6. Representations by the Determinant and Mean Values of L-Functions

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Introduction and Statement of Results

The determinant equation

$$\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = \Delta \tag{1}$$

appears in the context of various problems in analytic number theory. One would like to have, for a fixed integer $\Delta \neq 0$, a good asymptotic formula for the number of solutions of (1) as the entries vary over general sequences of integers.

To get a hold on the problem we specify the sizes:

$$\begin{aligned} M_1 < m_1 \leq 2M_1, \quad N_1 < n_1 \leq 2N_1, \\ M_2 < m_2 \leq 2M_2, \quad N_2 < n_2 \leq 2N_2. \end{aligned} \tag{2}$$

If we make no further restrictions on the entries, this is a problem from the spectral theory of $\mathrm{GL}_2(\mathbb{Z})$ automorphic forms. We should like to be able to treat the case where all four entries are from arbitrary sequences or, what amounts to the same thing, to evaluate the weighted sum

$$S_\Delta(M,N) = \sum\sum\sum\sum_{m_1 n_2 - m_2 n_1 = \Delta} f(m_1) g(m_2) a_{n_1} b_{n_2} \tag{3}$$

with f, g, a, b general functions supported in the box (2). In this generality the problem seems quisquose and would, just for example, have inopinate implications to the twin prime problem.

In this paper we are able to treat the case where the lower row has general weights a_{n_1}, b_{n_2}. For the upper row we require f, g to be smooth functions supported on $[M_1, 2M_1]$, $[M_2, 2M_2]$ with derivatives satisfying

$$f^{(j)} \ll \eta^j M_1^{-j}, \qquad g^{(j)} \ll \eta^j M_2^{-j}, \tag{4}$$

for all $j \geq 0$, some $\eta \geq 1$, and with the implied constant depending on j.

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The most interesting case is when the entries in the lower row are larger than the smoothed ones in the upper row. In the opposite case an analysis via known Fourier techniques directly gives the asymptotic formula. Here we prove:

Theorem 1. *Let $\Delta \neq 0$, a_{n_1}, b_{n_2} be complex numbers for*

$$N_1 < n_1 \leq 2N_1, \quad N_2 < n_2 \leq 2N_2$$

and f, g smooth functions supported on $[M_1, 2M_1]$, $[M_2, 2M_2]$ and with derivatives satisfying (4). *Then*

$$\begin{aligned} S_\Delta(M,N) = \sum\sum_{(n_1,n_2)|\Delta} \frac{(n_1,n_2)}{n_1 n_2} a_{n_1} b_{n_2} \int f\Big(\frac{x+\Delta}{n_2}\Big) g\Big(\frac{x}{n_1}\Big)\, dx \\ + O\bigg(\eta^{\frac{19}{8}} \|a\|\,\|b\| \Big(\frac{M_1 N_2}{M_2 N_1} + \frac{M_2 N_1}{M_1 N_2}\Big)^{\frac{19}{8}} \\ \times (N_1 N_2)^{\frac{3}{8}} (N_1 + N_2)^{\frac{11}{48}} (M_1 M_2 N_1 N_2)^{\varepsilon}\bigg), \end{aligned} \tag{5}$$

where the implied constant depends only on ε, and $\|\ \ \|$ denotes the L_2 norm.

This result should be compared with the trivial bound

$$S_\Delta(M,N) \ll \|a\|\,\|b\| (M_1 M_2)^{\frac{1}{2}} (M_1 M_2 N_1 N_2)^{\varepsilon}$$

that follows from Cauchy's inequality.

In the special case $\eta = 1$, $M_1 = M_2 = M$, $N_1 = N_2 = N$, the theorem simplifies to

$$\begin{aligned} S_\Delta(M,N) = \sum\sum_{(n_1,n_2)|\Delta} \frac{(n_1,n_2)}{n_1 n_2} a_{n_1} b_{n_2} \int f\Big(\frac{x+\Delta}{n_2}\Big) g\Big(\frac{x}{n_1}\Big)\, dx \\ + O\Big(\|a\|\,\|b\| N^{\frac{47}{48}} (MN)^{\varepsilon}\Big). \end{aligned}$$

Here the error term is smaller than the above trivial bound provided that $M > N^{47/48}$.

The main tool in the proof of Theorem 1 is a new estimate [1] for bilinear forms of Kloosterman fractions. The particular variant needed, which follows quickly from Theorem 2 of that paper, we record here as

Proposition. *Let α_m for $M < m \leq 2M$ and β_n for $N < n \leq 2N$ be arbitrary complex numbers, and $k \neq 0$ an integer. Then, for any $\varepsilon > 0$, we have*

$$\sum_{(m,n)=1}\sum \alpha_m \beta_n \, e\left(k\frac{\overline{m}}{n} + \frac{X}{mn}\right)$$
$$\ll \|\alpha\| \, \|\beta\| \left(1 + \frac{|X|}{MN}\right)(|k| + MN)^{\frac{3}{8}}(M+N)^{\frac{11}{48}+\varepsilon}$$

where $e(t) = e^{2\pi i t}$, $\overline{m}m \equiv 1 \pmod n$, and the implied constant depends only on ε.

We apply Theorem 1 to obtain a mean-value theorem for character sums and L-functions. The classical mean-value theorem for Dirichlet polynomials asserts that

$$\sum_{\chi \pmod q} \left|\sum_{n \leq N} \lambda_n \chi(n)\right|^2 = (\varphi(q) + O(N)) \sum_{\substack{n \leq N \\ (n,q)=1}} |\lambda_n|^2,$$

where the implied constant is absolute; cf. Theorem 6.2 in [3]. The result is best possible when $N < q$, in which case the error term is superfluous. In the case $N > q$ the result was improved [2] for sequences of triple convolution type $\lambda = \alpha * \beta * f$ with smooth f and with α, β quite general but having support specially located. Theorem 1 allows us to remove this last restriction and treat the convolution $\lambda = a * f$ for a general sequence a.

More precisely, we consider an arbitrary sequence a_n of complex numbers defined for $N < n \leq 2N$, $(n,q) = 1$, and a smooth function $f(m)$ supported on $M < m \leq 2M$ and having derivatives satisfying (4).

Theorem 2. *We have*

$$\sum_{\chi \neq \chi_0 \pmod q} \left|\sum_m f(m)\chi(m)\right|^2 \left|\sum_n a_n \chi(n)\right|^2 \ll \eta^{\frac{19}{8}} q^{\varepsilon}\left(q + N^{\frac{95}{48}}\right) M \|a\|^2.$$

Here the point is that we obtain a non-trivial bound with N as large as $q^{48/95}$, and $\frac{48}{95} > \frac{1}{2}$.

Since Dirichlet L-functions may be well approximated by sums of the type occurring in Theorem 2, we may deduce a mean-value theorem for the former.

Theorem 3. *For* $\operatorname{Re} s = \frac{1}{2}$ *we have*

$$\sum_{\chi \pmod q} |L(s,\chi)|^2 \left|\sum_n a_n \chi(n)\right|^2 \ll |s|^{\frac{19}{8}} q^{\varepsilon}\left(q + N^{\frac{95}{48}}\right)\|a\|^2.$$

The remaining terms give a contribution

$$S^* = \varphi(q) \sum_{l|q} \mu(l) \sum_{r \neq 0} \sum\sum\sum\sum_{m_1 n_2 - m_2 n_1 = qr/l} f(lm_1)\overline{f}(lm_2)\overline{a}_{n_1} a_{n_2}$$

$$= \varphi(q) \sum_{l|q} \mu(l) \sum_{1 \leq |r| \leq R} \Big\{ \sum\sum_{(n_1,n_2)|r} \frac{(n_1,n_2)}{n_1 n_2} \overline{a}_{n_1} a_{n_2} \int f\Big(\frac{lx+qr}{n_2}\Big)\overline{f}\Big(\frac{lx}{n_1}\Big)\,dx$$

$$+ O\Big(\eta^{\frac{19}{8}} \|a\|^2 N^{\frac{47}{48}} (MN)^{\varepsilon}\Big)\Big\} \qquad (6)$$

by Theorem 1, where we have put $R = 3MN/lq$. Here the contribution to S^* from the error term is

$$\ll \eta^{\frac{19}{8}} \|a\|^2 M N^{\frac{95}{48}+\varepsilon}.$$

For the main term we write $d = (n_1, n_2)$, $r = ds$ and sum first over s getting by Poisson summation

$$\sum_{s \neq 0} f\Big(\frac{lx + dqs}{n_2}\Big) = \int f\Big(\frac{lx + dqy}{n_2}\Big)\,dy + O(\eta).$$

Hence the contribution to S^* from the main term in (6) is

$$\varphi(q) \sum_{l|q} \mu(l) \sum_d \sum\sum_{(n_1,n_2)=1} \frac{\overline{a}_{dn_1} a_{dn_2}}{n_1 n_2} \iint f\Big(\frac{lx+qy}{n_2}\Big)\overline{f}\Big(\frac{lx}{n_1}\Big)\,dx\,dy$$

$$+ O\big(\eta \|a\|^2 q M N^{\varepsilon}\big)$$

$$= \Big(\frac{\varphi(q)}{q}\Big)^2 |\hat{f}(0)|^2 \Big(\sum_n a_n\Big)^2 + O\big(\eta \|a\|^2 q M N^{\varepsilon}\big).$$

Combining the above results, we conclude that

$$S = \Big(\frac{\varphi(q)}{q}\Big)^2 |\hat{f}(0)|^2 \Big(\sum_n a_n\Big)^2 + O\Big(\eta^{\frac{19}{8}} \|a\|^2 M \big(q + N^{\frac{95}{48}}\big)(MN)^{\varepsilon}\Big).$$

The contribution of the principal character is

$$S(\chi_0) = \Big|\sum_{(m,q)=1} f(m)\Big|^2 \Big(\sum_n a_n\Big)^2$$

and

$$\sum_{(m,q)=1} f(m) = \frac{\varphi(q)}{q} \hat{f}(0) + O\big(\eta \tau(q)\big).$$

Subtracting this contribution, we conclude the proof of Theorem 2.

Proof of Theorem 3

To prove Theorem 3 we first note that the contribution from χ_0 is bounded by

$$\ll \left(|s|\tau(q)\right)^2 N\|a\|^2,$$

which is admissible. For the non-principal characters we wish to apply Theorem 2. We approximate

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

with the aid of a smooth partition of unity getting sums of the type

$$M^{-\frac{1}{2}} \sum_m f(m)\chi(m),$$

where f satisfies (4) with $\eta = |s|$ and where $M < q^{10}$. For $M \geq q^{10}$ a trivial bound suffices. The number of partial sums is thus $\ll \log q$ and an application of Theorem 2 to each yields Theorem 3.

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References

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