int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} G(\tau, e^{i\theta})d\tau.$$

As A' decreases, the next nonzero contribution occurs when $\tau = \frac{-1}{z+1}$ or $\tau = \frac{z}{z+1}$. Since these points have real part $-1/2$ and $1/2$, respectively, we add a small circular arc to each of the vertical contours of integration in the usual way. The result is a contribution of

$$\frac{e^{-\pi i m}}{(2\cos(\theta/2))^k}\, e^{-\pi m(2\sin\theta - \tan(\theta/2))}$$

from this pole. However, if θ is close to $\pi/2$, the pole at $\frac{-z}{z-1}$ will be nearby. To avoid this, we choose A' so that the contribution from this pole appears only if θ is not close to $\pi/2$. Specifically, if $1.9 \le \theta < 2\pi/3$, we choose

$$A' = .65 < \mathrm{Im}(\tfrac{-1}{e^{i\theta}+1}),$$

so that the quantity we are bounding equals

$$\frac{e^{-\pi i m}}{(2\cos(\theta/2))^k} e^{-\pi m(2\sin\theta - \tan(\theta/2))} + e^{ik\theta/2}e^{-2\pi m\sin\theta} \int_{-1/2}^{1/2} G(x + .65i, e^{i\theta})dx.$$

Alternatively, if $\pi/2 < \theta < 1.9$, we choose

$$A' = .75 > \mathrm{Im}\left(\tfrac{-1}{e^{i\theta}+1}\right),$$

and the quantity we are bounding will equal

$$e^{ik\theta/2}e^{-2\pi m\sin\theta} \int_{-1/2}^{1/2} G(x + .75i, e^{i\theta})dx.$$

We deal with these cases separately.

In the first case, suppose that $1.9 \le \theta < 2\pi/3$. We assume that $m \ge |\ell| - \ell$, and deal first with the case where $\ell \ge 0$. Applying absolute values, we find that

$$\left|e^{ik\theta/2}e^{-2\pi m\sin\theta}f_{k,m}(e^{i\theta}) - 2\cos\left(\tfrac{k\theta}{2} - 2\pi m\cos(\theta)\right)\right|$$

is bounded above by

$$\frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} + e^{-2\pi m\sin\theta} \int_{-1/2}^{1/2} \left|G(x + .65i, e^{i\theta})\right| dx.$$

Looking at the first term,

$$1 < 2\cos(\theta/2) < \sqrt{2}$$

for $\theta \in [1.9, 2\pi/3)$, and
$$-m(2\sin\theta - \tan(\theta/2)) \leq 0$$
for these θ. We can thus bound the first term by 1, and need only show that
$$e^{-2\pi m \sin\theta} \int_{-1/2}^{1/2} \left|G(x + .65i, e^{i\theta})\right| dx < 0.985.$$

To do this, we first note that the length of the contour of integration is 1, so we have
$$e^{-2\pi m \sin\theta} \int_{-1/2}^{1/2} \left|G(x + .65i, e^{i\theta})\right| dx \leq \max_{|x|\leq\frac{1}{2}} e^{-2\pi m \sin\theta} \left|G(x + .65i, e^{i\theta})\right|.$$
Expanding G, this becomes
$$\max_{|x|\leq\frac{1}{2}} e^{-2\pi m(\sin\theta - .65)} \left|\frac{\Delta(e^{i\theta})}{\Delta(x + .65i)}\right|^{\ell} \left|\frac{E_{k'}(e^{i\theta})E_{14-k'}(x + .65i)}{\Delta(x + .65i)(j(x + .65i) - j(e^{i\theta}))}\right|.$$
To eliminate the dependence on ℓ and m, we note that for all $|x| \leq 1/2$ and $\theta \in [1.9, 2\pi/3)$,
$$\left|\frac{\Delta(e^{i\theta})}{\Delta(x + .65i)}\right| \leq 1,$$
so that
$$e^{-2\pi m(\sin\theta - .65)} \left|\frac{\Delta(e^{i\theta})}{\Delta(x + .65i)}\right|^{\ell} \leq \left|\frac{\Delta(e^{i\theta})}{\Delta(x + .65i)}\right|$$
for all x and θ in the appropriate intervals. (If $\ell = 0$, then either $m > 0$ and the exponential term is smaller than the ratio of the Δ terms, or else $m = 0 = D$ and there are no zeros to find.) Thus, we need only show that
$$\max_{|x|\leq\frac{1}{2}} \left|\frac{\Delta(e^{i\theta})}{\Delta(x + .65i)}\right| \left|\frac{E_{k'}(e^{i\theta})E_{14-k'}(x + .65i)}{\Delta(x + .65i)(j(x + .65i) - j(e^{i\theta}))}\right| < 0.985.$$
Close examination of this quantity for all six choices of k' shows that this is indeed the case. This proves Lemma 3 and hence Theorem 1 for the case $m, \ell \geq 0$.

Remark. For most choices of k', this quantity is closer to 0 than to 1. However, taking $k' = 0$ and looking at values of x near 0 and values of θ near $2\pi/3$ shows that replacing the integral with $\max_{|x|\leq .5}$ does not leave much margin for error in proving this quantity to be less than 1. This sensitivity prevents us from replacing the quotient of the Δ terms by 1, and factors into our choice of A' to be .65.

Now suppose that $\ell = -n$, for some integer $n \geq 1$, and that $m \geq 2n$. The first term becomes
$$(2\cos(\theta/2))^{12n-k'} e^{-2\pi m(\sin\theta - \tan(\theta/2))}.$$
Because $m \geq 2n$, this is again bounded by 1 for $\theta \in [1.9, 2\pi/3]$.

Working as before, we find that we need to bound
$$\max_{|x|\leq\frac{1}{2}} e^{-2\pi m(\sin\theta - .65)} \left|\frac{\Delta(x + .65i)}{\Delta(e^{i\theta})}\right|^{n} \left|\frac{E_{k'}(e^{i\theta})E_{14-k'}(x + .65i)}{\Delta(x + .65i)(j(x + .65i) - j(e^{i\theta}))}\right|.$$
Since $m \geq 2n$, this is less than or equal to
$$\max_{|x|\leq\frac{1}{2}} \left|e^{-4\pi(\sin\theta - .65)} \frac{\Delta(x + .65i)}{\Delta(e^{i\theta})}\right|^{n} \left|\frac{E_{k'}(e^{i\theta})E_{14-k'}(x + .65i)}{\Delta(x + .65i)(j(x + .65i) - j(e^{i\theta}))}\right|.$$

But since

$$\left| e^{-4\pi(\sin\theta - .65)} \frac{\Delta(x+.65i)}{\Delta(e^{i\theta})} \right| < 1$$

for all $|x| \le 1/2$ and $\theta \in [1.9, 2\pi/3)$, we need only bound

$$\max_{|x|\le\frac{1}{2}} \left| e^{-4\pi(\sin\theta - .65)} \frac{\Delta(x+.65i)}{\Delta(e^{i\theta})} \right| \left| \frac{E_{k'}(e^{i\theta})E_{14-k'}(x+.65i)}{\Delta(x+.65i)(j(x+.65i)-j(e^{i\theta}))} \right|$$

for $\theta \in [1.9, 2\pi/3)$, and, again, for every choice of k' this is less than .985. This completes the proof of the first case.

A similar calculation shows that if $\pi/2 < \theta < 1.9$, then

$$\left| e^{ik\theta/2} e^{-2\pi m \sin\theta} \int_{-1/2}^{1/2} G(x+.75i, e^{i\theta}) dx \right| < 1.985,$$

and this finishes the proof of Theorem 1.

6. Concluding remarks on the zeros of $f_{k,m}$

It is clear from (6) and the well-known mapping properties of j that $f_{k,m}$ has all of its zeros in $\mathcal{F}$ on the unit circle if and only if the Faber polynomial $F_{k,D}$ has all of its zeros in the interval $[0, 1728]$. In the case $k' = 0, D = 1$, we directly compute

$$F_{12\ell,1}(x) = x - (744 - 24\ell).$$

It is obvious that if $24\ell > 744$ or if $984 < -24\ell$, then the root of this linear polynomial is not in $[0, 1728]$, and so $f_{12\ell,1-\ell}$ will have a zero in $\mathcal{F}$ off the unit circle. Similar computations can be carried out for $D = 2$ or $D = 3$, providing further examples. On the other hand, a computation shows that for the following weights $k = 12\ell + k'$, all basis elements $f_{k,m}$ have all of their zeros in $\mathcal{F}$ on the unit circle.

$k' = 0$	$\ell \in [-41, 10]$
$k' = 4$	$\ell \in [-31, 23]$
$k' = 6$	$\ell \in [-62, 10]$
$k' = 8$	$\ell \in [-21, 36]$
$k' = 10$	$\ell \in [-50, 20]$
$k' = 14$	$\ell \in [-38, 30]$

As we mentioned, when $k > 0$ the forms $f_{k,-m}$ for $1 \le m \le \ell$ are cusp forms. It is interesting to compare our examples with the results of Rankin [9] and Gun [6], which give lower bounds for the number of zeros of certain linear combinations of cuspidal Poincaré series $P_{k,m}$ that are on the arc $\{e^{i\theta} : \theta \in (\frac{\pi}{2}, \frac{2\pi}{3})\}$. In a different direction, we remark that the zeros of Hecke eigenforms of weight k are expected to become equidistributed in $\mathcal{F}$ with respect to hyperbolic measure as $k \to \infty$ (see [11] for precise statements).

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UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
E-mail address: wdduke@ucla.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
E-mail address: jenkins@math.ucla.edu