Rational points on the sphere

W. Duke *

Department of Mathematics, University of California, Los Angeles, CA 98888.

Dedicated to the memory of Robert Rankin

Abstract

Using only basic tools from the theory of modular forms, the rational points of bounded height on the sphere are counted and shown to be uniformly distributed. The more difficult case of points with a given height is also treated.

1 Introduction.

The object of this paper is to give a short and reasonably self-contained treatment of the distribution properties of the rational points on the sphere

$$S^2 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$$

using only basic properties of modular forms, including the Shimura lift and the fundamental work of Rankin. Consider the set $\mathcal{R}$ of rational points $\mathbb{Q}^3 \cap S^2$. By the height $h(x)$ of a point $x \in \mathcal{R}$ we shall mean simply the least common denominator of its coordinates in reduced form. First we shall count the rational points of height $\le T$ on S^2 and show that they become uniformly distributed with respect to (normalized) Lebesgue measure μ on S^2 as $T \to \infty$. Then we shall show that the rational points of a given height become uniformly distributed as the height tends to infinity through odd values.

2 Rational points of bounded height.

For a function ψ on S^2 define

$$A(T, \psi) = \sum_{x \in \mathcal{R},\, h(x) \le T} \psi(x).$$

Thus $A(T, 1)$ is the number of rational points on S^2 with height $\le T$. The following theorem counts these points and shows that they are uniformly distributed.

*Research supported by NSF grant DMS-98-01642.

Theorem 1 *As $T \to \infty$ we have that $A(T,1) \sim \frac{3}{2\kappa} T^2$, where $\kappa = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots \simeq .9159$ is Catalan's constant. For any continuous function $\psi : S^2 \to \mathbb{C}$ we have*

$$A(T,\psi)/A(T,1) \to \int_{S^2} \psi d\mu$$

as $T \to \infty$.

Proof: Define

$$a(n,\psi) = \sum_{x \in \mathcal{R},\, h(x)=n} \psi(x) \tag{1}$$

and consider the Dirichlet series

$$\phi(s,\psi) = \sum_{n \geq 1} a(n,\psi) n^{-s}. \tag{2}$$

This Dirichlet series enjoys nice analytic properties in case $\psi = P_d$ is a homogeneous harmonic polynomial of degree d. That it is enough to restrict to such ψ for the proof of the second statement of the theorem is a generalization of Weyl's [6] famous criterion for uniform distribution on a torus. In our present context it follows from the classical fact that finite linear combinations of spherical harmonics are dense in the space of continuous functions on S^2 with respect to the L^∞ norm (Cor. 2.3 p. 141 of [5]).

Consider the set of integral vectors

$$\mathcal{N} = \{(x_1,x_2,x_3,y) \in \mathbb{Z}^4; x_1^2 + x_2^2 + x_3^2 - y^2 = 0, y > 0 \text{ and } gcd(x_1,x_2,x_3,y) = 1\}$$

and observe that the map

$$(x_1,x_2,x_3,y) \to (\frac{x_1}{y}, \frac{x_2}{y}, \frac{x_3}{y})$$

gives a bijection from $\mathcal{N}$ onto $\mathcal{R}$, where y is the height of the image of (x_1,x_2,x_3,y). Applying this bijection in (1) we derive that

$$r(n^2, P_d) = \sum_{\ell | n} a(\frac{n}{\ell}, P_d)$$

where

$$r(n, P_d) = \sum_{x \in \mathbb{Z}^3,\, \|x\|^2 = n} P_d(x/\|x\|) \tag{3}$$

and $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$. Hence from (2) we have the identity

$$\phi(s, P_d) = \zeta(s)^{-1} \sum_{n \geq 1} r(n^2, P_d) n^{-s}. \tag{4}$$

Suppose first that $P_d = 1$. A formula due to Hurwitz (see p. 751 of [1]) for $r(n^2, 1) = r_3(n^2)$ is equivalent to the identity

$$\sum_{n\geq 1} r(n^2,1)n^{-s} = 6(1-2^{1-s})\frac{\zeta(s)\zeta(s-1)}{L(s,\chi_{-4})}$$

where $\chi_{-4}(p) = (\frac{-4}{p})$ is the Kronecker symbol. This was thus an early precursor of the Shimura lift and gives

$$\phi(s,1) = 6(1-2^{1-s})\frac{\zeta(s-1)}{L(s,\chi_{-4})}, \tag{5}$$

which is holomorphic for $\mathrm{Re}(s) > 1$ except for a simple pole at $s = 2$ with residue $3/\kappa$. By a standard application of the Wiener-Ikehara theorem [2] one derives the asymptotic relation

$$A(T,1) \sim \frac{3}{2\kappa}T^2$$

as $T \to \infty$. Of course, an estimate for the remainder term can be found as well.

To finish the proof of Theorem 1 we need only show that for any P_d with $d > 0$ we have that

$$T^{-2}A(T,P_d) = T^{-2}\sum_{n\leq T} a(n,P_d) \to 0 \tag{6}$$

as $T \to \infty$, since for $d > 0$

$$\int_{S^2} P_d \, d\mu = 0.$$

Also, observe that we may assume that d is even for otherwise $A(T,P_d) = 0$. It is classical (see [4]) that the theta series with spherical harmonic

$$f(z) = \sum_{x\in\mathbb{Z}^3} P_d(x)e(\|x\|^2 z),$$

where as usual $e(z) = e^{2\pi i z}$, is a holomorphic cusp form of weight $3/2 + d$ for $\Gamma_0(4)$. It follows from (3) that

$$f(z) = \sum_{n\geq 1} r(n,P_d)n^{d/2}e(nz).$$

The Shimura lift [4] implies the identity

$$f(z) = \sum_{n\geq 1} b(n)n^{-s} = L(s-d,\chi_{-4})\sum_{n\geq 1} r(n^2,P_d)n^{-s}$$

where

$$F(z) = \sum_{n\geq 1} b(n)e(nz)$$

is a cusp form of weight $k = 2d + 2$ for $\Gamma_0(2)$. Thus we get from (4) that

$$\phi(s, P_d) = \sum_{n \geq 1} a(n, P_d) n^{-s} = \frac{L(s - \frac{1}{2}, F)}{\zeta(s) L(s, \chi_{-4})} \tag{7}$$

where

$$L(s, F) = \sum_{n \geq 1} b(n) n^{-\frac{k-1}{2}} n^{-s} \tag{8}$$

is the Dirichlet series associated to F. By the Rankin-Selberg method [3]

$$\sum_{n \leq x} |b(n)|^2 = cx^k + O(x^{k-2/5}). \tag{9}$$

for some positive constant c. By Cauchy's inequality it follows that the Dirichlet series (8) converges absolutely for $\mathrm{Re}(s) > 1$, hence that for $\phi(s, P_d)$ in (7) converges absolutely for $\mathrm{Re}(s) > 3/2$. Now (6) follows easily.

3 Rational points of equal height.

The analysis of the previous section shows that in fact we may treat the distribution of the rational points of a *given* height as the height gets large.

Theorem 2 *For any continuous function $\psi : S^2 \to \mathbb{C}$ we have*

$$a(n, \psi)/a(n, 1) \to \int_{S^2} \psi d\mu$$

as $n \to \infty$ through odd values.

Proof: As before it is enough to show that for $d > 0$

$$\frac{a(n, P_d)}{a(n, 1)} \to 0 \tag{10}$$

as $n \to \infty$ through odd values. From (5) we have

$$a(n, 1) = 6 \sum_{\ell | n, \, \ell \, \mathrm{odd}} \ell \, \chi_{-4}(n/\ell) \mu(n/\ell).$$

This "singular series" vanishes if and only if n is even, while for odd n it satisfies

$$a(n, 1) \gg n. \tag{11}$$

By (7) we have for $d > 0$ that

$$a(n, P_d) = \sum_{n = abc} b(n/ab)(n/ab)^{-d} \chi_{-4}(a) \mu(b) \mu(c). \tag{12}$$

From (9) Rankin derived the non-trivial bound

$$b(n) \ll n^{k/2-1/5} \tag{13}$$

which, when used in (12), yields

$$a(n, P_d) \ll n^{4/5}.$$

Combining this with (11) gives (10) and hence Theorem 2.

Remarks:

1. Note that Hecke's bound $b(n) \ll n^{k/2}$ is not sufficient to prove Theorem 2. Any substantial improvement, for example Rankin's estimate (13), is sufficient. Deligne gave the ultimate bound $b(n) \ll_\varepsilon n^{k/2+\varepsilon}$.

2. The method of proof in Theorem 1 can be generalized to ellipsoids of the form

$$Q(x) = 1$$

where Q is a positive integral quadratic form in $m \geq 2$ variables. In case m is even one must employ the symmetric square L-function in place of $L(s, F)$ from (8). Similarly, the method of proof in Theorem 2 generalizes if $m \geq 3$, with Hecke's bound being sufficient if $m \geq 4$.

References

[1] Hurwitz, A. Math. Werke, Bd. II, Birkhauser, Basel, 1933.

[2] Ikehara, S. An extension of Landau's theorem in the analytical theory of numbers, J. of Math and Physics, 10, (1931), 1–12.

[3] Rankin, R. A. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. I. The zeros of the function $\sum_{n=1}^{\infty} \tau(n)/n^s$ on the line $\Re s = 13/2$. II. The order of the Fourier coefficients of integral modular forms. Proc. Cambridge Philos. Soc. 35, (1939). 351–372.

[4] Shimura, Goro. On modular forms of half integral weight. Ann. of Math. (2) 97 (1973), 440–481.

[5] Stein, Elias M.; Weiss, Guido. Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

[6] Weyl, H. , Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77, (1916) 313–352 also in Gesammelte Bahandlungen, Band I, Springer 1968 563-599.