RIESZ MEANS OF CERTAIN ARITHMETIC FUNCTIONS

WILLIAM DUKE AND HA NAM NGUYEN

Abstract. We give examples of completely multiplicative arithmetic functions that assume only the values ± 1 and that have bounded first Cesàro means. The method of proof also yields some interesting identities involving special values of Dirichlet L-functions. In particular, we present some new class number formulas for quadratic fields.

1. Introduction

An arithmetic function $a : \mathbb{Z}^+ \to \{-1, 1\}$ that is completely multiplicative is tantamount to the assignment of ± 1 to each prime. The value at any integer is then determined by its unique factorization into primes. For example, if we assign -1 to each prime we get the Liouville function

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity.

A conjecture of Erdős (Problem #9 of [5]) proven a few years ago by Tao [13] implies that for *any* such arithmetic function $a(n)$ the partial sums

$$s(n) = \sum_{1 \le k \le n} a(k) \tag{1}$$

are unbounded. A potential strengthening of this result would be the statement that the (first) Cesàro mean

$$c(n) = \tfrac{1}{n} \sum_{1 \le m \le n} \sum_{1 \le k \le m} a(k) \tag{2}$$

is also unbounded. This is true for the Liouville function. We will show in this paper that there exist infinitely many completely multiplicative ± 1 arithmetic functions for which the Cesàro mean is bounded, hence that such a strengthening does not hold in general.

It is easy to describe these functions explicitly. Let $q > 2$ be a prime and $\left(\frac{\cdot}{q}\right)$, the Legendre symbol. Define $a_q(q) = -1$ while for $p \neq q$ set $a_q(p) = \left(\frac{p}{q}\right)$ and extend $a_q(n)$ to be completely multiplicative. Clearly $a_q(n) \in \{\pm 1\}$ for all n. We see directly that $s(n)$ as defined in (1) is unbounded, since for any positive integer m

$$s(1 + q^2 + q^4 + \cdots + q^{2m}) = m.$$

Date: November 21, 2019.

1991 *Mathematics Subject Classification.* 11N60, 11M06.

Key words and phrases. Dirichlet L-functions, arithmetic functions, class number formulas, q-additive.

Supported by NSF grant DMS 1701638 and the Simons Foundation.

Theorem 1. *For any prime $q > 2$ there exists a constant $A_q > 0$ such that*

$$\Big|\tfrac{1}{n}\sum_{1\leq m\leq n}\sum_{1\leq k\leq m} a_q(k)\Big| \leq A_q$$

for all $n \in \mathbb{Z}^+$.

Suppose now that q is any positive integer with $q > 1$ and that $\psi(n)$ is a periodic arithmetic function with period q. Say that ψ is *admissible* for q if either $\psi = \chi$ where χ is a primitive Dirichlet character mod q or $\psi = \tilde{\chi}$, where $\tilde{\chi}$ is defined by setting $\tilde{\chi}(n) = 1$ for $q \nmid n$ while otherwise $\tilde{\chi}(n) = 1 - q$. The method of proof of Theorem 1 yields some remarkable identities for special values of the L-function

$$L(s, \psi) = \sum_{n\geq 1} \psi(n) n^{-s}$$

when ψ is admissible for q. This series is absolutely convergent for $\mathrm{Re}(s) > 1$ and has an analytic continuation in s to an entire function of order one (see below in §3). Note that

$$L(s, \tilde{\chi}) = (1 - q^{1-s})\zeta(s). \tag{3}$$

Let $(x)_k = x(x+1)\cdots(x+k-1)$ be the Pochhammer symbol and $H_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ be the k^{th} harmonic number.

Theorem 2. *Let ψ be admissible for $q > 1$. For any $k \in \mathbb{Z}^+$ and $\alpha = \frac{\pi i}{\log q}$ we have*

$$\sum_{\substack{n\in\mathbb{Z}\\ n\text{ odd}}} \frac{k!\, L(n\alpha, \psi)}{(n\alpha)_{k+1}} = -\log q \sum_{0\leq j\leq k-1} (-1)^j \tbinom{k}{j} \tfrac{L(-j,\psi)}{q^j+1} \quad \text{and}$$

$$\sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ n\text{ even}}} \frac{k!\, L(n\alpha, \psi)}{(n\alpha)_{k+1}} = -L'(0,\psi) - \left(\tfrac{1}{2}\log q - H_k\right) L(0,\psi) - \log q \sum_{1\leq j\leq k-1} (-1)^{j+1} \tbinom{k}{j} \tfrac{L(-j,\psi)}{q^j-1},$$

where the infinite sums are absolutely convergent.

These identities yield some new class number formulas for quadratic fields. Suppose that $D \neq 1$ is a fundamental discriminant and

$$\mathbb{K} = \mathbb{Q}(\sqrt{D}).$$

Let $\sigma : \mathbb{K} \to \mathbb{K}$ generate the Galois group of $\mathbb{K}/\mathbb{Q}$ and for $\beta \in \mathbb{K}$ let $N(\beta) = \beta\beta^\sigma$. Let Cl_D^+ be the group of (narrow) fractional ideal classes in $\mathbb{K}$. Thus two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in the same class if there is $\beta \in \mathbb{K}$ with $N(\beta) > 0$ so that $\mathfrak{a} = (\beta)\mathfrak{b}$. Let

$$h(D) = \#\mathrm{Cl}_D^+$$

be the class number and $w = w_D$ be the number of roots of unity in $\mathbb{K}$. Thus $w = 2$ unless $D = -3, -4$ when $w = 6, 4$, respectively. If $D > 1$ let ϵ_D be the smallest unit of norm 1 in the ring of integers of $\mathbb{K}$ with $\epsilon_D > 1$. Finally, let $\chi_D(\cdot)$ be the Kronecker symbol, which is a primitive Dirichlet character mod $|D|$.

The next corollary follows from Theorem 2 with $k = 1$ together with standard class number formulas (see e.g. [4]).

Corollary 1. *For a fundamental discriminant $D \neq 1$ let $\alpha = \frac{\pi i}{\log|D|}$. Then*

$$w_D^{-1} h(D) \log|D| = -\sum_{\substack{n\in\mathbb{Z} \\ n \text{ odd}}} \frac{L(n\alpha,\chi_D)}{(n\alpha)(n\alpha+1)} \quad \textit{when } D<0 \textit{ and}$$

$$\tfrac{1}{2} h(D) \log \epsilon_D = -\sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ n \text{ even}}} \frac{L(n\alpha,\chi_D)}{(n\alpha)(n\alpha+1)} \quad \textit{when } D>0.$$

The following consequence of Theorem 2 and (13) below is of interest in connection with the Chowla-Selberg formula.

Corollary 2. *For $D<0$*

$$\sum_{1\le n\le |D|} \chi_D(n) \log\Gamma(\tfrac{n}{|D|}) = -\sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ n \text{ even}}} \frac{L(n\alpha,\chi_D)}{(n\alpha)(n\alpha+1)} + (\tfrac{1}{2}\log|D|+1)\, L(0,\chi_D).$$

2. Exact formulas for Riesz means

Theorems 1 and 2 follow from formulas for certain Riesz means. For any sequence $a(n)$ and any non-negative integer k define the k^{th} Riesz **(arithmetic)** mean of $a(n)$ by

$$s_k(x) = \sum_{n\le x} (1-\tfrac{n}{x})^k a(n)$$

(see [7, §5.16]). When $k=1$ this is essentially the first Cesàro mean (2) in that

$$c(n) = \tfrac{n+1}{n}\, s_1(n+1). \tag{4}$$

We give an explicit formula for $s_k(n)$ when $a(n) = a_\psi^\pm(n)$ is defined through the formula

$$a_\psi^\pm(n) = \sum_{q^m | n} (\pm 1)^m \psi(\tfrac{n}{q^m}), \tag{5}$$

where ψ is admissible for q and a choice of $\pm$ is made.

Proposition 1. *Let ψ be admissible for $q>1$. For a positive integer k and $x\ge 1$ we have*

$$\sum_{n\le x}(1-\tfrac{n}{x})^k a_\psi^-(n) = \tfrac{k!}{\log q} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ odd}}} \tfrac{L(n\alpha,\psi)}{(n\alpha)_{k+1}}\, x^{n\alpha} + \sum_{0\le j\le k-1} (-1)^j \tbinom{k}{j} \tfrac{L(-j,\psi)}{q^j+1}\, x^{-j} + O(x^{\frac{3}{4}-k}) \quad \textit{and} \tag{6}$$

$$\begin{aligned}\sum_{n\le x}(1-\tfrac{n}{x})^k a_\psi^+(n) &= \tfrac{1}{\log q}(\log x + \tfrac{1}{2}\log q - H_k)\, L(0,\psi) + \tfrac{1}{\log q} L'(0,\psi) \\ &+ \tfrac{k!}{\log q} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ n \text{ even}}} \tfrac{L(n\alpha,\psi)}{(n\alpha)_{k+1}}\, x^{n\alpha} + \sum_{1\le j\le k-1} (-1)^{j+1} \tbinom{k}{j} \tfrac{L(-j,\psi)}{q^j-1}\, x^{-j} + O(x^{\frac{3}{4}-k}),\end{aligned} \tag{7}$$

where the infinite series are absolutely convergent and $\alpha = \frac{\pi i}{\log q}$. When x is an integer these hold as identities without an error term. The $\log x$ term occurs in (7) if and only if either $\psi = \tilde{\chi}$ or $\psi = \chi$ and $\chi(-1) = -1$.

Theorem 1 is an immediate consequence of (6) in Proposition 1 when we take $q > 2$ prime, $\psi(\cdot) = \left(\frac{\cdot}{q}\right)$ and $k = 1$, since then $a_\psi^-(n) = a_q(n)$. We remark that another infinite set of examples is provided by $a_\psi^+(n)$ for this ψ, provided we assume that $p \equiv 1 \pmod 4$. This follows by the second formula of Proposition 1. The multiplicative ± 1 functions $a_\psi^+(n)$ were studied in [1], where their partial sums to n were expressed in terms of the digits in the base q expansion of n.

The exactness of the formulas of Proposition 1 when x is an integer is not important for the proof of Theorem 1, but is crucial for that of Theorem 2 and its corollaries. In fact Theorem 2 follows from Proposition 1 right away by taking $x = 1$.

Exact formulas of this type for arithmetic functions are unusual, but examples are well-known when $k = 1$. Note that by (4)

$$S(m) \stackrel{\text{def}}{=} \sum_{n \le m} (1 - \tfrac{n}{m}) a_\psi^+(n) = \tfrac{1}{m} \sum_{1 \le n \le m-1} s(n) \quad \text{where} \quad s(n) = \sum_{\ell \le n} a_\psi^+(\ell). \tag{8}$$

When $\psi = \tilde{\chi}$ we have that $s(n)$ gives the sum of the digits in the base q expansion of n. Actually, (7) with $k = 1$ and $\psi = \tilde{\chi}$ is equivalent to the well-known exact formula found by Trollope [14] and by Delange [3] for $S(m)$. In general, the partial sum $s(n)$ of $a_\psi^\pm(n)$ is an example of a q-additive function and, as in (8), a formula for the first Riesz mean of $a_\psi^\pm(n)$ amounts to a formula for the partial sums of $s(n)$. Exact formulas for the partial sums of many q-additive functions are known (see e.g. [9], [8]) but apparently consequences such as the corollaries to Theorem 2 have not been noticed. Also, formulas like those of Proposition 1 when $k > 1$ seem to be new.

We use the Mellin transform and standard analytic number theory for the proof of Proposition 1. This method was applied in [6] to several examples, including the Delange-Trollope formula, and does a good job of explaining the mechanism behind the exact formulas for the first Riesz means. The method is a bit more involved when $k > 1$.

3. Dirichlet L-functions

Throughout this section assume that $q > 1$ is fixed and that ψ is admissible for q. Note that for any such ψ

$$\sum_{1 \le n \le q} \psi(n) = 0. \tag{9}$$

We require some basic properties of the associated L-function

$$L(s, \psi) = \sum_{n \ge 1} \psi(n) n^{-s}. \tag{10}$$

This series converges absolutely and uniformly on compact subsets of $\{s \in \mathbb{C}; \mathrm{Re}(s) > 1\}$. The L-function has there the Euler product expansion

$$L(s, \psi) = \begin{cases} \prod_p (1 - \chi(p) p^{-s})^{-1}, & \psi = \chi \\ (1 - q^{1-s}) \prod_p (1 - p^{-s})^{-1}, & \psi = \tilde{\chi}. \end{cases}$$

A classical result is that $L(s,\psi)$ has an analytic continuation to an entire function in s of order one. In case $\psi=\chi$ set

$$\xi(s,\psi)=\left(\tfrac{\pi}{q}\right)^{-\frac{s+a_\chi}{2}}\Gamma(\tfrac{s+a_\chi}{2})L(s,\chi),$$

where $a_\chi\in\{0,1\}$ is such that $\chi(-1)=(-1)^{a_\chi}$. If $\psi=\tilde{\chi}$ set

$$\xi(s,\psi)=(1-q^s)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})L(s,\tilde{\chi}).$$

Then we have the functional equation

$$\xi(1-s,\psi)=\varepsilon_\psi\xi(s,\psi) \tag{11}$$

where ε_χ is a certain root of unity with $\varepsilon_{\tilde{\chi}}=1$. For proofs see [2]. We have the following immediate consequence of the functional equation (11) together with the well-known fact [2] that $L(1+it,\psi)\neq 0$ for all $t\in\mathbb{R}$.

Lemma 1. *For non-zero $t\in\mathbb{R}$ we have that $L(it,\psi)\neq 0$. For $j=0,1,2,\ldots$ we have that $L(-j,\psi)=0$ when $\psi=\chi$ if and only if $a_\chi+j$ is even, while $L(-j,\tilde{\chi})=0$ if and only if $j=-2,-4,\ldots$.*

The functional equation and the Phragmén-Lindelöf theorem imply the following explicit estimate [12].

Lemma 2. *For $0<\epsilon\leq\frac{1}{2}$ we have*

$$|L(\sigma+it,\psi)|\ll\begin{cases}(|t|+1)^{\frac{1}{2}-\sigma}, & \sigma<-\frac{1}{2}\\ \zeta(1+\epsilon)(|t|+1)^{\frac{1-\sigma+\epsilon}{2}}, & -\epsilon\leq\sigma\leq 1+\epsilon\end{cases}$$

where the implied constant depends only on q.

A useful tool to study these L-functions at non-positive integers is the Hurwitz zeta function, which is defined for $x>0$ and $\mathrm{Re}(s)>1$ by

$$\zeta(s,x)=\sum_{n\geq 0}(n+x)^{-s}.$$

See e.g. [10, §1.4] for the properties we quote below. For fixed $x>0$ it has an analytic continuation in s to an entire function except for a simple pole at $s=1$. For $j\in\mathbb{Z}^+$ we have

$$\zeta(1-j,x)=-\tfrac{1}{j}B_j(x),$$

where $B_j(x)$ is the Bernoulli polynomial. Also

$$\partial_s\zeta(x,0)=\log\left((2\pi)^{-\frac{1}{2}}\,\Gamma(x)\right).$$

Clearly

$$L(s,\psi)=q^{-s}\sum_{1\leq r\leq q}\psi(r)\zeta(s,\tfrac{r}{q})$$

so that we get the following lemma.

Lemma 3. *For integral $j \geq 1$ we have*

$$L(1-j,\psi) = -\tfrac{q^{j-1}}{j} \sum_{1\leq n\leq q} \psi(n) B_j(\tfrac{n}{q}), \tag{12}$$

where $B_j(x)$ is the Bernoulli polynomial. Also we have

$$L'(0,\psi) = -L(0,\psi)\log q + \sum_{1\leq n\leq q} \psi(n)\log\Gamma(\tfrac{n}{q}). \tag{13}$$

Note that we are using (9) to derive (13).

For convenience we record some properties of the Bernoulli polynomial $B_k(x)$ that we will need. A good reference is [10, §1.5.1]. For integral $k \geq 0$ the following hold:

$$B'_{k+1}(x) = (k+1)B_k(x) \tag{14}$$

$$B_k(x) - B_k(x-1) = k(x-1)^{k-1} \tag{15}$$

$$\sum_{0\leq j\leq k} \tbinom{k}{j} B_j(x) y^{k-j} = B_k(x+y). \tag{16}$$

4. Proof of Proposition 1

We will give details for (6) and simply indicate the small changes needed to prove (7). Using (5) it is easy to check that for $L(s,\psi)$ from (10)

$$\sum_{n\geq 1} a_\psi^-(n) n^{-s} = (1+q^{-s})^{-1} L(s,\psi). \tag{17}$$

We use the following summation formula. For $x \geq 1$ and $c > 1$ we have the absolutely convergent integral representation [11, p.142]

$$\tfrac{1}{k!}\sum_{n\leq x}(1-\tfrac{n}{x})^k a_q^-(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{L(s,\psi)x^s}{(1+q^{-s})(s)_{k+1}} ds,$$

where we have applied (17).

Next, the idea is to apply the standard method of shifting the contour to the left, in our case to the line $\mathrm{Re}(s) = \frac{3}{4} - k$. Note that it is valid to do this in view of Lemma 2 and the term $(s)_{k+1}$ in the denominator of the integrand. In the process we will pass over simple poles at zeros of $1+q^{-s}$, namely $s = \frac{\pi i n}{\log q}$ for odd $n \in \mathbb{Z}$. These are actual poles by Lemma 1. The residues at the poles are easily computed. This yields the asymptotic formula of (6), upon estimating

$$\frac{1}{2\pi i}\int_{\frac{3}{4}-k-i\infty}^{\frac{3}{4}-k+i\infty} \frac{L(s,\psi)(qx)^s}{(1+q^s)(s)_{k+1}} ds \tag{18}$$

by Lemma 2. The absolute convergence of the series is also guaranteed by Lemma 2.

The statement that the formula holds without the error term when x is an integer comes from the following lemma, after using

$$\frac{q^s}{1+q^s} = \sum_{m\geq 1} (-1)^{m+1} q^{ms}$$

in (18) and integrating term by term, this being easily justified.

Lemma 4. *For integers q, k, r with $q > 1$ and $k, r \geq 1$ and for χ as above*

$$\frac{1}{2\pi i}\int_{\frac{3}{4}-k-i\infty}^{\frac{3}{4}-k+i\infty} \frac{L(s,\psi)(qr)^s}{(s)_{k+1}} ds = 0.$$

Proof. As before, for $c > 1$ we have the absolutely convergent integral representation

$$\tfrac{1}{k!}\sum_{n\leq qr}(1-\tfrac{n}{qr})^k\psi(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{L(s,\psi)(qr)^s}{(s)_{k+1}}ds.$$

Shifting the contour to the line $\mathrm{Re}(s) = \frac{3}{4} - k$, a calculation of residues shows that the proof will be finished once we establish the following lemma. □

Lemma 5. *Assumptions as above.*

$$\sum_{1\leq n\leq qr}\psi(n)(1-\tfrac{n}{qr})^k = \sum_{1\leq j\leq k}(qr)^{1-j}(-1)^{j+1}\tbinom{k}{j-1}L(1-j,\psi). \tag{19}$$

Proof. Apply the binomial expansion and formula (12) to compute the coefficient of $\psi(n)$ for $1 \leq n \leq q$ on either side of (19), also using the q-periodicity of ψ. By (9) and (12) we reduce the proof to the polynomial identity in x:

$$\sum_{0\leq \ell\leq r-1}(x-r+\ell)^k = -\sum_{1\leq j\leq k}(-r)^{k-j+1}\tbinom{k}{j-1}\tfrac{1}{j}B_j(x) + C_{k,r}, \tag{20}$$

where $C_{k,r}$ is a constant. Then (19) follows by taking $x = \frac{n}{q}$.

To establish (20), we need only show that the derivatives of both sides with respect to x coincide. Thus we must show that

$$k\sum_{0\leq \ell\leq r-1}(x-r+\ell)^{k-1} = -\sum_{0\leq j\leq k-1}(-r)^{k-j}\tbinom{k}{j}B_j(x), \tag{21}$$

where we have applied (14) and then shifted indices in j. Now apply the identity (16) to the right hand side of (21) to reduce the needed identity to

$$k\sum_{0\leq \ell\leq r-1}(x-r+\ell)^{k-1} = B_k(x) - B_k(x-r). \tag{22}$$

Then (22) follows by applying the identity (15) r times to the right hand side of (22). This finishes the proof of (19).

Although we do not need it, the constant $C_{k,r}$ can be evaluated as $C_{k,r} = -\frac{(-r)^{k+1}}{k+1}$. □

This completes the proof of (6) in Proposition 1. The proof of (7) is similar except that we must account for a double pole at $s = 0$ since

$$\sum_{n\geq 1}a_\psi^+(n)n^{-s} = (1-q^{-s})^{-1}L(s,\psi).$$

The final statement of Proposition 1 follows by Lemma 1. □

Remark: It can be shown that the union of all ψ that are admissible for some divisor $q' > 1$ of a fixed $q > 1$ forms a $\mathbb{C}$-basis for the space of all q-periodic arithmetic functions $\psi(n)$ that satisfy

$$\sum_{1 \le n \le q} \psi(n) = 0.$$

It is therefore possible to extend much of our analysis to such arithmetic functions.

References

[1] Borwein, P.; Choi, S. K. K.; Coons, M., *Completely multiplicative functions taking values in* $\{-1, 1\}$, Trans. Amer. Math. Soc. 362 (2010), no. 12, 6279–6291.

[2] Davenport, H., *Multiplicative number theory*, Third edition. Revised and with a preface by Hugh L. Montgomery. Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000. xiv+177 pp.

[3] Delange, H., *Sur la fonction sommatoire de la fonction "somme des chiffres"* , Enseign. Math. (2) 21 (1975), no. 1, 31–47.

[4] Duke, W.; Imamoḡlu, Ö.; Tóth, Á., *Kronecker's first limit formula, revisited*, Res. Math. Sci. 5 (2018), no. 2, Paper No. 20, 21 pp.

[5] Erdős, P., *Some unsolved problems*, Michigan Math. J. 4 (1957), 291–300.

[6] Flajolet, P.; Grabner, P.; Kirschenhofer, P.; Prodinger, H.; Tichy, R. F., *Mellin transforms and asymptotics: digital sums*, Theoret. Comput. Sci. 123 (1994), no. 2, 291–314.

[7] Hardy, G. H. *Divergent Series*, Oxford, at the Clarendon Press, (1949). xvi+396 pp.

[8] Kamiya, Y.; Murata, L., *Relations among arithmetical functions, automatic sequences, and sum of digits functions induced by certain Gray codes*, J. Théor. Nombres Bordeaux 24 (2012), no. 2, 307–337.

[9] Murata, L.; Mauclaire, J.-L., *An explicit formula for the average of some q-additive functions*, Prospects of mathematical science (Tokyo, 1986), 141–156, World Sci. Publishing, Singapore, 1988.

[10] Magnus, W.; Oberhettinger, F.; Soni, R. P., *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52 Springer-Verlag New York, Inc., New York 1966 viii+508 pp.

[11] Montgomery, H. L.; Vaughan, R. C., *Multiplicative number theory. I. Classical theory.* Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, (2007). xviii+552 pp.

[12] Rademacher, H., *On the Phragmén-Lindelöf theorem and some applications*, Math. Z 72 (1959/1960), 192–204.

[13] Tao, T., *The Erdős discrepancy problem*, Discrete Anal. (2016), Paper No. 1, 29 pp.

[14] Trollope, J. R. , *An explicit expression for binary digital sums*, Math. Mag. 41 (1968) 21–25.

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA
Email address: wdduke@ucla.edu

LACC Mathematics Department 855 N. Vermont Ave. Los Angeles, CA. 90029, USA
Email address: nguyenhn@laccd.edu