Invent. math. 115, 209–217 (1994)

Inventiones mathematicae

A quadratic divisor problem

W. Duke*, J.B. Friedlander and H. Iwaniec*****

Department of Mathematics, Rutgers University, New Brunswich, NJ 08903, USA
Department of Mathematics, University of Toronto, Toronto, Canada MS5 1A1

Oblatum 28-IV-1993

1. Introduction

A great many problems in analytic number theory lead indirectly to counting 2×2 matrices having fixed determinant and integral entries over various ranges. More explicitly, one often encounters sums of the type

$$\sum_{auv \mp brs = h} f(u, v, r, s),$$

where a, b, h are fixed positive integers and f is a smooth function whose partial derivatives are under control. This may be viewed as counting the representations by a quadratic form $Q(u, v, r, s)$, a study initiated by H. Kloostermann [Kl]. A special case of the above sum, known as the additive divisor problem

$$\sum_{n \leqq x} \tau(n)\tau(n+h),$$

where $\tau(n)$ stands for the number of positive divisors of n was investigated by A. Ingham [In] who first gave an asymptotic formula and then by T. Estermann [Es] who established the asymptotic expansion

$$\sum_{n \leqq x} \tau(n)\tau(n+h) = xP_h(\log x) + O(x^{11/12} \log^3 x),$$

where $P_h(T)$ is a quadratic polynomial with leading coefficient $6\pi^{-2}\sigma_{-1}(h)$. The key ingredient in Estermann's paper is an estimate for Kloostermann sums

$$S(m, n; q) = \sideset{}{^*}\sum_{d \,(\mathrm{mod}\, q)} e\left(\frac{dm + \bar{d}n}{q}\right),$$

* partially supported by NSF grant DMS-9202022
** partially supported by NSERC grant A5123
*** partially supported by each of the above

where, as usual, the asterisk restricts the summation to reduced classes. After A. Weil established the best bound

$$S(m, n; q) \ll (m, n, q)^{1/2} q^{1/2} \tau(q)$$

substantial improvements on the error term of Estermann became possible, cf. [HB], [Wi]. Further advances have been made by J.-M. Deshouillers and H. Iwaniec [DeI_1] by exploring the spectral theory of automorphic forms; see also N.V. Kuznetsov [Ku]. The state of the art of this approach is reached in works of M. Jutila [Ju_2] and Y. Motohashi [Mo].

As usual in practice one needs asymptotic formulas with a good error term which are valid uniformly in the parameters a, b, h of considerable size. These can be very difficult problems indeed. Motohashi's result [Mo] is very strong with respect to h, but unfortunately for us he considers only the case $a = b = 1$. For other relevant results see [He], [Sm].

Motivated by specific applications in mind [DFI_2] in this paper we investigate sums of type

$$D_f(a, b; h) = \sum_{am \mp bn = h} \tau(m)\tau(n) f(am, bn), \tag{1}$$

where f is nice smooth function on $R^+ \times R^+$. Not only do we allow the coefficients a, b to be large but also f to oscillate mildly. In fact all the properties of f to be used are expressed in the following estimate for partial derivatives

$$x^i y^j f^{(ij)}(x, y) \ll \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} P^{i+j}, \tag{2}$$

with some $P, X, Y \geqq 1$ for all $i, j \geqq 0$, the implied constant depending on i, j alone. We shall use Weil's bound for Kloosterman sums rather than the spectral theory of automorphic forms since the latter approach would require us to deal with the congruence group $\Gamma_0(ab)$ facing intrinsic difficulties with small eigenvalues. The results obtained this way would not be good enough for large a, b. However, if an averaging over a, b was included then the density theorems for small eigenvalues might help (see [DeI_2]). Such a result is given by N. Watt [Wa].

Our objective here is to quickly get results useful for applications without straining for the best from available technologies. As in [DFI_1], [DuI] we have chosen to use the δ-method which is a simple alternative for the circle method. A direct approach starting from the definition of the divisor function is also a possibility, however it could not generalize to the corresponding problem with $\tau(m)$ replaced by the Fourier coefficients of cusp forms. We anticipate applying the δ-method for the latter problem elsewhere. Yet, in this case the results of J. Hafner [Ha] can be used as well.

2. Statement of results

The main term in our asymptotic formula will be expressed in terms of the series

$$\Lambda_{abh}(x, y) = \frac{1}{ab} \sum_{q=1}^{\infty} q^{-2}(ab, q) c_q(h)(\log x - \lambda_{aq})(\log y - \lambda_{bq}), \tag{3}$$

where $c_q(h) = S(h, 0; q)$ denotes the Ramanujan sum and λ_{aq}, λ_{bq} are constants given by

$$\lambda_{aq} = 2\gamma + \log \frac{aq^2}{(a, q)^2}. \tag{4}$$

Theorem 1. *Suppose $a, b \geqq 1$, $(a, b) = 1$, $h \neq 0$ and f satisfies (2). Then we have*

$$D_f(a, b; h) = \int_0^\infty g(x, \pm x \mp h)\,dx + O(P^{5/4}(X + Y)^{1/4}(XY)^{1/4+\varepsilon}), \tag{5}$$

where $g(x, y) = f(x, y)\Lambda_{abh}(x, y)$ and the implied constant depends on ε only.

Remarks. Since the main term has the order of magnitude of $(ab)^{-1}\min(X, Y)$ the result is valuable only if

$$ab < P^{-5/4}(X + Y)^{-5/4}(XY)^{3/4-\varepsilon}.$$

Notice that the error term in (5) does not depend on h but it is a trivial result whenever $|h| > (ab)^{-2}(X + Y)^{3/2}$.

The exponent 5/4 in (5) can be replaced by 3/4 by refining the argument and, we expect, by 1/2 with more elaborate refinements.

Corollary. *For $a, h, M \geqq 1$ we have*

$$\sum_{m \leq M} \tau(m)\tau(am + h) = \int_0^M \lambda(x, ax + h)\,dx + O(a^{1/9}(aM + h)^{2/9}M^{2/3+\varepsilon}), \tag{6}$$

where

$$\lambda(x, y) = \sum_{q=1}^\infty q^{-2}(a, q)c_q(h)\left(\log x - 2\gamma - 2\log\frac{q}{(a, q)}\right)(\log y - 2\gamma - 2\log q). \tag{7}$$

Remark. The error term in (6) is smaller than the main term provided $a < M^{1/3-\varepsilon}$ and $h < a^{-1/2}M^{3/2-\varepsilon}$.

Proof. Apply Theorem 1 for the test function $f(x, y) = f_1(x)f_2(y)$ where f_1, f_2 are single variable functions, smooth, non-negative, supported on $[0, X + XP^{-1}]$, $[0, 2Y]$ respectively, such that

$$f_1(x) = 1 \quad \text{if } 0 \leqq x \leqq X, \quad f_1^{(j)} \ll P^j X^{-j}$$

and

$$f_2(y) = 1 \quad \text{if } 0 \leqq y \leqq Y, \quad f_2^{(j)} \ll Y^{-j}. \tag{8}$$

We take $X = aM$ and $Y = aM + h$, so the sum (6) is majorized by $D_f(a, 1; h)$. Since f satisfies the hypothesis (2), by (5) we get

$$D_f(a, 1; h) = \int + O(P^{5/4}(aM + h)^{1/2}(aM)^{1/4+\varepsilon}).$$

Here the integral differs from that in (6) by $\ll P^{-1}M\log^2 M$. We make the optimal choice $P = a^{-1/9}M^{1/3}(aM + h)^{-2/9}$. This yields the upper bound in (6). The proof of the lower bound is similar.

3. The δ-symbol

Take a smooth, compactly supported function $w(u)$ on R such that $w(u) = w(-u)$ and $w(0) = 0.$. Normalize $w(u)$ by requiring

$$\sum_{q=1}^{\infty} w(q) = 1. \tag{9}$$

Then for any $n \in Z$ we have

$$\delta(n) = \sum_{q|n} \left(w(q) - w\left(\frac{n}{q}\right)\right) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Using additive characters to detect the divisibility $q|n$ we get

$$\delta(n) \sum_{q=1}^{\infty} \sum_{d \,(\mathrm{mod}\, q)}^{*} e\left(\frac{dn}{q}\right) \Delta_q(n), \tag{10}$$

where

$$\Delta_q(u) = \sum_{r=1}^{\infty} (qr)^{-1} \left(w(qr) - w\left(\frac{u}{qr}\right)\right). \tag{11}$$

Lemma 1. *For any $f \in C_0^{\infty}(R)$ we have*

$$\int_{-\infty}^{\infty} f(u)\Delta_q(u)\,\mathrm{d}u = f(0) \int_0^{\infty} w(r)\,\mathrm{d}r$$
$$- q^j \int_0^{\infty} \psi_j\left(\frac{r}{q}\right) \int_{-\infty}^{\infty} \left(f(u)\left(\frac{w(r)}{r}\right)^{(j)} - w(u)u^j f^{(j)}(ru)\right)\mathrm{d}u\,\mathrm{d}r \tag{12}$$

where $j \geqq 1$ and

$$\psi_j(z) = -\sum_{m=1}^{\infty} (2\pi i m)^{-j} (e(mz) + (-1)^j e(-mz)).$$

Proof. We split into two parts and change the variable in the second one getting

$$\int_{-\infty}^{\infty} f(u)\Delta q(u)\,\mathrm{d}u = \int_{-\infty}^{\infty} f(u)\left(\sum_{r=1}^{\infty} (qr)^{-1} w(qr)\right)\mathrm{d}u - \int_{-\infty}^{\infty} w(u)\left(\sum_{r=1}^{\infty} f(qru)\right)\mathrm{d}u.$$

Then we evaluate the sums over r by the Euler–Maclaurin formula (cf. [Ra, p. 14]). For the first sum it gives

$$\sum_{r=1}^{\infty} (qr)^{-1} w(qr) = \int_0^{\infty} \frac{w(qr)}{qr}\,\mathrm{d}r + \int_0^{\infty} \psi_j(r) \frac{\partial^j}{\partial r^j}\left(\frac{w(qr)}{qr}\right)\mathrm{d}r.$$

Since $w(u)$ is even we can write the second part as follows

$$f(0)\int_0^{\infty} w(u)\,\mathrm{d}u - \int_0^{\infty} w(u)\left(\sum_{r=-\infty}^{\infty} f(qru)\right)\mathrm{d}u.$$

Next by the Euler–Maclaurin formula we get

$$\sum_{r=\infty}^{\infty} f(qru) = \int_{-\infty}^{\infty} f(qru)\,\mathrm{d}r + \int_{-\infty}^{\infty} \psi_j(r) \frac{\partial^j}{\partial r^j} f(qru)\,\mathrm{d}r.$$

Combining these formulas we arrive at (12) after an obvious change of variables and observation of the cancellation of the leading integrals.

Now, suppose $w(u)$ is supported in $Q \leqq |u| \leqq 2Q$ and it has derivatives bounded by

$$w^{(j)} \ll Q^{-j-1}, \quad j \geqq 0. \tag{13}$$

Since $|\psi_j(z)| \leqq 1$ the terms on the right side of (12) are bounded by

$$f(0)(1 + O(Q^{-j-1})), \tag{14}$$

$$q^j Q^{-j-1} |\int f(u)\,\mathrm{d}u|, \tag{15}$$

$$q^j Q^{j-1} \int |f^{(j)}(u)|\,\mathrm{d}u, \tag{16}$$

respectively. We also have

$$f(0) \ll \int (|f(u)| + |f^{(j)}(u)|)\,\mathrm{d}u. \tag{17}$$

Then there follows from (12)–(17) the following

Corollary. *Let $j \geqq 1$. We have*

$$\int_{-\infty}^{\infty} f(u)\Delta_q(u)\,\mathrm{d}u = f(0) + O\left(Q^{-1} q^j \int (Q^{-j}|f(u)| + Q^j|f^{(j)}(u)|)\,\mathrm{d}u\right). \tag{18}$$

If $q < Q^{1-\varepsilon}$ this shows that $\Delta_q(u)$ approximates to the Dirac distribution very well on test functions such that $f^{(j)} \ll (qQ^{1+\varepsilon})^{-j}$.

Lemma 2. *We have*

$$\Delta_q(u) \ll (qQ + Q^2)^{-1} + (qQ + |u|)^{-1}. \tag{19}$$

Proof. By the Euler–Maclaurin formula we get

$$\Delta_q(u) = \int_0^{\infty} \left\{\frac{r}{q}\right\} \mathrm{d}\,\frac{w(r) - w(u/r)}{r}, \tag{20}$$

where $\{x\}$ denotes the fractional part of x. Hence we infer by (17) that

$$|\Delta_q(u)| \leqq \int_0^{\infty} \min\left(1, \frac{r}{q}\right)\left|\mathrm{d}\,\frac{w(r) - w(u/r)}{r}\right|$$

$$\leqq \int_0^{\infty} \min\left(1, \frac{2Q}{q}\right)\left|\mathrm{d}\,\frac{w(r)}{r}\right| + \int_0^{\infty} \min\left(\frac{1}{|u|}, \frac{1}{qQ}\right)|\mathrm{d}r\, w(r)|$$

$$\ll \min\left(\frac{1}{Q^2}, \frac{1}{qQ}\right) + \min\left(\frac{1}{|u|}, \frac{1}{qQ}\right)$$

which gives (19).

4. Applying the δ-symbol

We shall present only the case $am - bn = h$ since the other one of $am + bn = h$ is obtained by changing signs in relevant places of our arguments.

Using a smooth partition of unity for the proof of Theorem 1 we may assume that $f(x, y)$ is supported in the box $[X, 2X] \times [Y, 2Y]$. We may also attach to

$f(x, y)$ a redundant factor $\varphi(x-y-h)$, where $\varphi(u)$ is a smooth function supported on $|u| < U$ such that $\varphi(0) = 1$ and $\varphi^{(i)} \ll U^{-i}$. This, of course, does not alter $D_f(a, b; h)$ nevertheless it will help to improve the forthcoming performance by taking U optimally. The new function $F(x, y) = f(x, y)\varphi(x-y-h)$ has partial derivatives bounded by

$$F^{(ij)} \ll \left(\frac{1}{U} + \frac{P}{X}\right)^i \left(\frac{1}{U} + \frac{P}{Y}\right)^j \ll U^{-i-j} \tag{21}$$

provided $U \leqq P^{-1}\min(X, Y)$ which condition we henceforth assume to hold. Next we apply (10) to detect the equation $am - bn = h$. For the test function $w(u)$ we choose $Q = U^{1/2}$, so $\Delta_q(u)$ vanishes if $|u| \leqq U$ and $q \geqq 2Q$. Therefore we get

$$\begin{aligned} D_f(a, b; h) &= D_F(a, b; h) \\ &= \sum_{1 \leq q < 2Q} \sum_{d(\mathrm{mod}\, q)}^{*} e\left(\frac{-dh}{q}\right) \sum_m \sum_n \tau(m)\tau(n) e\left(\frac{dam - dbn}{q}\right) E(m, n), \end{aligned} \tag{22}$$

where $E(x, y) = F(ax, by)\Delta_q(ax - by - h)$.

5. Applying poisson summation

We shall execute the summation over m, n in (22) by means of the following Poisson type formula (cf. Jutila [Ju$_1$], Theorem 1.7)

Proposition 1. *Let $g(x)$ be a smooth, compactly supported function on R^+ and let $(d, q) = 1$. Then we have*

$$\begin{aligned} \sum_{n=1}^{\infty} \tau(n) e\left(\frac{dn}{q}\right) g(n) &= \frac{1}{q} \int \log x + 2\gamma - 2\log q) g(x)\, \mathrm{d}x \\ &\quad + \sum_{+-} \sum_{n=1}^{\infty} \tau(n) e\left(\frac{\pm dn}{q}\right) g^{\pm}(n), \end{aligned}$$

where

$$g^-(y) = -\frac{2\pi}{q} \int g(x) Y_0\left(\frac{4\pi\sqrt{xy}}{q}\right) \mathrm{d}x,$$

$$g^+(y) = \frac{4}{q} \int g(x) K_0\left(\frac{4\pi\sqrt{xy}}{q}\right) \mathrm{d}x,$$

and $Y_0(z)$, $K_0(z)$ are the Bessel functions.

By Proposition 1 applied once to each variable, we get

$$\begin{aligned} \frac{q^2}{(ab, q)} \sum_m \sum_n &= I + \sum_{m=1}^{\infty} \tau(m) e\left(-m\frac{\overline{ad}}{q}\right) I_a(m) + \sum_{n=1}^{\infty} \tau(n) e\left(n\frac{\overline{bd}}{q}\right) I_b(n) \\ &\quad + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tau(m)\tau(n) e\left(-m\frac{\overline{ad}}{q} + n\frac{\overline{bd}}{q}\right) I_{ab}(m, n) + {*}{*}{*}{*}{*}, \end{aligned} \tag{23}$$

where a/q, b/q have to be put in reduced forms before taking the inverses, and

$$I = \int\int (\log ax - \lambda_{aq})((\log by - \lambda_{bq}) E(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$I_a(m) = -2\pi \int\int Y_0\left(\frac{4\pi(a, q)\sqrt{mx}}{q}\right)(\log by - \lambda_{bq}) E(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$I_b(n) = -2\pi \int\int (\log ax - \lambda_{aq})\, Y_0\left(\frac{4\pi(b, q)\sqrt{ny}}{q}\right) E(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$I_{ab}(m, n) = 4\pi^2 \int\int Y_0\left(\frac{4\pi(a, q)\sqrt{mx}}{q}\right) Y_0\left(\frac{4\pi(b, q)\sqrt{ny}}{q}\right) E(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

As it is evident from Proposition 1 there are five more terms ***** in (23) involving the K_0-Bessel function. These can be estimated by the same method as we use for the ones displayed.

Inserting (23) into (22) we get from the summation in $d (\mathrm{mod}\, q)$ complete Kloosterman sums. We obtain the following formula:

$$D(a, b; h) = \sum_{q < 2Q} q^{-2}(ab, q) \Bigg\{ S(h, 0; q) I + \sum_{m=1}^{\infty} \tau(m) S(h.\, \bar{a}m; q) I_a(m)$$
$$+ \sum_{n=1}^{\infty} \tau(n) S(h, -\bar{b}n; q) I_b(n)$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tau(m)\tau(n) S(h, \bar{a}m - \bar{b}n; q) I_{ab}(m, n) + {*}{*}{*}{*}{*} \Bigg\}. \tag{24}$$

To the Kloosterman sums in this formula we shall apply Weil's bound

$$S(h, \bar{a}m - \bar{b}n; q) \ll (h, q)^{1/2} q^{1/2} \tau(q). \tag{25}$$

In case $m = n = 0$ we get the Ramanujan sum for which we have a simple formula and a better bound

$$S(h, 0; q) = \sum_{\nu | (h, q)} \nu \mu\left(\frac{q}{\nu}\right) \ll (h, q). \tag{26}$$

6. Evaluating the main term

First we evaluate the integral I. We have

$$abI = \int\int C(x, y) \Delta_q(x - y - h)\, \mathrm{d}x\, \mathrm{d}y$$
$$= \int\int C(x, x - h + u) \Delta_q(u)\, \mathrm{d}u\, \mathrm{d}x$$

where $C(x, y) = (\log x - \lambda_{aq})(\log y - \lambda_{bq}) F(x, y)$. By (18) and (21) we get

$$\int C(x, x - h + u) \Delta_q(u)\, \mathrm{d}u = C(x, x - h) + O\left(\left(\frac{q}{Q}\right)^j\right).$$

Assuming $q < Q^{1-\varepsilon}$ we make the error term above very small by taking j large. Hence we obtain

$$abI = \int C(x, x-h)\,\mathrm{d}x + O(Q^{-A}).$$

We also have the bound $abI \ll (X+Y)^{-1}XY\log Q$, which is valid for all q, see (30). Therefore the first part of (24) yields

$$(ab)^{-1}\sum_{q=1}^{\infty} q^{-2}(ab,q)c_q(h)\int C(x, x-h)\,\mathrm{d}x + O\left((ab)^{-1}\frac{XY}{X+Y}Q^{-1+\varepsilon}\right) \tag{27}$$

where the error term takes care of the tail $q > Q^{1-\varepsilon}$.

7. Estimating the error term

We need estimates for I_a, I_b, I_{ab}. To this end we integrate by parts in x, y using the bound

$$E^{(ij)} \ll \frac{1}{qQ}\left(\frac{ab}{qQ}\right)^{i+j} \tag{28}$$

and the recurrence formula $(z^\nu Y_\nu(z))' = z^\nu Y_{\nu-1}(z)$. In this way we show that these integrals are very small unless

$$m < aXQ^{-2+\varepsilon}, \qquad n < bYQ^{-2+\varepsilon}. \tag{29}$$

For m, n in this range we estimate the integrals trivially using the bound $Y_0(z) \ll z^{-1/2}$, which gives

$$I_a(m) \ll \left(\frac{aq^2}{mX}\right)^{1/4}(\log Q)\int\int,$$

$$I_b(n) \ll \left(\frac{bq^2}{nY}\right)^{1/4}(\log Q)\int\int,$$

$$I_{ab}(m,n) \ll \left(\frac{abq^4}{mnXY}\right)^{1/4}(\log Q)\int\int,$$

where

$$\begin{aligned}\int\int &= \int\int |F(ax, by)\Delta_q(ax - by - h)|\,\mathrm{d}x\,\mathrm{d}y \\ &= (ab)^{-1}\int\int |F(x, x-h-u)\Delta_q(u)|\,\mathrm{d}x\,\mathrm{d}u \\ &\ll (ab)^{-1}\min(X,Y)\int_{-U}^{U}|\Delta_q(u)|\,\mathrm{d}u \ll (ab)^{-1}(X+Y)^{-1}XY\log Q\end{aligned} \tag{30}$$

by (19). Next summing over m, n in the range (29) we obtain

$$\sum_m \tau(m)|I_a(m)| \ll \frac{q^{1/2}}{b}\frac{X^{3/2}Y}{X+Y}Q^{-3/2+\varepsilon},$$

$$\sum_n \tau(n)|I_b(n)| \ll \frac{q^{1/2}}{a}\frac{XY^{3/2}}{X+Y}Q^{-3/2+\varepsilon},$$

$$\sum_m\sum_n \tau(m)\tau(n)|I_{ab}(m,n)| \ll q\frac{(XY)^{3/2}}{X+Y}Q^{-3+\varepsilon}.$$

Introducing these bounds into (24) we get (5) with the error term

$$(ab)^{-1}\frac{XY}{X+Y}Q^{-1+\varepsilon}+\frac{(XY)^{3/2}}{X+Y}Q^{-5/2+\varepsilon}.$$

On taking $U = Q^2 = P^{-1}(X+Y)^{-1}XY$ the above error term becomes that of (5). This completes the proof of Theorem 1 in case $f(x, y)$ is supported in a dyadic box.

Finally Theorem 1 in its general form is derived from the dyadic version by breaking smoothly the summation in (1) into boxes $[X', 2X'] \times [Y', 2Y']$ and using (2) to determine that the worst case is $X' \sim X$, $Y' \sim Y$.

References

[DeI_1] Deshouillers J.-M., Iwaniec, H: An additive divisor problem. J. London Math. Soc. **26**, 1–14 (1982)

[DeI_2] Deshouillers J.-M. Iwaniec, H: Kloosterman sums and Fourier coefficients of cusp forms. Invent. Math **70**, 219–288 (1982)

[DFI_1] Duke, W., Friedlander, J., Iwaniec, H: Bounds for automorphic L-functions. Invent. Math. **112**, 1–8 (1993)

[DFI_2] Duke, W., Friedlander, J., Iwaniec H: Bounds for automorphic L-functions II Invent. Math. **115**, 219–239 (1994)

[DuI] Duke, W., Iwaniec, H: Convolution L-series (to appear)

[Es] Estermann, T: Über die Darstellung einer Zahl als Differenz von swei Produkten. J. Reine Angew. Math. **164**, 173–182 (1931)

[Ha] Hafner, J.L.: Explicit estimates in the arithmetic theory of Poincaré series, Math. Ann. **264**, 9–20 (1983)

[HB] Heath-Brown, D.R.: The fourth power moment of the Riemann zeta-function, Proc. London Math. Soc. **38**, 385–422 (1979)

[He] Hejhal, D.: Sur certaines séries de Dirichlet dont les pôles sont sur les lignes critiques. CR Acad. Sci. Paris, Sér A **287**, 383–385 (1978)

[In] Ingham, A.E.: Some asymptotic formulae in the theory of numbers. J. London Math. Soc. **2**, 202–208 (1927)

[Ju_1] Jutila, M: A method in the theory of exponential sums, Tata Lect. Notes Math. **80**, Bombay (1987)

[Ju_2] Jutila, M: The additive divisor problem and exponential sums. In: Advances in Number Theory, 113–135. Proc Conf. Kingston Ont., 1991, Oxford (1993)

[Kl] Kloosterman, H.D.: On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$. Acta Math. **49**, 407–464 (1926)

[Ku] Kuznetsov, N.V: Convolution of the Fourier coefficients of the Eisenstein–Maass series. Zap. Nauk Sem. LOMI **129**, 43–84 (1983)

[Mo] Motohashi, Y: The binary additive divisor problem (to appear).

[Ra] Rademacher, H: Topics in Analytic Number Theory, New York: Springer 1973

[Sm] Smith, R.A: The circle problem in an arithmetic progression, Can. Math. Bull. **11**, 175–184 (1968)

[Wa] Watt, N: (preprint).

[Wi] Wirsing, E: (unpublished manuscript).