Contemporary Mathematics
Volume 143, 1993

A RELATION BETWEEN CUBIC EXPONENTIAL AND KLOOSTERMAN SUMS

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Dedicated to the memory of E. Grosswald

Cubic exponential sums of the type

$$\sum_{x \,(\mathrm{mod}\, q)} \exp\left(2\pi i \frac{ax^3+bx}{q}\right) \tag{1}$$

occur in numerous problems of additive number theory. Weil's estimate for these sums is basic in their application. R. Livné [L1] has conjectured that these sums satisfy the Sato-Tate distribution with respect to prime q provided that $ab \neq 0$. He established in [L2] Birch's conjecture that they satisfy the Sato-Tate distribution with respect to the parameters $a, b\,(\mathrm{mod}\, q), a \not\equiv 0$, as q tends to infinity over primes. Livné's conjecture is perhaps out of the range of current methods. However, one should be able to estimate sums of sums of type (1) over all moduli q. A similar situation is well known for Kloosterman sums in which case the spectral theory of automorphic forms yields good estimates (see [I] and the references there).

In fact, we can show that the cubic sum (1) *is* a Kloosterman sum with cubic character. This result should provide a spectral resolution of sums over q of sums (1) in terms of automorphic forms on the three-fold cover of $GL(2)$. The special case $b = 0$ is closely related to Kummer's problem about the distribution of cubic Gauss sums. Here the spectral theory (the Eisenstein part) has already yielded important results (see [P]). The identity we give in this

1991 *Mathematics Subject Classification.* Primary 11L05; Secondary 11T23.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Research supported by NSF Grant DMS–8902992. First named author also supported in part by a Henry Rutgers Research Fellowship.

note is analogous to Nicholson's formula for the Airy integral which states that (see [W, p.190.]):

$$\int_0^\infty \cos\left(t^3+ty\right)dt = \frac{y^{1/2}}{3}K_{1/3}\left(2\,(y/3)^{3/2}\right), \quad y>0.$$

Here $K_\nu(z)$ is the usual Bessel function which has the following integral representation (see e.g. [W, p.183.]):

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-z^2/(4t)}t^{-\nu}\frac{dt}{t}, \quad z>0\,.$$

Thus Nicholson's formula can be written as

$$\int_{-\infty}^\infty e^{i\,(ax^3+x)}dx = 3^{-1/2}\int_0^\infty (x/a)^{1/3}\,e^{-x-(3^3ax)^{-1}}\frac{dx}{x}, \quad a>0\,.$$

We give the finite field analogue of this identity.

Theorem. *Let $F=GF(q)$, $q=p^r$, with $q\equiv 1 \pmod 3$ and $e(x)=e(\operatorname{tr}x/p)$. Then for ψ a multiplicative character of order three and $a\in F^*=F-\{0\}$ we have*

$$\sum_{x\in F} e(a\,x^3+x) = \sum_{x\in F^*}\psi(x\,\bar a)\,e(x-\overline{3^3ax}) \tag{2}$$

where $\overline{x}=x^{-1}$ for $x\in F^*$.

To prove this, let for $a\in F^*$

$$B(a)=\sum_{x\in F} e(a\,x^3+x)$$

and compute for multiplicative characters χ the Fourier transform

$$\begin{aligned}\hat B(\chi) &= \sum_{a\in F}\bar\chi(a)\,B(a)\\ &= \sum_a\sum_x e(ax^3+x)\,\bar\chi(a)\\ &= \sum_x\sum_a \bar\chi(a)\,e(ax^3)e(x)\\ &= (q-1)\,\delta(\chi)+\tau(\bar\chi)\sum_x \chi^3(x)\,e(x)\,,\end{aligned}$$

where $\tau(\chi) = \sum_{x\in F} \chi(x)e(x)$ and

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\hat{B}(\chi) = (q-1)\,\delta(\chi) + \tau(\bar{\chi})\tau(\chi^3)\,.$$

Now the triplication formula for Gauss sums states that for any χ

$$\tau(\chi^3) = \frac{1}{q}\,\chi^3(3)\tau(\chi)\tau(\chi\psi)\tau(\chi\psi^2)\,,$$

where as above ψ has order three. This is a special case of the Davenport-Hasse relation which was first obtained in [D–H]. However, in case $q = p$ the triplication formula was given an elementary proof in [G–S]. We refer to the discussion there. Thus we have

$$\begin{aligned} \hat{B}(\chi) &= (q-1)\delta(\chi) + \frac{1}{q}\,\chi^3(3)\tau(\bar{\chi})\tau(\chi)\tau(\chi\psi)\tau(\chi\psi^2) \\ &= \chi(-3^3)\tau(\chi\psi)\tau(\chi\psi^2)\,. \end{aligned}$$

after using the classical relation

$$\tau(\chi)\tau(\bar{\chi}) = \chi(-1)q - \delta(\chi)(q-1).$$

Hence

$$\begin{aligned} B(a) &= \frac{1}{q-1}\sum_{\chi}\chi(a)\hat{B}(a) = \frac{1}{q-1}\sum_{\chi}\chi(-3^3a)\tau(\chi\psi)\tau(\chi\psi^2) \\ &= \frac{\bar{\psi}(-3^3a)}{q-1}\sum_{\chi}\chi(-3^3a)\tau(\chi)\tau(\chi\psi) \\ &= \bar{\psi}(a)\sum_{xy=-\bar{3}^3\bar{a}}\psi(x)e(x+y) \\ &= \bar{\psi}(a)\sum_{x\in F^*}\psi(x)e(x-\overline{3^3ax}) \end{aligned}$$

proving (2).

Remark:

For $q = p$ and $(b,p) = 1$ sum (1) reduces to (2). If q is square-free the cubic sum (1) can be treated through multiplicativity.

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