HIGHER RADEMACHER SYMBOLS

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Abstract. Certain higher Rademacher symbols are defined that give class functions on the modular group. Their basic properties are derived via a two-variable reformulation of Eichler-Shimura cohomology. This reformulation better explains the role of cycle integrals and leads to new evaluations. The Rademacher symbols determine the values at non-positive integers of the zeta function for a narrow ideal class of a real quadratic field. This result is equivalent to one of Siegel, but is proven in a new way by using an identity for the value of such a zeta function at a positive integer greater than one as a sum of certain double zeta values defined for the quadratic field.

1. Introduction

The main purpose of this paper is to define and study certain higher Rademacher symbols that give class functions defined on the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. These symbols generalize the classical Rademacher symbol and are defined in terms of general Dedekind sums. While some versions of these symbols are well-known and have been for a long time, our approach is intended to unify and clarify various aspects of their theory. Some of their basic properties are obtained by using a two-variable reformulation of Eichler-Shimura cohomology. This reformulation explicates the appearance of cycle integrals in the classical theory of modular forms by interpreting them as giving a kind of character of a cohomology class. Their structure yields some new formulas for the symbols.

The Rademacher symbols determine the values at non-positive integers of the zeta function for a narrow ideal class of a real quadratic field. This is perhaps their most important property and is equivalent to a result of Siegel. We give a novel proof, which uses an identity for the value of such a zeta function at a positive integer greater than one as a sum of certain double zeta values defined for the quadratic field. These zeta values are somewhat analogous to the usual double zeta values studied by Euler.

The rest of this Introduction gives a review of some of the properties of the classical Rademacher symbol. The following two sections contain our results. The remainder of the paper contains the proofs.

The Rademacher symbol. The Rademacher symbol is defined in terms of the Dedekind symbol. The Dedekind symbol was introduced in [8] in connection with the transformation law under the modular group of the logarithm of the Dedekind eta function. This symbol is defined for a matrix $A = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm\left(\begin{smallmatrix} a_A & b_A \\ c_A & d_A \end{smallmatrix}\right) \in \Gamma$ by setting $\Phi(A) = \frac{b}{d}$ if $c = 0$, and otherwise

$$\Phi(A) = \tfrac{a+d}{c} - 12\,\mathrm{sgn}(c)s(a,c), \tag{1.1}$$

where $s(a,c)$ is the Dedekind sum, defined for $\gcd(a,c) = 1$, $c \neq 0$ by

$$s(a,c) = \sum_{n=1}^{|c|} \left(\left(\tfrac{n}{c}\right)\right)\left(\left(\tfrac{na}{c}\right)\right). \tag{1.2}$$

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Here $((x)) = 0$ if $x \in \mathbb{Z}$ and otherwise $((x)) = x - \lfloor x \rfloor - 1/2$.

Rademacher discovered that if we set

$$\Psi(A) = \Phi(A) - 3\,\mathrm{s}(A) \tag{1.3}$$

where $\mathrm{s}(A) = \mathrm{sgn}(c(a+d))$, then $\Psi(A)$ is well defined on Γ and for all $A, B \in \Gamma$ and $k \in \mathbb{Z}$ the following hold:

$$\Psi(A^k) = k\,\Psi(A) \quad \text{for } A \text{ not elliptic} \tag{1.4}$$

$$\Psi(A) = \Psi(BAB^{-1}) \tag{1.5}$$

$$\Psi(A) \in \mathbb{Z}. \tag{1.6}$$

By (1.5), Ψ gives rise to a class function on Γ, called the Rademacher symbol.

The modular group Γ is a Fuchsian group; it has the presentation

$$\Gamma = \langle T, S, U \mid TSU^2 = S^2 = U^3 = I \rangle,$$

where

$$T = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad S = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad U = \pm\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad \text{and } I = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Therefore Γ is the free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, whose generators can be taken to be S and U, respectively. It follows that Γ has no non-trivial homomorphisms to $\mathbb{Q}$. However, the Rademacher symbol induces a quasimorphism from Γ to $\mathbb{Z}$:

$$\begin{aligned} \Psi(AB) &= \Psi(A) + \Psi(B) + 3\,\omega(A,B), \text{ where} \\ \omega(A,B) &= -\mathrm{sgn}(c_A c_B c_{AB}) - \mathrm{s}(AB) + \mathrm{s}(A) + \mathrm{s}(B), \end{aligned} \tag{1.7}$$

with $\mathrm{s}(A) \in \{0, \pm 1\}$. Here $s(A)$ was defined below (1.3). In [26] and [27], Rademacher gave proofs of (1.4)–(1.6) and (1.7) using elementary (but nontrivial) arithmetic properties of the Dedekind sum (see also [13]). For example, (1.7) with $A = S$ amounts to the reciprocity formula for the Dedekind sum.

The Rademacher symbol gives a class invariant of primitive binary quadratic forms of a given (non-square) discriminant $D > 1$. For $A = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ define

$$\mathrm{u}(A) = \gcd(c, b, d-a) \quad \text{and} \quad \mathrm{v}(A) = \tfrac{c\,\mathrm{sgn}(a+d)}{\mathrm{u}(A)}. \tag{1.8}$$

Associate to a hyperbolic $A = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ the primitive form

$$q(x,y) = [a', b', c'] = \mathrm{u}(A)^{-1}\mathrm{sgn}(a+d)(cx^2 + (d-a)xy - by^2). \tag{1.9}$$

Let (t_0, u_0) be the fundamental solution of $t^2 - u^2 D = 4$. The map $A \mapsto q$ induces a bijection from the set of conjugacy classes of primitive hyperbolic A with

$$|\,\mathrm{Tr}(A)| = t_0 \quad \text{and} \quad \mathrm{u}(A) = u_0 \tag{1.10}$$

and the set of Γ-classes of primitive integral binary quadratic forms with discriminant D. By (1.5), we see that $\Psi(q) = \Psi(A)$ is a class invariant of q.

Since $T = US$ and $V = U^{-1}S$, where $V = \pm\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, it follows from the structure of Γ that any hyperbolic or parabolic conjugacy class is represented by an A where

$$A = T^{n_1}V^{m_1} \cdots T^{n_r}V^{m_r}, \tag{1.11}$$

with $r, n_j, m_j \in \mathbb{Z}^+$ unless A is parabolic, in which case $r = 1$ and one of n_1, m_1 is zero. The representation is unique up to cyclic permutations of the blocks $T^{n_j}V^{m_j}$. A useful aspect of this decomposition is that all of the entries in a hyperbolic A are strictly positive.

Also, if $\omega' < \omega$ are the fixed points of A, then ω has a purely periodic simple continued fraction with period $(n_1, m_1, \ldots, n_r, m_r)$:

$$w = n_1 + \cfrac{1}{m_1 + \cfrac{1}{n_2 + \cfrac{1}{m_2 + \cdots}}}. \tag{1.12}$$

In addition, the continued fraction of $\frac{a_A}{c_A}$ is simply the termination of (1.12) at m_r. By (1.7), if A, B are both of the form (1.11) then

$$\Psi(AB) = \Psi(A) + \Psi(B).$$

Since $\Psi(T) = -\Psi(V) = 1$, for A from (1.11) we have the formula

$$\Psi(A) = \sum n_j - \sum m_j. \tag{1.13}$$

Using that $V = TST$ we can derive from (1.11) another well-known decomposition; every hyperbolic conjugacy class is represented by an A where

$$A = T^{n_1} S T^{n_2} \cdots S T^{n_r} S \tag{1.14}$$

with $r \in \mathbb{Z}^+$ and $n_j \geq 2$, except that $n_1 > 2$ if $r = 1$. Now we have the purely periodic minus continued fraction[1] with period $(\!(n_1, \ldots, n_r)\!)$ for the fixed point $\omega > \omega'$ of A:

$$\omega = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \cdots}}$$

and the formula

$$\Psi(A) = \sum_j n_j - 3r. \tag{1.15}$$

The Rademacher symbol has proven to be of fundamental importance in arithmetic, geometry and topology. It evaluates at $s = 0$ the zeta function of a (narrow) ideal class of a real quadratic field. The symbol has applications to modular knots [12], the topology of Hilbert modular surfaces for a real quadratic field [18] and occurs in several other related topics (e.g. [1], [2], [5], [20], [28]).

2. Higher Rademacher symbols

In this section I will define for each $n \in \mathbb{Z}^+$ a certain higher Rademacher symbol Ψ_n and present for it generalizations of the properties (1.4)–(1.5), (1.13) and (1.15) of Ψ. Our definition of Ψ_n directly generalizes that of Ψ; it is given in terms of general Dedekind sums and $\Psi_1 = \Psi$. When $n > 1$ it appears to be difficult to obtain the basic properties of Ψ_n using only elementary methods, as Rademacher did for Ψ, and connections with modular forms and cohomology become invaluable.

For $a, c, r, s \in \mathbb{Z}$ with $c \neq 0$, $\gcd(a, c) = 1$ and $r, s \geq 0$, the general Dedekind sum is defined by

$$S_{r,s}(a, c) = \sum_{h \,(\mathrm{mod}\, |c|)} P_r(\tfrac{ah}{c}) P_s(\tfrac{h}{c}), \tag{2.1}$$

where $P_r(t)$ is the periodic Bernoulli function:

$$P_r(x) = B_r(x - \lfloor x \rfloor),$$

[1] Study of the “minus” continued fraction goes back at least to Möbius [25].

except that $P_1(x)=0$ for $x\in\mathbb{Z}$. Here B_r is the usual Bernoulli polynomial

$$B_r(t)=\sum_{0\le n\le r}\tbinom{r}{n}B_n t^{r-n}$$

with B_n the Bernoulli number. Thus

$$B_0(t)=1,\ \ B_1(t)=t-\tfrac12,\ \ B_2(t)=t^2-t+\tfrac16,\ \ B_3(t)=t^3-\tfrac32 t^2+\tfrac12 t,\ldots.$$

Clearly, $S_{1,1}(a,c)=s(a,c)$ from (1.2).

Define the important integers $\imath_n,\jmath_n$ with $\jmath_n>0$ by writing

$$-\frac{B_{2n}}{2n}=\zeta(1-2n)=\frac{\imath_n}{\jmath_n}$$

in lowest form. Here $\zeta(s)$ is the usual Riemann zeta function. By von Staudt's *second* theorem [37] (see [6])

$$\jmath_n=\prod_{(p-1)|2n}p^{1+\nu_p(2n)}, \tag{2.2}$$

where $\nu_p(2n)$ is the highest power of p dividing $2n$.

TABLE 1. Values of $\imath_n$ and $\jmath_n$

n	1	2	3	4	5	6	7	8	9	10
$-\frac{B_{2n}}{2n}=\frac{\imath_n}{\jmath_n}$	$\frac{-1}{12}$	$\frac{1}{120}$	$\frac{-1}{252}$	$\frac{1}{240}$	$\frac{-1}{132}$	$\frac{691}{32760}$	$\frac{-1}{12}$	$\frac{3617}{8160}$	$\frac{-43867}{14364}$	$\frac{174611}{6600}$

For $r+s=2n$ with $n\ge1$ and integers $r,s\ge0$ define the hypergeometric polynomial

$$F_{r,s}(t)=\tfrac{\Gamma(2n-1)}{\Gamma(s+1)\Gamma(r+1)}\,{}_2F_1(1-s,1-r;\tfrac32-n,\tfrac t4+\tfrac12). \tag{2.3}$$

Clearly $F_{r,s}(t)$ has rational coefficients and $F_{r,s}=F_{s,r}$. For example when $n=3$

$$F_{0,6}(t)=-\tfrac{1}{360}(t^5-10t^3+30t),\qquad F_{1,5}(t)=\tfrac15,\qquad F_{2,4}(t)=-\tfrac t4,\qquad F_{3,3}(t)=\tfrac19(t^2+2).$$

We are now ready to define the higher Rademacher symbol $\Psi_n(A)$ for $n\in\mathbb{Z}^+$ and

$$A=\pm\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma.$$

Recall that $\mathrm{v}(A)$ was defined in (1.8).

Definition. *Set $\Psi_n(I)=0$. If $c\neq0$ let*

$$\Psi_n(A)=-\mathrm{sgn}(c)\,\jmath_n\,\mathrm{v}(A)^{n-1}\sum_{\substack{r+s=2n\\ r,s\ge0}}F_{r,s}(a+d)S_{r,s}(a,c), \tag{2.4}$$

where, if $n=1$, we must subtract $3\,\mathrm{s}(A)$ from the RHS. If $c=0$ so that $A=T^k$ set

$$\Psi_n(A)=(-1)^n\Psi_n(V^k).$$

It is easy to see that Ψ_n is well-defined on Γ since for r,s as in (2.4)

$$F_{r,s}(-t)=(-1)^{r+1}F_{r,s}(t)\quad\text{and}\quad S_{r,s}(-a,-c)=(-1)^rS_{r,s}(a,c).$$

A calculation gives the values

$$\Psi_n(V^k)=k\,\imath_n\quad\text{for } k>0 \text{ and}\quad \Psi_n(V^{-1})=|\imath_n|. \tag{2.5}$$

It follows from (2.4) and (1.1) that $\Psi_1=\Psi$ by using the evaluations

$$F_{2,0}(t)=F_{0,2}(t)=-\tfrac t4,\quad F_{1,1}(t)=1\quad\text{and}\quad S_{2,0}(a,c)=S_{0,2}(a,c)=\tfrac{1}{6|c|}.$$

Theorem 1. *For $A, B \in \Gamma$ and $n \in \mathbb{Z}^+$ the following hold:*

$$\Psi_n(A^k) = k\,\Psi_n(A) \quad \text{for integral } k \geq 0 \text{ and non-elliptic } A \text{ when } n = 1 \tag{2.6}$$

$$\Psi_n(A^{-1}) = (-1)^n \Psi_n(A)$$

$$\Psi_n(A) = \Psi_n(BAB^{-1})$$

$$\Psi_n(A) > 0 \quad \text{for } n \text{ even.} \tag{2.7}$$

By (2.6), if $n > 1$ then $\Psi_n(A) = 0$ for elliptic A. Experimental and theoretical evidence suggests that the generalization of (1.6) holds for all Ψ_n.

Conjecture. For each fixed $n \in \mathbb{Z}^+$ and all $A \in \Gamma$ we have that $\Psi_n(A) \in \mathbb{Z}$.

For any fixed n there is a procedure using a finite calculation to verify that this holds *for all A.* This procedure involves the application of analogues of the formulas (1.13) and (1.15). In addition, it appears to be true that the set of values of Ψ_n for a fixed n have no common divisor > 1.

Analogues of (1.13) and (1.15) for a fixed $n > 1$. Clearly we need only consider hyperbolic or parabolic classes. First we define a "main term" $\Psi_n^{(0)}$ of Ψ_n. Suppose that a conjugacy class is represented by A as in (1.11):

$$A = T^{n_1} V^{m_1} \cdots T^{n_r} V^{m_r} \quad \text{with } t = \sum_{1 \leq i \leq r} (n_i + m_i) \quad \text{letters.}$$

Let A_j be the j^{th} cyclic permutation of the word from left to right, letter by letter. For instance, when $A = \pm \left(\begin{smallmatrix} 7 & 4 \\ 5 & 3 \end{smallmatrix}\right) = TV^2TV$ we have

$$A_0 = A, \; A_1 = V^2TVT, \; A_2 = VTVTV, \; A_3 = TVTV^2, \; A_4 = VTV^2T.$$

Define

$$\Psi_n^{(0)}(A) = \sum_{0 \leq j \leq t-1} \Big(c_j^{n-1} + (-1)^n b_j^{n-1} \Big), \tag{2.8}$$

where $b_j = \frac{1}{\mathrm{u}(A)} b_{A_j}$, $c_j = \frac{1}{\mathrm{u}(A)} c_{A_j} \in \mathbb{Z}$ with $\mathrm{u}(A)$ from (1.8). Clearly $\Psi_n^{(0)}(A) \in \mathbb{Z}$. Furthermore, it follows from known results (and will be shown here) that for $n = 2, 3, 4, 5, 7$

$$\Psi_n(A) = \imath_n \Psi_n^{(0)}(A) = (-1)^n \Psi_n^{(0)}(A) \tag{2.9}$$

holds for all A, which shows that $\Psi_n(A)$ is integral in these cases. This motivates the definition of the "parabolic" symbol

$$\Psi'_n(A) = \Psi_n(A) - \imath_n \Psi_n^{(0)}(A), \tag{2.10}$$

for which it can be checked using (2.5) that $\Psi'_n(A) = 0$ for parabolic A.

For any given n, we can derive many explicit formulas for $\Psi'_n(A)$ of the same general shape as the RHS of (2.8). There is a large degree of freedom in the choice of formula in general, and it appears that we may always choose one that is clearly integral valued and has a nice form.[2] Define for each $n > 1$ (for brevity I leave out the subscripts j):

$$\Psi_n^{(1)}(A) = \sum_j (c^{n-3} + (-1)^n b^{n-3}) (\tfrac{n-2}{2}\, a^2 - (n-1)ad + \tfrac{n-2}{2}\, d^2 + 1). \tag{2.11}$$

For $n < 12$ we have

$$\Psi'_n(A) = \kappa_n \Psi_n^{(1)}(A) \tag{2.12}$$

[2]This freedom is a result of the fact that the dimension of the space of polynomials available to construct Ψ'_n is in general larger than the dimension of a certain cohomology group determined by these polynomials.

with integers κ_n, where $\kappa_n = 0$ for $n = 2,3,4,5,7$. It is not surprising that beginning with $n = 12$ new secondary terms appear. For instance, we have

$$\begin{aligned}(2.13)\quad \Psi'_{12}(A) &= \kappa_{12}\Psi^{(1)}_{12}(A) + 50697900(c^7+b^7)\\ &\quad\times\big(6a^4 - 33a^3d + 55a^2d^2 - 33ad^3 + 6d^4 + 9a^2 - 20ad + 9d^2 + 1\big)\\ \Psi'_{14}(A) &= \kappa_{14}\Psi^{(1)}_{14}(A) + 168058800(c^9+b^9)\\ &\quad\times\big(55a^4 - 286a^3d + 468a^2d^2 - 286ad^3 + 55d^4 + 66a^2 - 144ad + 66d^2 + 1\big).\end{aligned}$$

For each n that I have checked, a formula with integral coefficients of this general type exists, although of course new secondary terms necessarily arise as n increases.

TABLE 2. Values of κ_n.

n	6	8	9	10	11	12	14
κ_n	180	1260	−18000	79380	−39375	165079530	2787335460

The correspondence from (1.9) allows us to write these formulas for primitive A as sums over the forms

$$a'_j x^2 + b'_j xy + c'_j y^2, \tag{2.14}$$

corresponding to the cyclic permutations A_j. The resulting set of forms consists of all the simple forms of their class, where $[a', b', c']$ is *simple* if $c' < 0 < a'$.

Suppose that $C_1, \ldots C_h$ are a complete set of representatives of hyperbolic conjugacy classes that satisfy the relations corresponding to (1.10), where $D > 1$ is a discriminant. For even n, we can derive the following identities using (2.8), (2.11) and these simple forms:

$$\begin{aligned}(2.15)\qquad \sum_{1\le j\le h}\Psi^{(0)}_n(C_j) &= 2\sum_{b\equiv D\,(\mathrm{mod}\,2)}\sigma_{n-1}(\tfrac{D-b^2}{4}),\\ \sum_{1\le j\le h}\Psi^{(1)}_n(C_j) &= \tfrac12\sum_{b\equiv D\,(\mathrm{mod}\,2)}\big((2n-3)b^2 - D\big)\sigma_{n-3}(\tfrac{D-b^2}{4}),\end{aligned}$$

where σ_{n-1} is the usual sum of divisors function. For the first identity, apply (2.8) together with (1.9), then translate to a sum over all simple forms of discriminant D from (2.14). This yields

$$\sum_{1\le j\le h}\Psi^{(0)}_n(C_j) = \sum_{\substack{c'<0<a'\\ D=b'^2-4a'c'}}(a')^{n-1} + (-c')^{n-1} = 2\sum_{b\equiv D\,(\mathrm{mod}\,2)}\sigma_{n-1}(\tfrac{D-b^2}{4}).$$

The second identity of (2.15) follows similarly from (2.11).

By employing the decomposition in (1.14) we can give a single explicit formula for $\Psi'_n(A)$ that holds for all n and that appears to always be integral. The formula is not, in general, as concise as those that can be derived for a fixed n as above, but it is uniform over n. For $0 \le r \le 2n-2$ and $x = (x_1, x_2)$ let

$$G_{n,r}(x) = \big((x_1x_2)^{\frac r2} + (-1)^{r+1}(x_1x_2)^{n-1-\frac r2}\big)C^{(1-n)}_r\big(\tfrac{x_1+x_2}{2\sqrt{x_1x_2}}\big), \tag{2.16}$$

where $C^{(1-n)}_r(t)$ is the Gegenbauer polynomial, defined through

$$(1-2tz+z^2)^{n-1} = \sum_{0\le r\le 2n-2} C^{(1-n)}_r(t)z^r. \tag{2.17}$$

It can be checked that $G_{n,r}$ is a symmetric polynomial that has integral coefficients. It satisfies

$$G_{n,r}(x) = (-1)^{r+1} G_{n,r^*}(x) \quad \text{where} \quad r^* = 2n-2-r.$$

For example when $n = 3$,

$$G_{3,0} = 1 - (x_1x_2)^2, \;\; G_{3,1} = -2(x_1+x_2)(1+x_1x_2), \;\; G_{3,2} = 0, \;\; G_{3,3} = G_{3,1}.$$

Now suppose a hyperbolic conjugacy class is represented by A in (1.14) and let $A^{(j)}$ be the j^{th} cyclic permutation of the blocks $T^{n_i}S$ from left to right. Define in terms of Bernoulli numbers

$$h_n(r) = \tfrac{\jmath_n}{2}(2-\delta_{r,n})\Big(\tfrac{B_{2n}}{2n}\big(\tfrac{B_r}{r} + \tfrac{B_{2n-r}}{2n-r}\big) - \tfrac{B_r}{r}\tfrac{B_{2n-r}}{2n-r}\Big) + \tfrac{\imath_n}{2n-2}\delta_{r,2}, \tag{2.18}$$

where $\delta_{r,n}$ is the Kronecker delta. Set

$$G_n(x) = \sum_{\substack{2\le r\le n \\ \text{r even}}} h_n(r) G_{n,r-1}(x). \tag{2.19}$$

Recall that $\Psi'_n(A)$ was defined in (2.10).

Theorem 2. *For each $n > 1$ and hyperbolic*

$$A = T^{n_1}ST^{n_2}\cdots ST^{n_r}S$$

we have the identity

$$\Psi'_n(A) = \sum_{0\le j\le r-1} c_j^{n-1} G_n(\omega_j, \omega'_j), \tag{2.20}$$

where (ω_j, ω'_j) are fixed by $A^{(j)}$ and $c_j = \mathrm{u}^{-1}c_{A^{(j)}} \in \mathbb{Z}$.

It appears that $G_n(x) \in \mathbb{Z}[x_1, x_2]$ for all $n \ge 2$. I have verified this for a large number of n, although the coefficients of the polynomials quickly become enormous.[3] It is straightforward to show that for any n where it holds we have that $\Psi'_n(A) \in \mathbb{Z}$, hence also that $\Psi_n(A) \in \mathbb{Z}$, for all A.

A calculation yields an expression for $\Psi^{(0)}_n(A)$ from (2.8) in terms of the decomposition (1.14). In the notation of Theorem 2 we have:

$$\Psi^{(0)}_n(A) = -\imath_n \sum_j c_j^{n-1} \sum_{1\le r\le 2n-1} \big(\tfrac{n_j^r}{r} - \tfrac{B_r}{r} - \tfrac{B_{2n-r}}{2n-r} + \nu_r\big) Q_{n,r-1}(\omega_j, \omega'_j), \tag{2.21}$$

where $\nu_r = -\frac{1}{2}(\delta_{r,1} + \delta_{r,2n-1}) + \frac{1}{2n-2}(\delta_{r,2} + \delta_{r,2n-2})$. By (2.8) this is integral valued, which is not at all obvious from the expression (2.21).

The formulas (2.20) and (2.21) can be written in terms of the entries of $A^{(j)}$. For example,

$$\begin{aligned}\Psi'_6(A) = 3\sum (a-d)(60 + 40a^2 {+} 2a^4 - 50b^2 - 25a^2b^2 + 24b^4 - 50c^2 \\ -25a^2c^2 + 24c^4 - 200ad {-} 48a^3d + 100ab^2d + 100ac^2d + 40d^2 \\ +152a^2d^2 - 25b^2d^2 {-} 25c^2d^2 - 48ad^3 + 2d^4),\end{aligned}$$

where again we sum over j and leave out the subscripts. Each formula can also be written as a sum over the associated quadratic forms $[a', b', c']$ from (1.9). These now comprise all reduced forms, where $[a', b', c']$ is reduced if

$$a', c' > 0 \quad b' > a' + c'.$$

[3] E.g. the coefficient of x_1 in $G_{24}(x)$ is 5932154033364156392062962058217938594635552972840.

3. Double zeta values for a real quadratic field

Another goal of the paper is to introduce some analogues for a real quadratic field $\mathbb{F}$ of the double zeta values studied by Euler. In addition to having independent interest, these zeta values are used to give a new proof of a formula, which is a more symmetric and somewhat simpler version of one first obtained by Siegel. The formula relates the higher Rademacher symbol defined in the previous section to the value of the zeta function of a narrow ideal class of $\mathbb{F}$ at a non-positive integer.

The double zeta value studied by Euler [11] is given by

$$\zeta(r,s) = \sum_{m_1>m_2>0} m_1^{-r} m_2^{-s},$$

where $r, s \in \mathbb{Z}^+$ with $r \geq 2$. Among the identities relating these values to values at integers of $\zeta(s)$ is

$$\zeta(2n) = 2 \sum_{n<r\leq 2n-1} \tbinom{n-1}{n-r} \zeta(r, 2n-r) \tag{3.1}$$

for integral $n > 1$. Loosely speaking, one can regard

$$\zeta(2s) = \zeta(s) \prod_p (1+p^{-s})^{-1}$$

as the zeta function of a "degenerate" real quadratic field in which every rational prime is inert.

The aim is to establish formulas for a genuine real quadratic field that are somewhat analogous to the formula (3.1). For $t \in \mathbb{Z}$ with $t > 2$ and $a, c \in \mathbb{Z}$ with $c > 0$ and $\gcd(a, c) = 1$ the double zeta value we need is given by

$$\zeta_t(r,s;a,c) = \sum_{\substack{m_1,m_2>0 \\ m_1^2+m_2^2<t\,m_1m_2 \\ m_2\equiv am_1 \,(\mathrm{mod}\, c)}} m_1^{-r} m_2^{-s}, \tag{3.2}$$

for suitable $r, s \in \mathbb{Z}^+$. Clearly

$$\zeta_t(s,r;a,c) = \zeta_t(r,s;\overline{a},c) \quad \text{where} \quad a\overline{a} \equiv 1 \,(\mathrm{mod}\, c).$$

Define for $n, r \in \mathbb{Z}^+$ with $0 < r < 2n$ the polynomial of degree $r-1$ in t by

$$c_{n,r}(t) = \tfrac{\Gamma(r)\Gamma(2n-r)}{\Gamma(n)^2} C_{r-1}^{(1-n)}(-\tfrac{t}{2}), \tag{3.3}$$

where again $C_{r-1}^{(1-n)}(x)$ is the Gegenbauer polynomial with negative index. It can be shown by induction using the recurrence relations satisfied by Gegenbauer polynomials that $c_{n,r}(t)$ has positive integral coefficients. Also

$$c_{n,r}(-t) = (-1)^{r+1} c_{n,r}(t) \qquad c_{n,r}(t) = c_{n,2n-r}(t).$$

For example,

$$c_{2,1}(t) = 2 \quad c_{2,2}(t) = t \quad c_{2,3}(t) = 2$$
$$c_{3,1}(t) = 6 \quad c_{3,2}(t) = 3t \quad c_{3,3}(t) = t^2+2 \quad c_{3,4}(t) = 3t \quad c_{3,5}(t) = 6$$
$$c_{4,1}(t) = 20 \quad c_{4,2}(t) = 10t \quad c_{4,3}(t) = 4t^2+4 \quad c_{4,4}(t) = t^3+6t \quad c_{4,5}(t) = 4t^2+4 \ldots.$$

Let $\mathbb{F}$ be a real quadratic field with discriminant D, ring of integers $\mathcal{O}_{\mathbb{F}}$ and totally positive fundamental unit

$$\varepsilon = \tfrac{1}{2}(t + u\sqrt{D}), \tag{3.4}$$

where $t, u \in \mathbb{Z}^+$. Let $\mathcal{A}$ be any (narrow) ideal class of $\mathbb{F}$. It is known that $\mathcal{A}$ contains (fractional) ideals $\mathfrak{a}$ with

$$\mathfrak{a} = \mathbb{Z} + \omega\mathbb{Z}$$

where $\omega = \alpha + \beta\sqrt{D} \in \mathbb{F}$ satisfies $\omega > \omega' = \alpha - \beta\sqrt{D}$. Let $a, b, c, d \in \mathbb{Z}$ be determined by $\varepsilon\omega = d\omega + b$ and $\varepsilon = c\omega + a$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ with $c > 0$ and the values are given by

$$a = \frac{\varepsilon'\omega - \varepsilon\omega'}{\omega - \omega'} \qquad b = \frac{\varepsilon' - \varepsilon}{\omega - \omega'}\omega\omega' \qquad c = \frac{\varepsilon - \varepsilon'}{\omega - \omega'} \qquad d = \frac{\varepsilon\omega - \varepsilon'\omega'}{\omega - \omega'}. \tag{3.5}$$

The zeta function of the class $\mathcal{A}$ is

$$\zeta(s, \mathcal{A}) = \sum_{\mathfrak{b} \in A} N(\mathfrak{b})^{-s},$$

where $\mathfrak{b} \subset \mathcal{O}_{\mathbb{F}}$ runs over all nonzero integral ideals in $\mathcal{A}$. As is well-known, $\zeta(s, \mathcal{A})$ has a meromorphic continuation in s to an entire function with a simple pole at $s = 1$. Suppose that $\mathcal{J}$ is the class that contains the different $\mathfrak{d} = (\sqrt{D})$. Thus $\mathcal{J}^2 = \mathcal{I}$, where $\mathcal{I}$ is the principal class and $\mathcal{J} = \mathcal{I}$ if and only if ϵ is a square in $\mathbb{F}$.

The analogue of (3.1) is given in the following result, which is proven in §4.

Theorem 3. *Notation as above, for integral $n > 1$ we have the evaluation*

$$\zeta(n, \mathcal{J}\mathcal{A}) = u(\tfrac{c}{u})^n D^{\frac{1}{2}-n} \sum_{1 \le r \le 2n-1} c_{n,r}(t)\zeta_t(r, 2n-r; a, c).$$

Example. Suppose that $D = (t-2)(t+2)$ is fundamental so $u = 1$. The first 15 such t are given by

$$t = 3, 4, 5, 8, 9, 12, 13, 15, 17, 19, 21, 24, 28, 31, 32, \ldots.$$

For $\mathbb{F} = \mathbb{Q}(\sqrt{t^2 - 4})$, Theorem 3 gives for $n > 1$

$$D^{n-\frac{1}{2}}\zeta(n, \mathcal{J}) = \sum_{1 \le r \le 2n-1} c_{n,r}(t)\zeta_t(r, 2n-r) \quad \text{for} \quad \zeta_t(r, s) = \sum_{\substack{m_1, m_2 > 0 \\ m_1^2 + m_2^2 < t m_1 m_2}} m_1^{-r} m_2^{-s}. \tag{3.6}$$

In case $t = 3$ so $\mathbb{F} = \mathbb{Q}(\sqrt{5})$, for which $\mathcal{J} = \mathcal{I}$, we have that $\zeta(s, \mathcal{J}\mathcal{A}) = \zeta_{\mathbb{F}}(s)$, the Dedekind zeta function of $\mathbb{F}$. Thus we get

$$5^{n-\frac{1}{2}}\zeta_{\mathbb{F}}(n) = \sum_{1 \le r \le 2n-1} c_{n,r}(3)\zeta_3(r, 2n-r), \quad \text{where} \quad \zeta_3(r, s) = \sum_{\substack{m_1, m_2 > 0 \\ m_1^2 + m_2^2 < 3 m_1 m_2}} m_1^{-r} m_2^{-s}$$

satisfies $\zeta_3(r, s) = \zeta_3(s, r)$. In particular we have

$$5^{\frac{3}{2}}\zeta_{\mathbb{F}}(2) = 4\zeta_3(1, 3) + 3\zeta_3(2, 2) = \tfrac{2\pi^4}{15}$$
$$5^{\frac{5}{2}}\zeta_{\mathbb{F}}(3) = 12\zeta_3(1, 5) + 18\zeta_3(2, 4) + 11\zeta_3(3, 3) = 57.4417\ldots.$$

Perhaps the most important application of Ψ_n is that is gives a formula for $\zeta(1-n, \mathcal{A})$. As an application of Theorem 3 and its proof we will derive the following.

Theorem 4. *For any $\mathcal{A}$ and $n \in \mathbb{Z}^+$ we have*

$$\zeta(1-n, \mathcal{A}) = \tfrac{1}{\jmath_n}\Psi_n(A), \tag{3.7}$$

where the entries in $A = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ are determined by (3.5).

The case $n = 1$ is due to Hecke [14], [15] (see also [23]). Theorem 4 is essentially equivalent to a special case of Siegel's formula from [33]. Our proof differs from Siegel's and also from that given by Shintani [31, p. 410]. In fact, in his last mathematical paper [35], Siegel used an elementary method that is somewhat similar in spirit to our proof. Also, Siegel's and Shintani's formulas apply to zeta functions for ray classes. It is not difficult to generalize our arguments to cover these as well.

Example (continued). For $\mathbb{F} = \mathbb{Q}(\sqrt{D})$ with fundamental $D = t^2 - 4$ the formula (3.7) becomes

$$\zeta(1-n,\mathcal{I}) = -\sum_{\substack{r+s=n\\ r,s\geq 0}} F_{2r,2s}(t)B_{2r}B_{2s} \tag{3.8}$$

where, when $n = 1$, we must subtract $\frac{1}{4}$ from the RHS. In particular, we have the following simple evaluations of the zeta function of the principal class:

$$\begin{aligned}\zeta(0,\mathcal{I}) &= \tfrac{1}{12}\,(t-3)\\ \zeta(-1,\mathcal{I}) &= \tfrac{1}{720}\,t\,(t^2-1).\\ \zeta(-2,\mathcal{I}) &= \tfrac{1}{7560}\,t\,(t^2-1)(t^2-9).\end{aligned}$$

It is known ([4], see also [3]) that the (wide) class number of $\mathbb{F}$ is one exactly for the discriminants $D = 5, 12, 21, 77, 437$ corresponding to $t = 3, 4, 5, 9, 21$. In these cases (3.8) yields special values of the Dedekind zeta function of $\mathbb{F}$ and the simplest Hecke L-function with a genus character.

A calculation combining the formula of Theorem 2 and (2.21) results in Zagier's elegant formula for $\zeta(1-n,\mathcal{A})$ given in Corollaire on p. 149 of [40]. In terms of the decomposition (1.14) and in the notation of Theorem 2 this can be written

$$\zeta(1-n,\mathcal{A}) = \sum_j c_j^{n-1} \sum_{1\leq r\leq 2n-1} \Big(\tfrac{B_{2n}}{2n}\tfrac{n_j^r}{r} - \tfrac{B_r}{r}\tfrac{B_{2n-r}}{2n-r}\Big) Q_{n,r-1}(\omega_j,\omega_j'). \tag{3.9}$$

This result is derived in another way in the recent paper [38].[4] As discussed at the end of [40], it is not in general easy to estimate the denominator of $\zeta(1-n,\mathcal{A})$ for a fixed n by using the RHS of (3.9). On the other hand, by (2.10) and Theorem 4

$$\zeta(1-n,\mathcal{A}) = \zeta(1-2n)\Psi_n^{(0)}(A) + \tfrac{1}{\jmath_n}\Psi_n'(A). \tag{3.10}$$

Here $\Psi_n^{(0)}(A) \in \mathbb{Z}$ and it can be checked numerically for a given n whether of not G_n from (2.19) is integral. If it is, which seems to always be true, then $\Psi_n'(A) \in \mathbb{Z}$ for all $A \in \Gamma$. In this case we see that the denominator of $\zeta(1-n,\mathcal{A})$ always divides that of $\zeta(1-2n)$, namely $\jmath_n$ from (2.2).

For the Dedekind zeta function

$$\zeta_{\mathbb{F}}(s) = \sum_{\mathcal{A}} \zeta(s,\mathcal{A})$$

we have that $\jmath_n\zeta_{\mathbb{F}}(1-n) \in \mathbb{Z}$ for any $n \in \mathbb{Z}^+$ (see [41] and its references, e.g. [29]). In case $n \in \{2, 4, 6, 8, 10\}$ we can apply the formulas (2.15) and (2.12) to obtain the following

[4]Examples show that the related formulas given in Theorem 8 of [21] and in the example that follows it need to be corrected.

evaluation, which yields more information in these cases:

$$\jmath_n \zeta_{\mathbb{F}}(1-n) = 2\imath_n \sum_{b\equiv D \,(\mathrm{mod}\, 2)} \sigma_{n-1}\left(\tfrac{D-b^2}{4}\right) + \kappa_n b_n(D), \quad \text{where} \tag{3.11}$$

$$b_n(D) = \tfrac{1}{2} \sum_{b\equiv D \,(\mathrm{mod}\, 2)} \big((2n-3)b^2 - D\big)\sigma_{n-3}\left(\tfrac{D-b^2}{4}\right).$$

For $n = 2, 4$ when $\kappa_n = 0$ and $\imath_n = 1$, this is a well-known formula of Siegel [34]. For these n it was obtained in a different way in [9].

Remarks. i) When n is even, after Siegel's work [32] the value $\zeta_{\mathbb{F}}(1-n)$ is a positive multiple of the co-volume of a discrete subgroup of an orthogonal group which, after Theorem 4, also explains the positivity of $\Psi_n(A)$ asserted in (2.7) of Theorem 1.

ii) When $n = 6, 8, 10$ the numbers $b_n(D)$ are the Fourier coefficients of a cusp form of weight $n + \frac{1}{2}$, a multiple of which gives the well known Shimura correspondent to the unique cusp form (Hecke eigenform) of weight $2n$ for Γ, giving an explicit and attractive example of a general and well-known phenomenon.

In the next section I will prove Theorems 3 and 4. Then, in §5, I will give a reformulation of Eichler-Shimura cohomology in terms of polynomials in two variables. This yields a proof of Theorem 1, the method for deriving the formulas (2.12), (2.13) and the proof of Theorem 2.

4. Proofs of Theorems 3 and 4

The derivation of (3.1) uses partial fractions. Our proof of Theorem 3 relies on the following decomposition, which applies naturally to a real quadratic field, where nontrivial units must be accounted for.

Lemma 1. *Fix any $\varepsilon > 1$ and set $t = \varepsilon + \varepsilon^{-1}$. Choose $\mu, \nu \in \mathbb{C}^*$ and $n \in \mathbb{Z}^+$ with $n \geq 2$. If $\varepsilon^j \mu + \varepsilon^{-j}\nu \neq 0$ for all j then*

$$\frac{1}{(\mu\nu)^n} = (\varepsilon - \varepsilon^{-1}) \sum_{\substack{r+s=2n \\ r,s\geq 1}} c_{n,r}(t) \sum_{j\in\mathbb{Z}} \frac{1}{(\varepsilon^j\mu + \varepsilon^{-j}\nu)^r(\varepsilon^{j+1}\mu + \varepsilon^{-j-1}\nu)^s}. \tag{4.1}$$

If $\mu + \nu = 0$ then $\varepsilon^j\mu + \varepsilon^{-j}\nu \neq 0$ for $j \neq 0$ and

$$\begin{aligned}\frac{1}{(\mu\nu)^n} = (\varepsilon - \varepsilon^{-1}) &\sum_{\substack{r+s=2n \\ r,s\geq 1}} c_{n,r}(t) \sum_{\substack{j\in\mathbb{Z} \\ j\neq 0,-1}} \frac{1}{(\varepsilon^j\mu + \varepsilon^{-j}\nu)^r(\varepsilon^{j+1}\mu + \varepsilon^{-j-1}\nu)^s} \\ &+(\varepsilon^2 - \varepsilon^{-2}) \sum_{\substack{r+s=2n \\ r,s\geq 1}} c_{n,r}(t^2-2) \frac{1}{(\varepsilon^{-1}\mu + \varepsilon\nu)^r(\varepsilon\mu + \varepsilon^{-1}\nu)^s}.\end{aligned} \tag{4.2}$$

Proof. Assume first that $\mu, \nu > 0$. Define for $j \in \mathbb{Z}$

$$I_j(x, y; s) = \int_{C_j} (u_1v_1)^s e^{-u_1x - v_1y} du_1\, dv_1$$

where C_j is the sector in the first quadrant of the (u_1, v_1) plane bounded by the lines $u_1 = (\varepsilon^2)^{j+1}v_1$ and $u_1 = (\varepsilon^2)^j v_1$. Summing this integral over $j \in \mathbb{Z}$ we get

$$\sum_{j\in\mathbb{Z}} I_j(x, y; s) = \int_0^\infty \int_0^\infty (u_1v_1)^s e^{-u_1x - v_1y} du_1\, dv_1 = \Gamma(s+1)^2(xy)^{-s-1}. \tag{4.3}$$

Make the change of variables in $I_j(x,y;s)$:

$$\begin{aligned} u_1 =& \varepsilon^j u + \varepsilon^{j+1} v \\ v_1 =& \varepsilon^{-j} u + \varepsilon^{-j-1} v. \end{aligned}$$

Then $du_1\, dv_1 = (\varepsilon - \varepsilon^{-1}) du\, dv$ and

$$\begin{aligned} &I_j(x,y;s) = \\ &\quad(\varepsilon-\varepsilon^{-1})\int_0^\infty\int_0^\infty (u^2+v^2+tuv)^s e^{-u(\varepsilon^j y+\varepsilon^{-j}x)-v(\varepsilon^{j+1}y+\varepsilon^{-j-1}x)}du\, dv. \end{aligned}$$

The formula (4.1) follows from (4.3) using (2.17) and (3.3). It holds by analytic continuation for all $\mu,\nu\in\mathbb{C}$ that satisfy $\varepsilon^j\mu+\varepsilon^{-j}\nu\neq 0$ for all $j\in\mathbb{Z}$.

If $\mu+\nu=0$ then clearly $\varepsilon^j\mu+\varepsilon^{-j}\nu\neq 0$ for $j\neq 0$ since $\varepsilon>1$. Now we give the same proof except that we make a single change of variable for $C_{-1}\cup C_0$. □

Let $\mathfrak{a}$ be a non-zero ideal in $\mathbb{F}$. The complementary ideal is given by

$$\mathfrak{a}^* = \{\alpha\in\mathbb{F}; \mathrm{Tr}(\alpha\beta)\in\mathbb{Z} \text{ for all } \beta\in\mathfrak{a}\}.$$

It is known (see e.g. [36, p.135]) that $\mathfrak{a}\mathfrak{a}^*$ is independent of $\mathfrak{a}$ and that $\mathfrak{a}\mathfrak{a}^*=\mathfrak{d}^{-1}$, where $\mathfrak{d}$ is the different of $\mathbb{F}$, which, as was already mentioned, is explicitly given by $\mathfrak{d}=(\sqrt{D})$. Now let ε be the unit from (3.4). If $\mathfrak{a}=[\alpha_1,\alpha_2]=\alpha_1\mathbb{Z}+\alpha_2\mathbb{Z}$ then

$$\mathfrak{a}^* = \left[\tfrac{\alpha_2'}{\alpha_1\alpha_2'-\alpha_2\alpha_1'}, \tfrac{-\alpha_1'}{\alpha_1\alpha_2'-\alpha_2\alpha_1'}\right]. \tag{4.4}$$

A calculation gives

$$N(\mathfrak{a}) = |D|^{-\frac12}|\alpha_1\alpha_2'-\alpha_2\alpha_1'|. \tag{4.5}$$

We may assume that $\mathfrak{a}=\mathbb{Z}+\omega\mathbb{Z}$ where $\omega=\frac{d-a+\sqrt{t^2-4}}{2c}$ with $t=a+d$ and $t^2-4=u^2D$. Here a,c,d coincide with those from (3.5). Then by (4.5)

$$N(\mathfrak{a}) = \tfrac{u}{c}. \tag{4.6}$$

By (4.4) we have

$$\mathfrak{a}^* = \left[\tfrac12+\tfrac{a-d}{2u\sqrt{D}}, \tfrac{c}{u\sqrt{D}}\right].$$

For $\alpha\in\mathfrak{a}^*$

$$\alpha = x_1\left(\tfrac12+\tfrac{a-d}{2u\sqrt{D}}\right)+x_2\left(\tfrac{c}{u\sqrt{D}}\right) = \tfrac{x_1}{2}+\tfrac{2x_2c+(a-d)x_1}{2u\sqrt{D}} \tag{4.7}$$

for some $x_1,x_2\in\mathbb{Z}$. Set $m_1=\mathrm{Tr}(\alpha)=x_1$ and

$$m_2=\mathrm{Tr}(\alpha\varepsilon)=\mathrm{Tr}\Big(\big(\tfrac{x_1}{2}+\tfrac{2cx_2+x_1(a-d)}{2u\sqrt{D}}\big)\big(\tfrac{t+u\sqrt{D}}{2}\big)\Big)=\tfrac12(t+a-d)x_1+cx_2=ax_1+cx_2.$$

Thus

$$m_2\equiv am_1 \pmod{c}.$$

The system is easily solved for x_1,x_2 in terms of m_1,m_2 as

$$x_1=m_1 \quad\text{and}\quad x_2=\frac{m_2-am_1}{c}. \tag{4.8}$$

Furthermore, a computation using (4.7) and (4.8) gives that $N(\alpha)=\frac{-m_1^2-m_2^2+tm_1m_2}{u^2D}$. Thus we have the following result.

Lemma 2. *Assumptions as above, let* $\mathfrak{a} = \left[1, \frac{d-a+\sqrt{t^2-4}}{2c}\right]$ *where* $t = a+d$ *and* $t^2-4 = u^2D$. *Then the map* $\alpha \mapsto (\operatorname{Tr}\alpha, \operatorname{Tr}\alpha\varepsilon)$ *gives a bijection from* $\mathfrak{a}^*$ *to*

$$\{(m_1, m_2) \in \mathbb{Z}^2; m_2 \equiv am_1 \pmod{c}\}.$$

In addition

$$N(\alpha) = \frac{-m_1^2 - m_2^2 + tm_1m_2}{u^2D}.$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Since $\mathfrak{d} \in \mathcal{J}$, we have a bijection induced by $\alpha \mapsto \mathfrak{b} = (\alpha)\mathfrak{a}\mathfrak{d}$ from

$$\{\alpha \in \mathfrak{a}^*; N(\alpha) > 0\}/U^+ \quad \text{to} \quad \{\mathfrak{b} \in \mathcal{J}\mathcal{A};\ \mathfrak{b} \text{ integral and nonzero}\}.$$

Here $N(\alpha) = \alpha\alpha'$ and U^+ is the group of units with norm 1. Therefore we have

$$\zeta(s, \mathcal{J}\mathcal{A}) = N(\mathfrak{a})^{-s} D^{-s} \sum_{\substack{\alpha\in\mathfrak{a}^*/U^+ \\ N(\alpha)>0}} N(\alpha)^{-s}. \tag{4.9}$$

By (4.9) and the first statement (4.1) of Lemma 1 we deduce that for integral $n > 1$

$$\begin{aligned}\zeta(n, \mathcal{J}\mathcal{A}) &= N(\mathfrak{a})^{-n} D^{-n} (\varepsilon - \varepsilon^{-1}) \sum_{\substack{r+s=2n \\ r,s\geq 1}} c_{n,r}(t) \\ &\qquad \times \sum_{j\in\mathbb{Z}} \sum_{\substack{\alpha\in\mathfrak{a}^*/U^+ \\ N(\alpha)>0}} \frac{1}{(\varepsilon^j\alpha + \varepsilon^{-j}\alpha')^r (\varepsilon^{j+1}\alpha + \varepsilon^{-j-1}\alpha')^s} \\ &= N(\mathfrak{a})^{-n} D^{-n} (\varepsilon - \varepsilon^{-1}) \sum_{1\leq r\leq 2n-1} c_{n,r}(t) \sum_{\substack{\alpha\in\mathfrak{a}^* \\ N(\alpha)>0}} \frac{1}{(\operatorname{Tr}\alpha)^r (\operatorname{Tr}\alpha\varepsilon)^{2n-r}}. \qquad (4.10)\end{aligned}$$

From (3.2) we have

$$\zeta_t(r, 2n-r; a, c) = \sum_{\substack{m_1,m_2>0 \\ m_1^2+m_2^2<t\,m_1m_2 \\ m_2\equiv am_1 \,(\mathrm{mod}\, c)}} m_1^{-r} m_2^{r-2n}.$$

Theorem 3 now follows from Lemma 2 applied in (4.10), together with (4.6) and the fact that

$$\varepsilon - \varepsilon^{-1} = u\sqrt{D}.$$

□

Next we further develop the method just used to prove Theorem 3 in order to prove Theorem 4.

Proof of Theorem 4. A derivation similar to that of (4.9) yields

$$\zeta(s, \mathcal{A}) = N(\mathfrak{a})^{-s} D^{-s} \sum_{\substack{\alpha\in\mathfrak{a}^*/U^+ \\ N(\alpha)<0}} |N(\alpha)|^{-s}. \tag{4.11}$$

Assume that $n > 1$. From (4.9) and (4.11) we have

$$\zeta(n, \mathcal{J}\mathcal{A}) + (-1)^n \zeta(n, \mathcal{A}) = N(\mathfrak{a})^{-n} D^{-n} \sum_{\substack{\alpha\in\mathfrak{a}^*/U^+ \\ \alpha\neq 0}} N(\alpha)^{-n}. \tag{4.12}$$

By Lemmas 1 and 2 we deduce

$$N(\mathfrak{a})^{-n}D^{-n}\sum_{\substack{\alpha\in\mathfrak{a}^*/U\\ \alpha\neq 0}}N(\alpha)^{-n}=\tfrac{1}{2}u^{1-n}D^{\frac{1}{2}-n}\Big(c^n\sum_{\substack{r+s=2n\\ r,s\geq 1}}c_{n,r}(t)\sum_{\substack{m_1,m_2\in\mathbb{Z}\\ m_2\equiv am_1\,(\mathrm{mod}\,c)}}{}^{\prime}\; m_1^{-r}m_2^{-s}$$
$$+2\zeta(2n)tc^{-n}\sum_{\substack{r+s=2n\\ r,s\geq 1}}(-1)^r c_{n,r}(t^2-2)\Big),\tag{4.13}$$

where as usual the prime indicates that terms resulting in division by zero are omitted from the sum. Here the terms in the first sum over r when $r=1$ or $r=2n-1$ are to be interpreted as limits. Now by using basic Fourier analysis with $e(z)=e^{2\pi iz}$

$$\sum_{\substack{m_1,m_2\in\mathbb{Z}\\ m_2\equiv am_1\,(\mathrm{mod}\,c)}}{}^{\prime}\; m_1^{-r}m_2^{-s}=\tfrac{1}{c}\sum_{h(c)}\sum_{m_1}e(-\tfrac{ahm_1}{c})\sum_{m_2}e(\tfrac{m_2h}{c})m_1^{-r}m_2^{-s}\tag{4.14}$$
$$=\frac{(2\pi i)^{2n}}{r!s!\,c}\sum_{h(c)}P_r(-\tfrac{ah}{c})P_s(\tfrac{h}{c})=(-1)^r\tfrac{(2\pi i)^{2n}}{r!s!\,c}S_{r,s}(a,c),$$

where $S_{r,s}(a,c)$ was given in (2.1). Here we have applied the following well-known Fourier expansion

$$P_r(u)=-r!\sum_{k\neq 0}e(ku)(2\pi ik)^{-r},$$

where, when $r=1$ we collect the terms indexed by k and $-k$. Thus by (4.12) – (4.14) we get

(4.15)

$$\zeta(n,\mathcal{A})+(-1)^n\zeta(n,\mathcal{JA})=\tfrac{1}{2}u^{1-n}D^{\frac{1}{2}-n}c^{n-1}(2\pi)^{2n}\Big(\sum_{\substack{r+s=2n\\ r,s\geq 1}}c_{n,r}(t)(-1)^r\tfrac{1}{r!s!}S_{r,s}(a,c)$$
$$+2(-1)^{n+1}(2\pi)^{-2n}\zeta(2n)tc^{1-2n}\sum_{\substack{r+s=2n\\ r,s\geq 1}}(-1)^r c_{n,r}(t^2-2)\Big).$$

The following elementary identities are easily proven.

Lemma 3. *Let $F_{r,s}(t)$ be defined in (2.3) and $c_{n,r}(t)$ defined in (3.3). For $r+s=2n$, when $r,s>0$ we have*

$$F_{r,s}(t)=\tfrac{(-1)^{r+1}\Gamma^2(n)}{\Gamma(r+1)\Gamma(s+1)}c_{n,r}(t).$$

In addition,

$$F_{2n,0}(t)=F_{0,2n}(t)=\tfrac{\Gamma^2(n)}{2\Gamma(2n+1)}t\sum_{\substack{r+s=2n\\ r,s\geq 1}}(-1)^r\,c_{n,r}(t^2-2).$$

Using Lemma 3 we get from (4.15) the identity

$$\zeta(n,\mathcal{A})+(-1)^n\zeta(n,\mathcal{JA})=-\tfrac{1}{2}u^{1-n}D^{\frac{1}{2}-n}c^{n-1}(2\pi)^{2n}\Gamma(n)^{-2}\tag{4.16}$$
$$\times\Big(\sum_{\substack{r+s=2n\\ r,s\geq 1}}F_{r,s}(t)S_{r,s}(a,c)+2(-1)^{n+1}(2\pi)^{-2n}\zeta(2n)c^{1-2n}2\Gamma(2n+1)F_{r,s}(t)\Big).$$

Next we apply the easily established result

$$2\Gamma(2n+1)(2\pi)^{-2n}(-1)^{n+1}c^{1-2n}\zeta(2n)=c^{1-2n}B_{2n}=S_{0,2n}(a,c)=S_{2n,0}(a,c)$$

so from (4.16)

$$\zeta(n,\mathcal{A})+(-1)^n\zeta(n,\mathcal{JA})=-\tfrac{1}{2}u^{1-n}D^{\frac{1}{2}-n}c^{n-1}(2\pi)^{2n}\tfrac{1}{\Gamma^2(n)}\sum_{\substack{r+s=2n\\ r,s\geq 0}}F_{r,s}(t)S_{r,s}(a,c). \tag{4.17}$$

After Hecke (see e.g. [10, (59),(6)] and [36]) the completions

$$D^{\frac{s}{2}}\pi^{-s}\Gamma(\tfrac{s}{2})^2\big(\zeta(s,\mathcal{A})+\zeta(s,\mathcal{JA})\big)\ \text{ and }\ D^{\frac{s}{2}}\pi^{-s}\Gamma(\tfrac{s+1}{2})^2\big(\zeta(s,\mathcal{A})-\zeta(s,\mathcal{JA})\big). \tag{4.18}$$

are invariant under $s\mapsto 1-s$. In particular we have that

$$\zeta(1-n,\mathcal{A})=(-1)^n\zeta(1-n,\mathcal{JA}).$$

To finish the proof of Theorem 4 use the functional equation (4.18) to derive that

$$\zeta(1-n,\mathcal{A})=2D^{n-\frac{1}{2}}(2\pi)^{-2n}\Gamma^2(n)\big(\zeta(n,\mathcal{A})+(-1)^n\zeta(n,\mathcal{JA})\big) \tag{4.19}$$

and apply this in (4.17). □

Remark. A version of Lemma 1 was given in [39], based on [7], but it does not seem to deal with or imply the crucial second part of Lemma 1.

5. A reformulation of Eichler-Shimura cohomology

We now present a reformulation of Eichler-Shimura cohomology for the modular group given in terms of polynomials in two variables and associate to each cohomology class a kind of character, which is a class function on Γ. For a particular (Eisenstein) cohomology class this character yields the higher Rademacher symbol. As will be apparent, many of the arguments we use apply to more general Fuchsian groups. This section contains a proof of Theorem 1.

For $A=\pm\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{PSL}(2,\mathbb{R})$ and $x=(x_1,x_2)\in\mathbb{R}^2$ let

$$J_A(x)=(cx_1+d)(cx_2+d).$$

Then for any $A,B\in\mathrm{PSL}(2,\mathbb{R})$ we have the cocycle relation

$$J_{BA}(x)=J_B(Ax)J_A(x) \tag{5.1}$$

where

$$Ax=(Ax_1,Ax_2)=(\tfrac{ax_1+b}{cx_1+d},\tfrac{ax_2+b}{cx_2+d}).$$

For a fixed $n\in\mathbb{Z}^+$ let $\mathcal{P}_n$ be the $(2n-1)$-dimensional $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1,x_2]$ spanned by the polynomials having degree $r^*=2n-2-r$ for $0\leq r\leq 2n-2$ determined by

$$\big((\tau-x_1)(\tau-x_2)\big)^{n-1}=\sum_{0\leq r\leq 2n-2}Q_{n,r}(x)\tau^r. \tag{5.2}$$

We have $Q_{0,0}(x)=1$ and for $n>1$

$$Q_{n,r}(x)=(x_1x_2)^{n-1-\frac{r}{2}}C_r^{(1-n)}(\tfrac{x_1+x_2}{2\sqrt{x_1x_2}}), \tag{5.3}$$

where $C_r^{(1-n)}(t)$ is the Gegenbauer polynomial. In general, $\mathcal{P}_n$ is a proper subspace of the space of all symmetric polynomials in x_1,x_2 of degree at most $2n-2$, while $\mathcal{P}_2$ contains all such polynomials. For instance $\mathcal{P}_3$ is spanned by

$$\{Q_{3,0},Q_{3,1},Q_{3,2},Q_{3,3},Q_{3,4}\}=\{(x_1x_2)^2,-2x_1x_2(x_1+x_2),x_1^2+4x_1x_2+x_2^2,-2(x_1+x_2),1\}.$$

The space $\mathcal{P}_n$ is closed under the action

$$Q(x)\mapsto Q|A(x):=J_A^{n-1}(x)Q(Ax) \tag{5.4}$$

of $A = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$. This is easily checked using the identity

$$(c\tau + d)^2 J_A(x)(A\tau - Ax_1)(A\tau - Ax_2) = (\tau - x_1)(\tau - x_2).$$

Let $\sigma \in Z(\Gamma, \mathcal{P}_n)$, the usual space of 1-cocycles so $\sigma : \Gamma \to \mathcal{P}_n$ with

$$\sigma_{AB}(x) = \sigma_A|B(x) + \sigma_B(x) \tag{5.5}$$

for all $A, B \in \Gamma$. For such σ and $A = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c_A \neq 0$ define

$$\psi_\sigma(A) = \mathrm{v}(A)^{n-1}\sigma_A(\alpha), \tag{5.6}$$

where $\alpha = \alpha_A = (\alpha_1, \alpha_2) \in \mathbb{C}^2$ are the fixed points of A and $\mathrm{v}(A)$ was defined in (1.8). Here $\alpha_1 = \alpha_2$ when A is parabolic. Set $\psi_\sigma(I) = 0$ and for $n \in \mathbb{Z}$

$$\psi_\sigma(T^n) = \psi_\sigma(V^{-n}).$$

Note that $V^{-1} = STS^{-1}$. It is clear that $\sigma_A(\alpha) = 0$ for any elliptic A, so $\psi_\sigma(A) = 0$ for elliptic A.

Lemma 4. *For all $A, B \in \Gamma$ and $n \in \mathbb{Z}$ we have for*

$$\mathrm{u}(A) = \mathrm{u}(BAB^{-1}) \tag{5.7}$$

$$\mathrm{v}(A^n) = \mathrm{sgn}(n)\mathrm{v}(A). \tag{5.8}$$

Proof. For (5.7), if $C = BAB^{-1}$ a computation shows that

$$\gcd(b_A, c_A, d_A - a_A) \mid \gcd(b_C, c_C, d_C - a_C).$$

The result follows by symmetry. The proof of (5.8) is similar. $\square$

Lemma 5. *For all $A, B \in \Gamma$ and $n \in \mathbb{Z}^+$ the following hold:*

$$\psi_\sigma(A^n) = n\psi_\sigma(A) \tag{5.9}$$

$$\psi_\sigma(A^{-1}) = (-1)^n\psi_\sigma(A) \tag{5.10}$$

$$\psi_\sigma(BAB^{-1}) = \psi_\sigma(A). \tag{5.11}$$

Proof. We may suppose that $c_A, c_{BAB^{-1}} \neq 0$. Let α satisfy $A\alpha = \alpha$.

A calculation shows that

$$c_{BAB^{-1}} = c_A J_B(\alpha). \tag{5.12}$$

In particular,

$$J_A(\alpha) = 1. \tag{5.13}$$

Next (5.9) and (5.10) follow from (5.8) and the fact that from the cocycle relation (5.5) and (5.13)

$$\sigma_{A^n}(\alpha) = n\sigma_A(\alpha)$$

for any $n \in \mathbb{Z}$.

To prove (5.11), another application of the cocycle relation yields

$$\begin{aligned} \sigma_{BAB^{-1}}(B\alpha) &= \sigma_B|AB^{-1}(B\alpha) + \sigma_{AB^{-1}}(B\alpha) \\ &= \sigma_B|AB^{-1}(B\alpha) + \sigma_A|B^{-1}(B\alpha) + \sigma_{B^{-1}}(B\alpha). \end{aligned} \tag{5.14}$$

By (5.4) and (5.1)

$$\begin{aligned} \sigma_B|AB^{-1}(B\alpha) + \sigma_{B^{-1}}(B\alpha) &= J^{n-1}_{AB^{-1}}(B\alpha)\sigma_B(A\alpha) - J_B^{1-n}(\alpha)\sigma_B(\alpha) \\ &= J_B^{1-n}(\alpha)\big(J_A^{n-1}(\alpha)\sigma_B(\alpha) - \sigma_B(\alpha)\big) = 0 \end{aligned}$$

since $J_A(\alpha) = 1$ and $J_B(\alpha) \neq 0$ by assumption and (5.12). Thus (5.14) gives

$$\sigma_{BAB^{-1}}(B\alpha) = \sigma_A|B^{-1}(B\alpha) = J_B(\alpha)^{1-n}\sigma_A(\alpha)$$

and so by (5.12)

$$(c_{BAB^{-1}})^{n-1}\sigma_{BAB^{-1}}(B\alpha) = (c_A)^{n-1}\sigma_A(\alpha).$$

Finally, $\alpha_{BAB^{-1}} = B\alpha$ and both $\mathrm{sgn}(a+d)$ and $\mathrm{u}(A)$ are invariant under conjugation. □

Let $H^1(\Gamma, \mathcal{P}_n)$ be the first cohomology group. For a fixed 1-cocycle σ, the function ψ_σ is well-defined on the class of $H^1(\Gamma, \mathcal{P}_n)$ represented by σ since from (5.6) and (5.13) we have that $\psi_\sigma(A) = 0$ for a co-boundary σ. By (5.11) the function ψ_σ induces a $\mathbb{Q}$-valued class function on Γ. Let $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ be the subgroup represented by parabolic cocycles. Thus each element of $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ can by represented by a cocycle σ that satisfies $\sigma_T(x) = 0$.

The map $Q \mapsto Q(x_1, x_1)$ is a vector space isomorphism from $\mathcal{P}_n$ onto the space of all polynomials in x_1 of degree at most $2n-2$ that maps the action $Q|A$ to the usual one on polynomials in one variable. Thus $H^1(\Gamma, \mathcal{P}_n)$ and $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ are isomorphic to the usual Eichler-Shimura cohomology groups on all polynomials of degree at most $2n-2$ in one variable. The advantage of working with two-variable polynomials $\mathcal{P}_n$ is the natural appearance of the class function ψ_σ, which we will refer to as the character of (the class represented by) σ.

For $k \in 2\mathbb{Z}$ let M_k be the $\mathbb{C}$-vector space of all holomorphic functions f on the upper half-plane $\mathcal{H}$ that satisfy

$$f|_k A(\tau) = (c\tau + d)^{-k} f(A\tau) = f(\tau),$$

for $\tau \in \mathcal{H}$ and all $A \in \Gamma$ and that have a Fourier expansion

$$f(\tau) = \sum_{r \geq 0} a(r) q^r \qquad q = e(\tau), \tag{5.15}$$

with $a(r) \in \mathbb{C}$ for all r. The space of cusp forms S_k consists of those $f \in M_k$ with $a(0) = 0$. Then for $k > 2$

$$\ell = \dim S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1, & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor, & \text{otherwise} \end{cases} \tag{5.16}$$

and $\dim M_k = \ell + 1$.

Any $f \in M_k$ gives rise to the 1-cocycle $\sigma \in Z(\Gamma, \mathcal{P}_{k/2})$ defined by

$$\sigma_A^{(f)}(x) = \sigma_A(x) = \int_{A^{-1}\tau_0}^{\tau_0} f(\tau)\big((\tau - x_1)(\tau - x_2)\big)^{\frac{k}{2}-1} d\tau, \tag{5.17}$$

where $\tau_0 \in \mathcal{H}$ is fixed. Note that

$$\sigma_A(x_1, x_1) = \int_{A^{-1}\tau_0}^{\tau_0} f(\tau)(\tau - x_1)^{k-2} d\tau$$

is the usual Eichler integral. For $A = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c \neq 0$ the character value at A of σ from (5.6) is given by

$$\psi_\sigma(A) = \mathrm{v}(A)^{\frac{k}{2}-1} \int_{A^{-1}\tau_0}^{\tau_0} f(\tau)\big((\tau - \omega)(\tau - \omega')\big)^{\frac{k}{2}-1} d\tau, \tag{5.18}$$

where (ω, ω') is fixed by A. Here $\mathrm{v}(A)$ was defined in (1.8). Also $\omega' = \overline{\omega}$ if ω is not real and $\omega = \omega'$ if A is parabolic. Since $\psi_\sigma(A)$ is well-defined on a cohomology class, this integral (5.18) is independent of τ_0. For hyperbolic A with associated binary quadratic form q from (1.9) the integral may be identified with a cycle integral:

$$\psi_\sigma(A) = \int_{A^{-1}\tau_0}^{\tau_0} f(\tau) q(\tau, 1)^{\frac{k}{2}-1} d\tau. \tag{5.19}$$

The Eisenstein series can be defined for any $k \in 2\mathbb{Z}$ by

$$E_k(\tau) = 1 + \tfrac{2}{\zeta(1-k)} \sum_{r \geq 1} \sigma_{k-1}(m) q^r \qquad \sigma_{k-1}(r) = \sum_{d|r} d^{k-1} \tag{5.20}$$

and $E_k \in M_k$ if $k > 2$.

Lemma 6. *Let ψ_k be the character determined by (5.18) when $f = E_k$ with $k > 2$. For $A \in \Gamma$ with $c_A \neq 0$ we have*

$$\mathrm{v}(A)^{-\frac{k}{2}+1} \psi_k(A) = \frac{\operatorname{sgn}(c)}{\zeta(1-k)} \sum_{\substack{r+s=k \\ r,s \geq 0}} F_{r,s}(t) S_{r,s}(a,c). \tag{5.21}$$

Proof. For hyperbolic A the formula (5.21) follows from [33, Hilfsatz 5 and §3]. In his proof Siegel applies the standard Riemann-Dedekind-Hecke method to compute the invariant integral

$$\int_{A^{-1}\tau_0}^{\tau_0} E_k(\tau) q(\tau,1)^{\frac{k}{2}-1} d\tau.$$

A similar argument gives the result for all A with $c_A \neq 0$. □

Proof of Theorem 1 for $n > 1$. For the higher Rademacher symbol by Lemma 6 we have the identity

$$\Psi_n(A) = \imath_n \psi_{2n}(A). \tag{5.22}$$

Thus all parts of Theorem 1 follow from Lemma 5 except for (2.7), which is a consequence of Theorem 4 and (4.19), since n is even. □

For a 1-cocycle σ denote by $\overline{\sigma}$ the 1-cocycle determined by conjugating coefficients:

$$\overline{\sigma}_A(x) = \overline{\sigma_A(\overline{x}_1, \overline{x}_2)}.$$

In view of the isomorphism mentioned at the end of §5 induced by taking $x_1 = x_2$, the following result is a direct consequence of the well-known Eichler-Shimura isomorphism.

Proposition 1. *The map $(f,g) \mapsto (\sigma^{(f)}, \overline{\sigma}^{(g)})$ determined by (5.17) induces isomorphisms from $M_k \oplus S_k$ to $H^1(\Gamma, \mathcal{P}_n)$ and from $S_k \oplus S_k$ to $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$.*

The next result is crucial for our method of establishing decomposition formulas for $\Psi_n(A)$.

Lemma 7. *If a character of $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ vanishes for $T^j V$ or $T^{j+1} S$ for $j = 1, \ldots$ then it vanishes for all $A \in \Gamma$.*

Proof. By Proposition 1 we see that a character of $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ can be expressed in the form (5.18) for a cusp form f and $A \in \Gamma$ with $c_A \neq 0$. Now by interpreting the character value as a cycle integral, the result follows from [17, Thm. 5], which is an improvement of earlier results from numerous papers cited there, e.g. [16], [22], [19]. In fact, it is shown that we only need to assume that the character vanishes for $\lfloor \frac{n}{2} \rfloor$ values of j to conclude that it is identically zero, if $n \geq 6$. □

Remarks. i) The proof by Siegel of his version of (5.21) is part of the derivation of his version of Theorem 4. The proof we gave of Theorem 4 does not use the cycle integral representation of $\Psi_n(A)$ given in Lemma 6. However, we certainly need it to prove our other results.

ii) Siegel treated the case $k = 2$ of Lemma 6 using a limit formula. Some of the differences between Ψ_n for $n = 1$ and $n > 1$ are reflected in the fact that E_2 as defined by (5.20) is not modular and that its modular version E_2^* is not holomorphic. For a discussion and references see e.g. [10].

iii) The $b_n(D)$ in (3.11) are sums of cycle integrals of a cusp form, that is sums of characters of a parabolic class. Their interpretation as Fourier coefficients of a cusp form of half-integral weight mentioned in ii) of the Remarks after (3.11) is originally due to Shintani [30].

6. Proofs of (2.12), (2.13) and Theorem 2

First we will justify our derivation of (2.12) and (2.13). For that we give a formula for any character $\psi_\sigma(A)$ in terms of the decomposition (1.11). Let

$$\mathcal{W}_n^S = \{Q \in \mathcal{P}_n; Q|S = -Q\}. \tag{6.1}$$

Lemma 8. *Fix $n \in \mathbb{Z}$ with $n > 1$.*

i) Every class in $H^1(\Gamma, \mathcal{P}_n)$ is represented by a cocycle σ where $\sigma_S \in \mathcal{W}_n^S$ and $\sigma_U = 0$.

ii) Conversely, given $Q \in \mathcal{W}_n^S$ there is a class of $H^1(\Gamma, \mathcal{P}_n)$ represented by the cocycle σ defined by setting $\sigma_S = Q$ and $\sigma_U = 0$.

iii) Suppose that the cocycle σ has $\sigma_S \in \mathcal{W}_n^S$ and $\sigma_U = 0$. For A given in (1.11), i.e.

$$A = T^{n_1}V^{m_1} \cdots T^{n_r}V^{m_r} \quad \textit{with } t = \sum_{1 \le i \le r} (n_i + m_i)$$

and A_j the j^{th} cyclic permutation, we have

$$\psi_\sigma(A) = \mathrm{u}^{1-n} \sum_{0 \le j \le t-1} c_j^{n-1} \sigma_S(\omega_j, \omega_j'), \tag{6.2}$$

where (ω_j, ω_j') is fixed by A_j and $c_j = c_{A_j}$.

Proof. The statement i) follows from Proposition 1 upon using the choice

$$\tau_0 = \rho = \tfrac{1}{2}(1 + \sqrt{-3})$$

in (5.17), since ρ is fixed by U and $\sigma_S(x) \in \mathcal{W}_n^S$. The statement ii) is straightforward.

Turning to iii), set

$$M_0 = A, \;\; M_1 = T^{n_1-1}V^{m_1} \cdots T^{n_r}V^{m_r}, \;\; M_2 = T^{n_1-2}V^{m_1} \cdots V^{m_r}, \ldots, M_{t-1} = V,$$

so that M_j is obtained from A by deleting the first j letters. Then for $0 \le j \le t-1$

$$A_j = M_j A M_j^{-1}. \tag{6.3}$$

Therefore by the cocycle relation (5.5) and the easily verified fact that $\sigma_T = \sigma_V = \sigma_S$ we have

$$\sigma_A(\omega_0, \omega_0') = \sigma_S|M_1(\omega_0, \omega_0') + \sigma_S|M_2(\omega_0, \omega_0') + \cdots + \sigma_S|M_{t-1}(\omega_0, \omega_0') + \sigma_S|M_0(\omega_0, \omega_0'). \tag{6.4}$$

Next for $0 \le j \le t-1$ the following hold:

$$M_j(\omega_0, \omega_0') = (\omega_j, \omega_j'), \quad c_{A_j} = c_A J_{M_j}(\omega_0, \omega_0') \quad \text{and} \quad \mathrm{v}(A) = \mathrm{u}^{-1} c_A. \tag{6.5}$$

The first equation is a consequence of (6.3). For the second, apply (5.12) and (6.3). For the third refer to the definition (1.8) and use that the entries of A are positive. Now (6.2) follows from (6.4), (6.5) and (5.6). □

Recall that $G_{n,r}(x)$ was defined for $0 \le r \le 2n-2$ in (2.16). Clearly $G_{n,r} \in \mathcal{W}_n^S$. In fact a basis for $\mathcal{W}_n^S$ is obtained by taking $0 \le r \le n-1$ when n is even and $0 \le r \le n-2$ when n is odd. By (ii) of Lemma 8 there is a cohomology class of $H^1(\Gamma, \mathcal{P}_n)$ represented by a 1-cocycle σ with $\sigma_S(x) = G_{n,r}(x)$ and $\sigma_U(x) = 0$. By (i) of Lemma 8 every class is represented this way. A calculation using (iii) of Lemma 8 shows that $\Psi_n^{(0)}$ defined in (2.8) gives the character that comes from $G_{n,0}$.

It can be checked that if $r > 0$ then the cocycle associated to $G_{n,r}$ is represents an element of $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$. Denote by $\Psi_n^{(r)}$ the associated character. After Proposition 1, a comparison of dimensions shows that in general the classes represented are not independent, and this is the source of the freedom to find desirable decomposition formulas for Ψ_n. In particular, it can be shown that $\Psi_n^{(r)} = 0$ for odd r.

In view of Lemma 7 we may limit ourselves to computing character values on

$$T^{t-2}V = T^{t-1}ST$$

for $t > 2$.

Lemma 9. *For even* $0 \le j \le n-2$ *and* $t > 2$

$$\Psi_n^{(j)}(T^{t-2}V) = 2 \sum_{\substack{j \le r \le 2n-2 \\ r \text{ even}}} (2n-r)\tbinom{r}{r-j} B_{r-j} F_{r,2n-r}(t) - 2 \sum_{\substack{2 \le r \le j+2 \\ r \text{ even}}} r\tbinom{2n-r}{j+2-r} B_{j+2-r} F_{r,2n-r}(t).$$

Proof. By (iii) of Lemma 8

$$\Psi_n^{(j)}(T^{t-2}V) = \sum_{1 \le g \le t-1} G_{n,j}(\omega_g, \omega_g'),$$

where

$$\omega_g = \tfrac{1}{2}\big(-2g + t + \sqrt{t^2-4}\big) \tag{6.6}$$

so ω_g, ω_g' are the fixed points of

$$A_g = T^g S = T^{t-g-1} V T^{g-1},$$

which has $\mathrm{Tr}(A_g) = t$.

Recall that $r^* = 2n-2-r$. We have for $Q_{n,j}$ defined in (5.2) or (5.3)

$$G_{n,r}(x) = Q_{n,r^*}(x) + (-1)^{r+1} Q_{n,r}(x). \tag{6.7}$$

We are reduced to showing that for an even integer j with $0 \le j \le 2n-2$

$$\sum_{1 \le g \le t-1} Q_{n,j}(\omega_g, \omega_g') = 2 \sum_{\substack{j \le r \le 2n-2 \\ \text{r even}}} (r-2n)\tbinom{r}{r-j} B_{r-j} F_{r,s}(t) - (j+1)(2n-j-1) F_{j+1,2n-j-1}(t). \tag{6.8}$$

It is convenient to define when $r, s > 0$

$$F^*_{r,s}(t) = r(2n-r)F_{r,s}(t) = \tfrac{\Gamma(2n-1)}{\Gamma(s)\Gamma(r)}\, {}_2F_1(1-s, 1-r; \tfrac{3}{2}-n, \tfrac{t}{4}+\tfrac{1}{2}). \tag{6.9}$$

We have

$$\big((\tau-\omega_g)(\tau-\omega_g')\big)^{n-1}=\sum_{1\le r\le 2n-1}(g-\tau)^{r-1}F^*_{r,2n-r}(t).$$

In particular, for even j with $0\le j\le 2n-2$

$$Q_{n,j}(\omega_g,\omega_g')=\sum_{1\le r\le 2n-1} g^{r-1-j}\tbinom{r-1}{j}\,F^*_{r,2n-r}(t).$$

After summing over g we apply the well-known identity of Faulhaber-Bernoulli: for integers k,t with $k\ge 1$ and $t>1$

$$\sum_{1\le g\le t-1} g^{k-1}=\tfrac{1}{k}\sum_{0\le h\le k-1}\tbinom{k}{h}B_h\,t^{k-h}-\delta_{k,1}.$$

In the resulting equation we employ the next readily proven identity in order to finish the proof of (6.8), hence of Lemma 9:

Lemma 10. *For even integers h,j with $0\le h\le 2n-2-j$ and $0\le j\le 2n-2$*

$$\sum_{r=h+j+1}^{2n-1} t^{r-j-h}\tbinom{r-j}{h}\tbinom{r-1}{j}\tfrac{1}{r-j}F^*_{r,2n-r}(t)=2(h+j-2n)\tbinom{j+h}{h}F_{h+j,2n-j-h}(t).$$

□

Next we apply Lemma 9 to evaluate $\Psi_n'(T^{t-2}V)=\Psi_n'(T^tS)$ as a polynomial in t.

Lemma 11. *The function $\Psi_n'(A)$ defined in (2.10) gives the character value of a parabolic cohomology class. For $t>2$ we have*

$$\Psi_n'(T^{t-2}V)=\Psi_n'(T^tS)=2\sum_{\substack{2\le r\le n\\ r\ \text{even}}}h_n(r)F^*_{r,2n-r}(t),\tag{6.10}$$

where $h_n(r)$ was defined in (2.18).

Proof. As mentioned before, $\Psi_n^{(0)}$ is the character that comes from $G_{n,0}$ by Lemma 8. That $\Psi_n'(A)$ is a character now follows from (2.10) and (5.22). Finally, (2.5) shows that $\Psi_n'(A)$ is the character of a parabolic class.

By (2.4) we have

$$\Psi_n(T^{t-2}V)=-\jmath_n\sum_{\substack{r+s=n\\ r,s\ge 0}}F_{2r,2s}(t)B_{2r}B_{2s}.\tag{6.11}$$

By Lemma 9 when $j=0$

$$\Psi_n^{(0)}(T^{t-2}V)=2\sum_{\substack{0\le r\le 2n-2\\ \text{r even}}}(2n-r)B_rF_{r,2n-r}(t)-4F_{2,2n-2}(t).$$

Therefore

$$\Psi_n'(T^{t-2}V)=-\jmath_n\Big(\sum_{\substack{r+s=n\\ r,s\ge 2}}F_{2r,2s}(t)B_{2r}B_{2s}+2B_{2n}\sum_{\substack{2\le r\le 2n-2\\ \text{r even}}}B_rF^*_{2,2n-2}(t)\Big)-4\imath_nF_{2,2n-2}(t).$$

from which (6.10) follows. □

Now it is clear how (2.12) and (2.13) are derived. We apply Lemma 7 to reduce the problem to linear algebra, i.e. equating polynomials in t by using Lemma 11 and Lemma 9 with some of the $j>0$. For the formulas we give one takes the minimal range of j required, which is $j=2$ for (2.12) and $j=2,4$ for (2.13).

Proof of Theorem 2. To prove Theorem 2 we make use of the "period polynomials" in two variables defined by

$$\mathcal{W}_n = \{Q \in \mathcal{P}_n; Q|(S+1) = 0 \text{ and } Q|(1+U+U^2) = 0\}.$$

We have the following analogue of Lemma 8:

Lemma 12. *Fix $n \in \mathbb{Z}$ with $n > 1$.*

i) Every class in $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ is represented by a cocycle σ where $\sigma_S \in \mathcal{W}_n$ and $\sigma_T = 0$.

ii) Conversely, given $Q \in \mathcal{W}_n$ there is a class of $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_n)$ represented by the cocycle σ defined by setting $\sigma_S = Q$ and $\sigma_T = 0$.

iii) Suppose that the cocycle σ has $\sigma_S \in \mathcal{W}_n$ and $\sigma_T = 0$. For A given in (1.14), i.e.

$$A = T^{n_1} S T^{n_2} \cdots S T^{n_r} S$$

and $A^{(j)}$ the j^{th} cyclic permutation of (1.14) from left to right, we have

$$\psi_\sigma(A) = \mathrm{u}^{1-n} \sum_{0 \le j \le r-1} c_j^{n-1} \sigma_S(\omega_j, \omega_j'),$$

where (ω_j, ω_j') is fixed by $A^{(j)}$ and $c_j = c_{A^{(j)}}$.

Proof. The statement i) follows from Proposition 1 upon using the choice $\tau_0 = i\infty$ in (5.17), which is valid if f is a cusp form. The statement ii) is straightforward after noting that the assumptions $\sigma_S \in \mathcal{W}_n$ and $\sigma_T = \sigma_{US} = 0$ imply $\sigma_U = \sigma_S$. The proof of statement iii) is similar to that of iii) of Lemma 8, where now we define

$$M_0 = A, \quad M_1 = T^{n_2} S \cdots T^{n_r} S, \quad \ldots, \quad M_{r-1} = S$$

so $A^{(j)} = M_j A M_j^{-1}$. Proceed using the cocycle relation and that $\sigma_T = 0, \operatorname{Tr} A > 0$. □

Now for even r with $2 \le r \le 2n-2$

$$G_{n,r-1}(\omega_0, \omega_0') = G_{n,r-1}\big(\tfrac{1}{2}(t + \sqrt{t^2-4}), \tfrac{1}{2}(t - \sqrt{t^2-4})\big) = 2F^*_{r,2n-r}(t).$$

Thus by Lemma 11 for $t > 2$

$$\Psi_n'(T^t S) = \sum_{\substack{2 \le r \le n \\ \text{r even}}} h_n(r) G_n(\omega_0, \omega_0').$$

Theorem 2 is a consequence of Lemmas 7, 12 and the following result. Recall that $G_n(x)$ was defined in (2.19).

Lemma 13. *For each $n > 1$ we have that $G_n(x) \in \mathcal{W}_n$.*

Proof. This follows by an elementary computation involving Bernoulli polynomials. A proof can be extracted from that of the first statement of Theorem 1' of [21] with, in the notation there, $n = w$.

□

Some Questions.

(1) What is the arithmetic significance of the integers κ_n from (2.12)? Do they have properties in common with the numerators $\imath_n$ of $\zeta(1-2n)$? What is their impact in the equations (3.11)?

(2) To what extent do the the values of Ψ_n for all n separate conjugacy classes of Γ, or equivalently classes of primitive binary quadratic forms? See [24] for a treatment of this question for the usual Rademacher symbol ($n = 1$), which certainly does not distinguish classes. One may ask the same question about all the characters of $H^1(\Gamma, \mathcal{P}_n)$.

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