RAMANUJAN AND MODULAR FORMS

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What follows is a selection of three topics from Ramanujan's work that involve modular forms, along with informal descriptions of some of their further developments.[1] The topics chosen range from being famous to being relatively unknown. Their presentations assume varying degrees of background knowledge of modular forms, from minimal to moderate.

1. The tau function

For $s \in \mathbb{C}$ and n a positive integer let $\sigma_s(n) = \sum_{d|n} d^s$ be the usual sum of divisors arithmetic function. An identity from [14] that might be surprising at first sight states that for each n,

$$12 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = (1-6n)\sigma_1(n) + 5\sigma_3(n). \tag{1}$$

This implies the elementary result that $\sigma_3(n) \equiv \sigma_1(n) \pmod{24}$ for n odd.

One of Ramanujan's most influential papers [25] opens with a generalization of (1). To state it, first extend the definition of $\sigma_s(n)$ to $n = 0$ when $s \neq -1$ by setting $\sigma_s(0) = \frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann zeta function. By the functional equation, when s is odd,

$$\sigma_s(0) = \left(\tfrac{i}{2\pi}\right)^{s+1}\Gamma(s+1)\zeta(s+1). \tag{2}$$

Theorem 1. [25] *Suppose that r and s are positive odd integers and that n is a non-negative integer. Then*

$$\sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m) = \tfrac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\tfrac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n) + \tfrac{\zeta(1-r)+\zeta(1-s)}{r+s}\, n\,\sigma_{r+s-1}(n) + a_{r,s}(n), \tag{3}$$

where $a_{r,s}(0) = 0$ and $a_{r,s}(n) = 0$ for all n when $r + s \in \{2, 4, 6, 8, 12\}$. In general,

$$a_{r,s}(n) = O\big(n^{\frac{2}{3}(r+s+1)}\big).$$

Ramanujan concluded from Theorem 1 that

$$\sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m) \sim \tfrac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\tfrac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n) \tag{4}$$

as $n \to \infty$, when r and s are positive odd integers.

[1] Several important areas of Ramanujan's research involving modular forms are not discussed here, including modular equations, partitions and mock theta functions.

The tau function arises when $r+s=10$, in which case $a_{r,s}(n)=a_{r,s}(1)\tau(n)$, where $\tau(n)\in\mathbb{Z}$ is defined by

$$\sum_{n\geq 1}\tau(n)q^n = \eta^{24}(z) = q\prod_{m\geq 1}(1-q^m)^{24} \tag{5}$$
$$= 1-24q+252q^2-1472q^3+4830q^4-6048q^5-16744q^6+\cdots.$$

For instance, when $r=s=5$ the identity (3) yields

$$174132\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m) = 65\sigma_{11}(n)+691\sigma_5(n)-756\tau(n), \tag{6}$$

which implies the elegant congruence

$$\tau(n)\equiv\sigma_{11}(n) \pmod{691}. \tag{7}$$

The identity of Theorem 1 is best understood as a relation between the Fourier coefficients of modular forms for the full modular group. I will give a proof.

A holomorphic modular form F of weight $k\in 2\mathbb{Z}^+$ for the modular group is a function on the upper half-plane $\mathcal{H}$ that satisfies

$$F(\tfrac{az+b}{cz+d}) = (cz+d)^k F(z) \tag{8}$$

for $a,b,c,d\in\mathbb{Z}$ with $ad-bc=1$ and has a convergent Fourier expansion

$$F(z)=\sum_{n\geq 0}a_F(n)q^n \quad\text{for}\quad q=e^{2\pi i z}. \tag{9}$$

By the structure of the modular group we may simplify the requirement in (8) to

$$F(-\tfrac{1}{z}) = z^k F(z). \tag{10}$$

The space M_k of all such forms is finite dimensional, which is the ultimate source of identities like (3). A well-known argument using Cauchy's theorem gives the inequality (see e.g. [30, p. 10]),

$$\dim M_k \leq \begin{cases} \lfloor\frac{k}{12}\rfloor \text{ if } k\equiv 2 \pmod{12} \\ \lfloor\frac{k}{12}\rfloor+1 \text{ otherwise.}\end{cases} \tag{11}$$

Modular forms may be constructed as Eisenstein series. For $k>2$ let

$$E_k(z)=\tfrac{1}{2\zeta(k)}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}(mz+n)^{-k}. \tag{12}$$

Clearly, $E_k\in M_k$. Its Fourier expansion is a consequence of the Lipschitz formula:

$$E_k(z)=1+\tfrac{2}{\zeta(1-k)}\sum_{n\geq 1}\tfrac{n^{k-1}q^n}{1-q^n} = 1+\tfrac{2}{\zeta(1-k)}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{13}$$

For the subspace S_k of cusp forms, which comprises those $F\in M_k$ with $a_F(0)=0$, we have that

$$\dim S_k = \dim M_k - 1. \tag{14}$$

In fact, the Eisenstein series can be used to show that equality holds in (11).

The important function

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$$

is not quite modular but satisfies

$$E_2(\tfrac{-1}{z}) = z^2 E_2(z) + \tfrac{6z}{\pi i}. \tag{15}$$

A simple proof of this was given by Hurwitz in his dissertation [17] by using the conditionally convergent version of (1),

$$E_2(z) = 1 + \tfrac{1}{2\zeta(2)} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} (mz+n)^{-2}.$$

If $F \in M_k$, then by differentiating (10) we see that $F'(z) := \frac{1}{2\pi i} \frac{d}{dz} F(z)$ satisfies

$$F'(\tfrac{-1}{z}) = z^{k+2} F'(z) + \tfrac{k}{2\pi i} z^{k+1} F(z).$$

Thus (15) gives

$$E_2' = \tfrac{E_2^2 - E_4}{12} \quad \text{and} \quad E_k' - \tfrac{k}{12} E_2 E_k \in M_{k+2} \quad \text{for } k > 2. \tag{16}$$

For positive odd s we have $\zeta(1-s) = 0$ unless $s = 1$, when $\zeta(0) = -\frac{1}{2}$. By (16) we see that for r, s odd

$$F(z) = E_{r+1}(z) E_{s+1}(z) - E_{r+s+2}(z) + \tfrac{24(\zeta(1-r)+\zeta(1-s))}{r+s} E_{r+s}'(z) \in S_{r+s+2}. \tag{17}$$

Therefore the first statement of Theorem 1 follows from (17), (11) and (14), after a computation of the Fourier coefficients of F using (13) and (2) is performed.

Turning next to (5), it follows from (11) and (14) that

$$\Delta(z) = \tfrac{1}{1728}\big(E_4^3(z) - E_6^2(z)\big),$$

which is not identically zero, spans S_{12}. Now (16) implies that $\frac{\Delta'(z)}{\Delta(z)} = E_2(z)$, which gives the fundamental identity

$$\Delta(z) = \eta^{24}(z) = q \prod_{m \geq 1} (1 - q^m)^{24}.$$

Ramanujan observed empirically that the L-function associated to $\Delta(z)$ has an Euler product:

$$\sum_{n \geq 1} \tau(n) n^{-s} = \prod_{p \text{ prime}} (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}. \tag{18}$$

This observation was quickly proven by Mordell [22] and then profoundly generalized by Hecke [16]. The Euler product (18) became a paradigm for L-functions attached to automorphic representations.

When $k = r + s + 2 = 12$, a computation using (17) shows that $F(z) = c_{r,s}\Delta(z)$ where

$$c_{1,9} = \tfrac{2^7\, 3^3\, 11}{5^2\, 7\cdot 13\, B_{12}}, \qquad c_{3,7} = -\tfrac{2^6\, 3^2\, 5^2}{7\cdot 13\, B_{12}}, \qquad c_{5,5} = \tfrac{2^5\, 3^4}{5\cdot 13\, B_{12}},$$

with $B_{12} = -\frac{691}{2730}$ the Bernoulli number and, of course, $c_{r,s} = c_{s,r}$. Now (3) gives

$$c_{1,9} = 2^6\, 3^2\, 11\, a_{1,9}(1), \quad c_{3,7} = 2^9\, 3^2\, 5^2\, a_{3,7}(1), \quad c_{5,5} = 2^6\, 3^4\, 7^2\, a_{5,5}(1). \tag{19}$$

Applying the value of $a_{5,5}(1)$ in the identity (3) leads to (6), hence (7). The other two cases of (19) also imply congruences. More generally, we have that $\dim S_k = 1$ precisely for

$$k = r + s + 2 \in \{12, 16, 18, 22, 26\}.$$

For such k this forces $a_{r,s}(n) = a_{r,s}(1)\tau_k(n)$, where

$$E_{k-12}(z)\Delta(z) = \sum_{n\geq 1} \tau_k(n)q^n$$

and so $\tau(n) = \tau_{12}(n)$. The associated L-functions have Euler products and it can be checked that the factorizations of $B_k\, a_{r,s}(1)$ still only involve small primes. The congruence (7), along with others satisfied by $\tau_k(n)$, helped motivate the theory connecting coefficients of modular forms to ℓ-adic representations, c.f. [28].

The second statement of Theorem 1 is equivalent to the estimate

$$a_F(n) = O\big(n^{\frac{2}{3}(k-1)}\big)$$

for the coefficient of the cusp form F in (17) for any k. Ramanujan's proof of this makes use of theta functions and elliptic function theory. Immediately after he stated Theorem 1, Ramanujan made his famous conjecture, that for all $\epsilon > 0$ we have

$$a_F(n) = O_\epsilon\big(n^{\frac{1}{2}(k-1)+\epsilon}\big).$$

For $\tau_k(n)$ the conjecture is more precise (due to the Euler product):

$$|\tau_k(n)| \leq d(n)n^{\frac{k-1}{2}}, \qquad d(n) = \sigma_0(n).$$

As is well-known, this conjecture was proven by Deligne [6] as a special case of the Riemann hypothesis for varieties over finite fields (Weil's conjecture). This culmination came after a series of improvements, by a number of people, of Ramanujan's original estimate. The methods developed to attack the original conjecture, and that only led to its approximation, are often the only ones known that can be used to obtain nontrivial results for certain of its natural generalizations. This is true for its generalization to Maass cusp forms [5] and to holomorphic cusp forms of weight half an odd integer [18]. The (generalized) Lindelöf hypothesis for automorphic L-functions, itself a consequence of the generalized Riemann hypothesis, can be viewed as an analogue of the Ramanujan conjecture, one that includes the half-integral weight example by Waldspurger's theorem. Obtaining a subconvexity bound for an L-function amounts to approximating this conjecture. An early paper on automorphic L-function subconvexity, which is now an active area of research, is [11]. An instructive overview can be found in the paper [19] of Iwaniec and Sarnak.

Sometimes an approximation to the conjecture implies an asymptotic formula. For instance, (4) is an example where a weak estimate suffices. Such an asymptotic formula can have interesting arithmetic consequences. This comment applies in particular to problems around the representation of integers by a quadratic form and the distribution of the representing vectors. As a relevant illustration, in a footnote in another influential paper [26], Ramanujan wrote that *the even numbers which are not of the form* $x^2 + y^2 + 10z^2$ *are the numbers* $4^\lambda(16\mu + 6)$, *while the odd numbers that are not of that form, viz.,*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391 \ldots \tag{20}$$

do not seem to obey any simple law. Dickson [7, p. 341] confirmed the observation about even numbers by an elementary argument, but the problem of whether or not there are infinitely many odd integers not represented remained open until [12], whose main result implies the following as a very special case.

Theorem 2. *The set of odd numbers not represented by Ramanujan's form*

$$x_1^2 + x_2^2 + 10x_3^2$$

is finite.

The proof of this relies on the nontrivial approximation to the Ramanujan conjecture for half-integral weight Fourier coefficients of Iwaniec [18] mentioned above, extended to weight $\frac{3}{2}$ using [8]. It also relies on Siegel's theorem, so the finiteness statement is ineffective, meaning that the proof does not give an explicit bound for the largest number not represented. Actually, the list (20) is incomplete and two more exist: 679 and 2719. Ono and Soundararajan [23] showed that if one assumes appropriate Riemann hypotheses, the new list is complete.

2. Special values of the Rogers-Ramanujan continued fraction

The most familiar infinite simple continued fraction is that of the golden ratio

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots := \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}.$$

In Question 352 posed in the Journal of the Indian Mathematical Society in 1912 and then in his first letter to Hardy [15], Ramanujan gave the following analogue:

$$\sqrt{\tfrac{5+\sqrt{5}}{2}} - \tfrac{1+\sqrt{5}}{2} = \frac{e^{-\frac{2\pi}{5}}}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\cdots . \tag{21}$$

This delightful formula and others like it continue to captivate mathematicians. Although he was strongly encouraged to do so by Hardy, apparently Ramanujan never presented a rigorous proof of (21) (see [4, pp. 77, 87]). He rediscovered the following identity, which is ultimately behind it and other such identities, and was first given by Rogers [27].

Theorem 3.

$$r(z) = \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots = q^{\frac{1}{5}}\prod_{n\geq 1}(1-q^n)^{\left(\frac{5}{n}\right)}, \tag{22}$$

where $\left(\frac{5}{n}\right)$ *is the Jacobi symbol.*

This result is a consequence of the Rogers-Ramanujan identities, which state that

$$\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n\geq 1} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$
$$\sum_{n\geq 0} \frac{q^{n(n+1)}}{(1-q)\cdots(1-q^n)} = \prod_{n\geq 1} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}.$$

Since

$$R(t) = \sum_{n\geq 0} t^n q^{n^2} [(1-q)\cdots(1-q^n)]^{-1}$$

satisfies the recurrence $R(t) = R(tq) + tqR(tq^2)$ we have that

$$r(z) = q^{\frac{1}{5}} R(q)/R(1),$$

which implies (22).

Using (22) it can be shown that $r(z)$ is closely related to $\eta(z)$ from (5), for instance through

$$r^{-1}(z) - 1 - r(z) = \tfrac{\eta(\frac{z}{5})}{\eta(5z)}. \tag{23}$$

Together with an identity of Jacobi, (23) was used by Watson [29] to prove (21). Most proofs of such evaluations use modular equations in some form (see [1]).

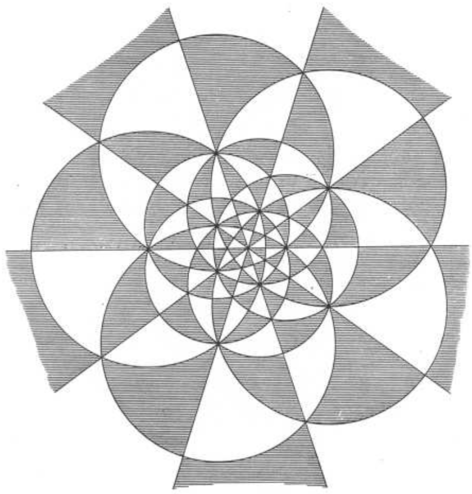

FIGURE 1. Icosahedral tessellation

There is a unified way to derive identities for special values of r, which is also in some ways simpler and more direct. This approach is explained in detail in [9].[2] The idea is to apply Klein's theory of the icosahedron [20] and his use of geometry and invariant theory. One shows that r is a Hauptmodul for $\Gamma(5)$, the principal congruence subgroup of level 5 of the modular group acting as a group of transformations on $\mathcal{H}$. The function r maps a fundamental domain for $\Gamma(5)$ to $\mathbb{C}$,

[2] In (7.4) of [9], $\tau/5$ should be replaced by τ.

resulting in an icosahedral tessellation of $\mathbb{C}$ by fundamental domains for the modular group (see Figure 1). Explicitly, we have

Theorem 4. *The function r defined in (22) satisfies the following transformations*

$$\begin{aligned} r(z+1) &= e^{\frac{2\pi i}{5}} r(z) \\ r(-\tfrac{1}{z}) &= \tfrac{-(1+\sqrt{5})r(z)+2}{2r(z)+1+\sqrt{5}} \end{aligned}$$

and the icosahedral equation

$$(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3 + j(z)r^5(r^{10} + 11r^5 - 1)^5 = 0, \tag{24}$$

where $j(z) = E_4^3(z)/\Delta(z)$ is the usual modular invariant.

From this we can easily derive (21), since $r(i)$ is a fixed point of the second transformation. The icosahedral equation (24) makes it easy to apply the classical theory of complex multiplication to show, for example, that if z is in an imaginary quadratic field, then $r(z)$ is an algebraic number that can be expressed in terms of radicals over $\mathbb{Q}$. That this is true was asserted by Ramanujan in his first letter, at least when $z^2 \in \mathbb{Q}$ and provided that his words *exactly found* are interpreted in this way. In his second letter to Hardy he provided the example

$$r(\sqrt{-5}) = \frac{\sqrt{5}}{1+\sqrt[5]{5^{\frac{3}{4}}(\frac{\sqrt{5}-1}{2})^{\frac{5}{2}}-1}} - \frac{\sqrt{5}+1}{2}. \tag{25}$$

This identity can be verified using (24) together with the evaluation from [13, p.399],

$$j(\sqrt{-5}) = 8(25 + 13\sqrt{5})^3.$$

In fact, (24) shows that $r(z)$ is an algebraic unit when z is imaginary quadratic, since then $j(z)$ is an algebraic integer. Another typical example, which is not in Ramanujan's work, is the evaluation

$$\begin{aligned} 4\,r(\tfrac{-1+\sqrt{-19}}{2}) = -8 - 3\sqrt{5} - \sqrt{125+60\sqrt{5}} \\ + \sqrt{250 + 108\sqrt{5} + (16+6\sqrt{5})\sqrt{125+60\sqrt{5}}}. \end{aligned}$$

This is a also a consequence of (24), upon using that from [13, p.400],

$$j(\tfrac{-1+\sqrt{-19}}{2}) = -2^{15}3^3.$$

There are other modular functions for genus zero subgroups of the modular group that have continued fraction expansions, found by Ramanujan and others, including Eisenstein and Selberg (see [9, §9]). Continued fractions representations of this kind for modular functions are rare. However, expansions stemming from higher order recurrences that naturally generalize continued fractions can sometimes be made. A modular function for the group $\Gamma(7)$ that occurs in a uniformization of the Klein quartic is treated in this manner in [9].

3. Integrals of Eisenstein series and hypergeometric functions

Hardy devoted a chapter of [15] to Ramanujan's work on hypergeometric series as presented in his notebooks. However, the notebooks contain some interesting hypergeometric identities connected to modular forms that are not covered there, nor in the standard reference of Bailey [2], and they are still not very well-known.

In [24, p.280 I] we find

$$\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{1}{n^2\left(e^{\frac{ny}{2}}+e^{-\frac{ny}{2}}\right)} = \frac{\sqrt{x}}{4}\,\frac{1+(\frac{2}{3})^2x+(\frac{2\cdot 4}{3\cdot 5})^2x^2+(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7})^2x^3+\cdots}{1+(\frac{1}{2})^2x+(\frac{1\cdot 3}{2\cdot 4})^2x^2+(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2x^3+\cdots} \tag{26}$$

where $0<x<1$ and the value of y is given by

$$y=\pi\,\frac{1+(\frac{1}{2})^2(1-x)+(\frac{1\cdot 3}{2\cdot 4})^2(1-x)^2+\cdots}{1+(\frac{1}{2})^2x+(\frac{1\cdot 3}{2\cdot 4})^2x^2+\cdots}. \tag{27}$$

Although it might not be obvious, this identity belongs to the theory of modular forms of level two. The usual theta constants are given by

$$\vartheta_2=\sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}, \qquad \vartheta_3=\sum_{n\in\mathbb{Z}} q^{n^2}, \qquad \vartheta_4=\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2},$$

where, following Jacobi, we let $q=e^{\pi i z}$ for $z\in\mathcal{H}$. This is the square-root of the q we used before. Then

$$\lambda(z)=(\tfrac{\vartheta_2}{\vartheta_3})^4 \quad \text{and} \quad 1-\lambda(z)=(\tfrac{\vartheta_4}{\vartheta_3})^4$$

are both Hauptmoduls for $\Gamma(2)$. The inverse of λ, when λ is restricted to a certain fundamental domain for $\Gamma(2)$, is given by the quotient of hypergeometric series

$$z(\lambda)=i\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\lambda)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\lambda)}, \tag{28}$$

extended through analytic continuation to $\mathbb{C}\setminus((-\infty,0)\cup(1,\infty))$. Further continuation of the quotient is accomplished by moving the variable λ along loops around 0 and 1. These correspond to two linear fractional transformations of z that freely generate $\Gamma(2)$. After setting $x=\lambda$ and $y=-\pi i z$ in (27), Ramanujan's identity (26) can be expressed in the following way.

Theorem 5. *For $q=e^{\pi i z}$ with $z=z(\lambda)$ given in (28) we have the identity*

$$\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{q^{\frac{n}{2}}}{n^2(1+q^n)} = \frac{\sqrt{\lambda}}{4}\,\frac{{}_3F_2(1,1,1;\frac{3}{2},\frac{3}{2};\lambda)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\lambda)}. \tag{29}$$

The LHS of (29) is an Eichler integral of an Eisenstein series of weight 3. It may be thought of as giving a uniformization in z of the analytic continuation of the RHS in λ. The three hypergeometric functions

$$\sqrt{\lambda}\,{}_3F_2(1,1,1;\tfrac{3}{2},\tfrac{3}{2};\lambda), \quad {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\lambda), \quad {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-\lambda),$$

give a basis of solutions to an inhomogeneous hypergeometric equation. The identity (29) can be used to relate their projective monodromy to cocycles of the Eichler integral, which in this case are linear polynomials.

Ramanujan presented no proof of (29). The first was given by Berndt [3, III p.153]. He deduced it from another entry in the notebooks (c.f. [3, II, p.88]), whose proof makes use of non-homogeneous hypergeometric equations. This approach generalizes nicely to prove the identities (32) and (34) given below.

Another proof of (29), which is perhaps closer to Ramanujan's style, is given in [10]. This proof, which I now recall, uses a continued fraction of Stieltjes that Ramanujan rediscovered and recorded in his second notebook shortly after the formula (26), namely

$$\frac{4F}{\sqrt{\lambda}}\sum_{\substack{n\geq 1\\ n\text{ odd}}}\frac{q^{\frac{n}{2}}}{(1+q^n)(tF^2+n^2)}=\frac{1}{t+}\,\frac{1^2}{1+}\,\frac{2^2\lambda}{t+}\,\frac{3^2}{1+}\,\frac{4^2\lambda}{t+}\cdots, \tag{30}$$

where $F={}_2F_1(\frac{1}{2},\frac{1}{2};1;\lambda)$. See [10] for detailed references. The even part is

$$\frac{1}{t+1^2-}\,\frac{1^22^2\lambda}{t+2^2\lambda+3^2-}\,\frac{3^24^2\lambda}{t+4^2\lambda+5^2-}\,\frac{5^26^2\lambda}{t+6^2\lambda+7^2-}\cdots$$

and this converges for $t=0$ to

$$\frac{1}{1-}\,\frac{(\frac{2}{3})^2\lambda}{1+(\frac{2}{3})^2\lambda-}\,\frac{(\frac{4}{5})^2\lambda}{1+(\frac{4}{5})^2\lambda-}\,\frac{(\frac{6}{7})^2\lambda}{1+(\frac{6}{7})^2\lambda-}\cdots. \tag{31}$$

A well-known identity of Euler gives for complex $a_1,\ldots,a_n$ that

$$1+\sum_{\ell=1}^{n}a_1a_2\cdots a_\ell=\frac{1}{1-}\,\frac{a_1}{1+a_1-}\,\frac{a_2}{1+a_2-\cdots}\,\frac{a_n}{1+a_n}.$$

This implies that the continued fraction in (31) equals

$$1+(\tfrac{2}{3})^2\lambda+(\tfrac{2\cdot 4}{3\cdot 5})^2\lambda^2+\cdots={}_3F_2(1,1,1;\tfrac{3}{2},\tfrac{3}{2};\lambda).$$

Thus (29) follows from (30). □

Some new identities where generalized hypergeometric quotients are uniformized by modular integrals are also proven in [10]. For instance, we have for the Eichler integral of an Eisenstein series of weight 4 the following formula.

Theorem 6.

$$\sum_{\substack{n\geq 1\\ n\ odd}}\frac{(-1)^{\frac{n-1}{2}}q^{\frac{n}{2}}}{n^3(1+q^n)}=\frac{\sqrt{\lambda(1-\lambda)}}{4}\,\frac{{}_4F_3\big(1,1,1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};4\lambda(1-\lambda)\big)}{{}_3F_2\big(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;4\lambda(1-\lambda)\big)}. \tag{32}$$

Note that by identities of Gauss and Clausen we have for $|\lambda|\leq\frac{1}{2}$

$${}_3F_2\big(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,1;4\lambda(1-\lambda)\big)={}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\lambda)^2.$$

For a recent application of (32) see [21].

Ramanujan apparently did not discover (32). He did attempt to evaluate the next case in [24, I p.280], but the entry has a faint line through it:

$$\sum_{\substack{n\geq 1\\ n\text{ odd}}}\frac{q^{\frac{n}{2}}}{n^4(1+q^n)}=\frac{\sqrt{\lambda}}{4}\,\frac{1+(\frac{2}{3})^2[1+(\frac{2}{3})^2\{1+(\frac{1}{2})^2\}]\lambda+\cdots}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\lambda)^3}. \tag{33}$$

This formula is not quite correct but it is close; its correct version is given in [10, (2.8)]. It is not hypergeometric.

There are some cases where the modular integrals of cusp forms uniformize hypergeometric quotients. One example is the following from [10]. Let a_n be defined by

$$\sum_{n\geq 1} a_n q^n = q \prod (1-q^{4n})^6.$$

Then $\sum_{n\geq 1} a_n e^{2\pi i n z}$ is the unique newform of weight 3 for $\Gamma_0(16)$ with character χ, defined by $\chi(n) = (-1)^{\frac{n-1}{2}}$ for n odd, $\chi(n) = 0$ otherwise. It is shown in [10] that

$$\sum_{n\geq 1} a_n q^{\frac{n}{4}} n^{-2} = \frac{\lambda^{\frac{1}{4}} \ {}_3F_2(\frac{3}{4}, \frac{3}{4}, 1; \frac{5}{4}, \frac{5}{4}; \lambda)}{2 \ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)}. \tag{34}$$

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