ON THE ARITHMETIC OF POLYNOMIALS OVER A NUMBER FIELD

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Abstract. Counterparts of several classical results of number theory are proven for the ring of polynomials with coefficients in a number field. A theorem of Milnor that determines the Witt ring of a function field is applied to prove an analogue of Gauss’s principal genus theorem for binary forms with polynomial coefficients. This is used to help understand when and why quadratic reciprocity fails in these polynomial rings. Another application is a count of the number of cyclic subgroups whose order is divisible by four in the primary decomposition of the torsion subgroup of the Jacobian of certain hyperelliptic curves. Invariant theory is applied to prove an analogue of a classical theorem of Fueter to give criteria for an elliptic curve with a polynomial discriminant and zero $j$-invariant to have no affine points over the associated function field.

1. Introduction

Let $k$ be a field. As a Euclidean domain with respect to polynomial degree, the ring $k[t]$ shares some number theoretic properties with the integers $\mathbb{Z}$, since it has a division algorithm and unique factorization into primes (monic irreducibles). When $k$ is finite the basic arithmetic theory of $k[t]$ and extensions of its field of fractions $k(t)$ is very well developed. When $k$ is a number field, this theory is also an active and attractive area of research (see e.g. [37] and its references). The aim of this paper is to prove some new counterparts of several old and very well-known results of classical number theory when $\mathbb{Z}$ is replaced by $k[t]$, where $k$ is a number field.

Given $p, q \in k[t]$ with $p(t)$ prime, write $(\frac{q}{p}) = 1$ if $p \nmid q$ and $q(t)$ is a square modulo $p(t)$. If $k$ is finite, after Artin [2] quadratic reciprocity holds between any pair of distinct primes $p$ and $q$, meaning that

\[(1.1) \quad (\frac{q}{p}) = 1 \quad \text{if and only if} \quad (\frac{p}{q}) = 1,
\]

where $p^* = (-1)^{|p|}p$ with $|p| := \deg p$ (see also [30]). When $k$ is a number field, it is easy to show that (1.1) does not always hold. The first problem we want to address is to characterize those $q$ with odd degree for which $(\frac{q}{p}) = 1$ implies $(\frac{p^*}{q}) = 1$ for all primes $p \neq q$.

Our approach is guided by Gauss’s second proof of classical quadratic reciprocity in [11, Art. 262] (or [12]), which is an application of the genus theory of binary quadratic forms. That proof relies on the finiteness of the class group. For any square-free $D \in k[t]$ of odd degree, let $J_D$ (or $J_D(k)$ if $k$ is not understood) be the Mordell-Weil group of $k$-rational divisor classes of degree zero on the hyperelliptic curve over $k$ determined by $s^2 = D(t)$. Naturally, $J_D$ plays the role of the class group. After Mordell-Weil, $J_D$ is a finitely generated abelian group that might, or might not be, finite.

Theorem 1. Let $k$ be a number field and $q(t) \in k[t]$ be a fixed prime with odd degree. Then the following holds if and only if $J_q$ is finite:

\[(1.2) \quad (\frac{q}{p}) = 1 \quad \text{implies} \quad (\frac{p^*}{q}) = 1 \quad \text{for all primes} \quad p(t) \neq q(t).
\]
In words, whether or not \( \mathcal{J}_q \) is finite determines whether or not (one way) quadratic reciprocity holds for a prime \( q \in k[t] \) of odd degree, when \( k \) is a number field.

It can happen that (1.2) holds for a prime \( q(t) \in k[t] \) of odd degree over \( k \) and all primes \( p \in k[t] \) with \( p \not\equiv q \), and where \( q \) is still irreducible over a finite extension \( k'/k \) but (1.2) fails for some \( p \in k'[t] \). In this case by Theorem 1 the rank of \( \mathcal{J}_q(k) \) is zero while that of \( \mathcal{J}_q(k') \) is at least one.

Example. The polynomial \( q(t) = t^3 + 16 \in \mathbb{Q}[t] \) is a prime of odd degree for which (1.2) holds for any prime \( p \not\equiv q \), since \( s^2 = q(t) \) has rank 0 as an elliptic curve over \( \mathbb{Q} \). Now \( q \) is still irreducible over \( k' = \mathbb{Q}(\sqrt{5}) \) but (1.2) fails for \( p(t) = t - 2(1 + \sqrt{5}) \). This follows since \( q(2 + 2\sqrt{5}) \in k'^2 \) so \( (\frac{2}{p}) = 1 \), but it can be shown that \( 2 + 2\sqrt{5} - \alpha \not\in k'^2 \), where \( \alpha^3 + 16 = 0 \) and \( K = \mathbb{Q}(\sqrt{5}, \alpha) = k'(\alpha) \), so \( (\frac{\sqrt{5}}{q}) \neq 1 \). In fact, the curve \( s^2 = q(t) \) has rank 1 over \( k' \) with infinite part generated by \((2 + 2\sqrt{5}, 8 + 4\sqrt{5})\).

For a fundamental discriminant \( D \in \mathbb{Z} \), the class group of binary quadratic forms over \( \mathbb{Z} \) with discriminant \( D \) can be identified with the (narrow) divisor class group \( H \) of the quadratic field \( \mathbb{Q}(\sqrt{D}) \). For any finitely generated abelian group \( G \), let \( e_n = e_n(G) \) be the number of subgroups of order \( 2^m \), where \( m \geq n \), in a representation of the torsion part of \( G \) as a direct product of cyclic subgroups of prime power order. After Gauss, \( 2^{e_1(H)} \) is the number of (unordered) factorizations \( D = D_1D_2 \) into fundamental discriminants \( D_1, D_2 \). An elegant supplement was published in [29]. It may be viewed as a consequence of the principal genus theorem of Gauss.

Theorem (Redei-Reichardt). There are \( 2^{e_2(H)} \) factorizations \( D = D_1D_2 \) into fundamental discriminants that satisfy \( (\frac{D_2}{p}) = 1 \) for each \( p|D_2 \) and \( (\frac{D_2}{p}) = 1 \) for each \( p|D_1 \).

I will derive a parallel result from a \( k[t] \)-version of the principal genus theorem.

Again, let \( \mathcal{J} = \mathcal{J}_D \) be the Mordell-Weil group of the hyperelliptic curve over a number field \( k \) given by \( s^2 = D(t) \). As is well-known (and also follows from Propositions 5 and 11 below), for \( D \in k[t] \) monic and square-free of odd degree, there are \( 2^{e_1(\mathcal{J})} \) factorizations \( D = D_1D_2 \) into co-prime monic polynomials, where \( D_2 \) has odd degree. Our second theorem determines \( e_2(\mathcal{J}) \).

Theorem 2. Let \( D \in k[t] \) be square-free and monic of odd degree. Then there are \( 2^{e_2(\mathcal{J})} \) factorizations \( D = D_1D_2 \) into monic polynomials each in \( k[t] \) with \( D_2 \) of odd degree, where \( (\frac{D_1}{p}) = 1 \) for each prime \( p|D_2 \) and \( (\frac{D_2}{p}) = 1 \) for each prime \( p|D_1 \).

Example. For any number field \( k \) suppose that \( f \in k[t] \) is monic of degree \( \geq 1 \) and \( f(0) = \alpha \neq 0 \) and that \( D_1(t) := f^2(t) + t \) is square-free. Set \( D_2(t) = t \) and let

\[
D(t) = D_1D_2 = (f^2(t) + t)t.
\]

Then \( D \) is monic, square-free and has odd degree. Clearly \( (\frac{D_1}{p}) = 1 \) since \( D_1(0) = \alpha^2 \in k'^2 \). Also, \( (\frac{1}{p}) = 1 \) for any prime \( p|D_1 \) since

\[
-t \equiv f^2(t) \quad (\text{mod } p(t)).
\]

Hence by Theorem 2 the Mordell-Weil group of the genus \( |f| \) curve \( s^2 = t(f^2(t) + t) \) over \( k \) has points of order 4.

The non-existence of classes of order 3 in the class group \( H \) of \( \mathbb{Q}(\sqrt{D}) \) has a consequence for the rational points on the elliptic curve \( y^2 = x^3 + D \), at least for some \( D \). The next classical result follows from a more general statement proven in [10].
Theorem (Fueter). Suppose that $D \neq 1$ is square-free. Then the equation $y^2 = x^3 + D$ either has an infinity of solutions $x, y \in \mathbb{Q}$ or it has none. It has none if $D < 0$ satisfies $D \equiv 2 \pmod{9}$, $D \neq 1 \pmod{4}$ and $H$ has no elements of order 3.

I will give an analogue of this result as well for the rational points on the elliptic curve over $k(t)$ determined by $y^2 = x^3 + D(t)$, where $D \in k[t]$ is square-free and non-constant. This equation can also be considered to give an elliptic surface over $k$ where, in general, it has many more points. For $D \in k[t]$ let

$$D'(t) = -27D(-3t).$$

Theorem 3. Suppose that $D \in k[t]$ is square-free and non-constant, where $k$ is a number field. Then either the equation

$$y^2 = x^3 + D(t)$$

has an infinity of solutions $x, y \in k(t)$ or it has none. It has none if $D$ is monic of odd degree and neither $J_D$ nor $J_{D'}$ has points of order 3.

Examples. i) Suppose that $a, b \in k$ satisfy $4a^3 + 27b^2 \neq 0$ and that

$$(1.4) \quad 3t^4 + 6at^2 + 12bt - a^2 = 0$$

has no roots $t \in k$. Then the equation

$$(1.5) \quad y^2 = x^3 + t^3 + at + b$$

has no solutions $x, y \in k(t)$. To see this, note that (1.4) determines the $t$-coordinates of points of order 3 on the elliptic curve $J_D : s^2 = t^3 + at + b$ and the change of variables $t \mapsto -\frac{t}{3}$ gives the corresponding equation for $s^2 = t^3 + 9at - 27b$, which is isomorphic to $J_{D'}$. Thus, if (1.4) has no rational roots then neither $J_D$ nor $J_{D'}$ has rational points of order 3 and we may apply Theorem 3. Suppose that $k = \mathbb{Q}$. Using (1.4), an easy counting argument shows that the number $N(X)$ of admissible $(a, b) \in \mathbb{Z}$ with

$$\max(4|a^3|, 27|b^2|) \leq X$$

and such that (1.5) has solutions $x, y \in \mathbb{Q}(t)$, satisfies $N(X) \ll X^{\frac{1}{2}}$. Therefore equations (1.5) with $a, b \in \mathbb{Z}$ and solutions $(x, y) \in \mathbb{Q}(t)$ are rare.

ii) The curve $y^2 = x^3 + t^3 - 3$ with $D(t) = t^3 - 3$ over $\mathbb{Q}(t)$ contains the point $(\frac{4-t^3}{t^2}, \frac{3t^3-8}{t^3})$. Now $J_D$ has trivial torsion, while $J_{D'}$ has points of order 3. Thus it is not enough in the second statement of Theorem 3 to assume only that $J_D(k)$ has no points of order 3 to conclude that $y^2 = x^3 + D(t)$ has no solutions. Similarly, it is not enough to assume only that $J_{D'}(k)$ has no points of order 3.

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2. Overview and discussion of proofs

To prove the theorems of this paper, binary forms with polynomial coefficients provide an efficient and convenient tool. The next section §3 ends with the proofs of Theorems 1 and 2. We use a set of binary quadratic forms with coefficients in $k[t]$ that are analogous to positive definite binary quadratic forms with integral coefficients. They split into a set of classes $C_D$ of forms with discriminant $D$. An adaptation of Dirichlet-Dedekind’s version of Gaussian composition makes $C_D$ into an abelian group. Following Gauss, we count classes of order two in terms of ambiguous forms. We define genus characters using certain well-known Steinberg
symbols studied in [21]. In the classical case, to prove quadratic reciprocity it is enough to
know, in addition to the number of classes of order two, that the class number is finite
and that a genus character is trivial when evaluated at a square. For us, the corresponding
argument is enough to prove the “if” part of Theorem 1. We actually prove the theorems
with $\mathcal{J}_D(k)$ replaced by $\mathcal{C}_D$ in the statements, and later show that $\mathcal{J}_D(k)$ is isomorphic to
$\mathcal{C}_D$.

For the proof of the converse we need an analogue of Gauss’s principal genus theorem.
This result identifies $\mathcal{C}_D^2$ with the classes killed by all genus characters. Here a new difficulty
arises since there are infinitely many polynomials of bounded degree. The fact that there are
only finitely many integers of bounded norm is behind most proofs of the classical principal
genus theorem, such as Gauss’s, that uses reduction of indefinite ternary quadratic forms,
and proofs using $L$-functions or Legendre’s theorem. To overcome this difficulty, we apply
the determination of the Witt ring of $k(t)$ given by Milnor [22] and formulated in Proposition
8. This result implies a kind of Hasse principle (see Proposition 9) that allows us to identify
genera with rational classes (i.e. $k(t)$-classes) of forms and prove the genus theorem. It is
to be observed that the theorem of Milnor is not formulated in terms of local fields but in
terms of their residue fields, which for us are number fields.

The proof of the “only if” part of Theorem 1 also requires that if $\mathcal{C}_D$ is infinite, it must
contain a class that is not a square. For this it is enough to know that $\mathcal{C}_D$ is finitely generated,
which is a consequence of the Mordell-Weil theorem. Toward the end of §3 we show that
$\mathcal{C}_D$ is isomorphic to the Mordell-Weil group $\mathcal{J}_D$. This result is based in part on work of
Jacobi [16], Mumford [26] and David Cantor [4]. Cantor translated divisor arithmetic into
computations with polynomials that also occur in the composition formulas. His algorithms
have been extensively applied when $k$ is a finite field. For my purpose, I need to make
the relation between divisor classes and the classes of binary quadratic forms that I have
defined, completely explicit. In terms of quadratic forms, the weak Mordell-Weil theorem is
equivalent to the finiteness of the number of genera. In view of this, the result we give in
Proposition 7 that forms are in the same genus if and only if they are rationally equivalent,
is of independent interest.

A third ingredient in the proof of the “only if” part of Theorem 1 is an application of
Hilbert’s irreducibility theorem over $k$. Thanks to this result, it is not hard to make a binary
quadratic form in $\mathbb{Q}_D$ represent primes.

Concerning Theorem 2, in view of [11, Art. 305-307] it seems likely that Gauss was aware
that the principal genus theorem gives information about elements of order 4 in the class
group of binary forms over $\mathbb{Z}$, along the lines of the theorem of Redei and Reichardt that we
stated. Such a proof is carried out in [34, p.163]. Once we have the principal genus theorem
over $k[t]$ and have characterized the classes of order two, Theorem 2 follows similarly.

The proof of the second statement in Theorem 3 is based on the theory of binary cubic
forms with coefficients in $k[t]$; the aspects of this theory that we need are given in §4. That
binary cubic forms over $\mathbb{Z}$ give information about classes of binary quadratic forms of order
3 was discovered by Eisenstein [9]. We make essential use of a $k(t)$-version of a familiar pair
of dual 3-isogenies connecting the elliptic curves $E_D$ and $E_{-27D}$, one that was employed by
Fueter (see [10, p.71]). The non-existence part of Fueter’s theorem was extended and parts
of his proof simplified by Mordell (see [24] and [25, Chap. 26]). These proofs of Fueter and
Mordell use algebraic number theory. As will become clear, it is also possible to prove such
results using classical binary cubic forms. Well-known results from algebraic number theory allow one to assume in the statement of such theorems over \( \mathbb{Q} \) the class number condition for only one of \( \mathbb{Q}(\sqrt{D}) \) and \( \mathbb{Q}(\sqrt{-3D}) \) (c.f. [32]). The corresponding simplification can not be made in general in Theorem 3, as Example ii) below its statement shows.

A consequence of some parts of the proof of Theorem 3 is the finiteness of the number of classes of binary cubic forms with coefficients in \( k[t] \) and with a fixed discriminant \( 4D \), where \( D \) is monic, square-free with odd degree (see Proposition 15).

The changes needed to include discriminants \( D \in k[t] \) with even degree in our results lead to interesting problems that are not addressed in this paper.

3. **Binary quadratic forms over \( k[t] \)**

The theory of binary quadratic forms over \( k[t] \) can be developed in a way that is quite similar to that over \( \mathbb{Z} \), especially when the discriminant has odd degree, where the forms are similar to positive definite ones.

I assume throughout that \( k \) is a fixed number field. However, except for the application of the Mordell-Weil theorem to \( \mathcal{O}_D \), most of the results presented here remain valid when \( k \) is any perfect field of characteristic not 2.

**Preliminaries.** The gcd of a finite set of polynomials in \( k[t] \) is the monic polynomial of largest degree that divides each. Two polynomials are coprime if their gcd is 1. If \( c = \text{gcd}(a, b) \) then there are \( x, y \in k[t] \) with \( c = ax + by \). For \( a \in k[t] \) the inequality \( |a| < 0 \) is equivalent to \( a = 0 \).

Suppose that \( D \in k[t] \) is square-free. Associated to \( Q = (a \ b) \) with \( a, b, c \in k[t] \) is the binary quadratic form

\[
Q(x, y) = (a, b, c) = a(t)x^2 + 2b(t)xy + c(t)y^2
\]

with discriminant (= Gaussian determinant) \( D = b^2 - ac \). Note that any form \( Q = (a, b, c) \) with \( b^2 - ac = D \) is primitive in that \( \text{gcd}(a, b, c) = 1 \). Let \( M \in \text{SL}_2(k[t]) \) act on \( Q \) by

\[
Q|M := MQM'.
\]

For \( M = \begin{pmatrix} m & n \\ n & r \end{pmatrix} \) we have \( Q|M(x, y) = Q(mx + ny, rx + sy) = Q'(x, y) \) and the coefficients of \( Q' = (a', b', c') \) are given by

\[
\begin{align*}
a' &= am^2 + 2bmn + cn^2 \\
b' &= amr + b(ms + rn) + cnr \\
c' &= ar^2 + 2brs + cs^2.
\end{align*}
\]

Clearly \( b^2 - a'c' = D \). The two forms \( Q \) and \( Q' \) are said to be equivalent, written \( Q \sim Q' \).

Say that \( n \in k[t] \) is represented by \( Q \) if \( Q(x, y) = n \) for \( x, y \in k[t] \) and that \( n \) is properly represented by \( Q \) if \( Q(x, y) = n \) for coprime \( x, y \in k[t] \). Equivalent forms represent and properly represent the same polynomials.

The integral version of the following useful result was given by Dirichlet [8, §60]. The proof for \( k[t] \) is similar.

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1On the other hand, it would be natural give a proof of the second statement of Theorem 3 along the lines of those of Fueter and Mordell using quadratic extensions of \( k(t) \). In fact, Zannier has kindly shown me such a proof of the first statement.
Lemma 1. Let $Q = (a, b, c)$ and $Q' = (a', b', c')$, both with discriminant $D$, be given. They are equivalent if there exist $m, n, r, s \in k[t]$ with
\begin{equation}
\begin{aligned}
a' &= am^2 + 2bmn + cn^2, \\
(b + b')n + am &= a's \quad \text{and} \\
(b' - b)m - cn &= a'r,
\end{aligned}
\end{equation}
in which case $Q|M = Q'$ with $M = (\begin{pmatrix} m & r \\ n & s \end{pmatrix}) \in \text{SL}_2(k[t])$. Conversely, if $Q, Q'$ satisfy $Q' = Q|M$ with this $M$, then (3.3) holds.

A proof of the next standard result may be found in [27, Thm VII.3]).

Lemma 2. $\text{SL}_2(k[t])$ is generated by 
\begin{align*}
C_\alpha &= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \\
T_n &= \begin{pmatrix} 1 & n(t) \\ 0 & 1 \end{pmatrix} \quad \text{and} \\
S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{align*}
for $n \in k[t]$ and $\alpha \in k^\times$.

We have the formulas
\begin{align}
(a, b, c)|C_\alpha &= (\alpha^2 a, b, \alpha^{-2} c) \\
(a, b, c)|T_n &= (a, b + na, c + 2nb + n^2 a) \\
(a, b, c)|S &= (c, -b, a).
\end{align}

The following Lemma is easily verified.

Lemma 3. Suppose that $D \in k[t]$ is square-free and that $Q = (a, b, c)$ has $D = b^2 - ac$. The group of automorphs of $Q$, that is those $M \in \text{SL}_2(k[t])$ with $Q|M = Q$, are all of the form
\begin{equation}
M = \pm \begin{pmatrix} x - by & ay \\ cy & x + by \end{pmatrix},
\end{equation}
for $x, y \in k[t]$ solutions of the Pell equation
\begin{equation}
x^2 - D(t)y^2 = 1.
\end{equation}
Transformations $M_1$ with $Q|M_1 = (-a, b, -c)$ exist if and only if the negative Pell equation
\begin{equation}
x^2 - D(t)y^2 = -1
\end{equation}
has solutions, in which case each such $M_1$ is given by
\begin{equation}
M_1 = \pm \begin{pmatrix} x - by & ay \\ cy & x + by \end{pmatrix}.
\end{equation}
For a solution $x, y$ of (3.7), the transformation $M$ from (3.5) has $\det M = -1$ and satisfies $Q|M = -Q$.

Solutions with $y \neq 0$ to (3.6) can exist only for even degree $D$ and then only in special cases. The problem of solving polynomial Pell equations has a long history going back to Abel and Chebyshev. See [28], [39] for more information.

Lemma 4. Let $Q$ have discriminant $D$.

i) If $a' \in k[t]$ is properly represented by $Q$ then there exists $Q' = (a', b', c')$ that is equivalent to $Q$, where $|b'| < |a'|$.

ii) $Q$ can properly represent a polynomial $n \in k[t]$ with $|n| \leq \frac{1}{2}|D|$.

iii) $Q$ can properly represent some $n \in k[t]$ that is prime to any fixed $f \in k[t]$.
Proof. i) If \( Q(m, n) = a' \) where \( m, n \) are coprime then there exist \( r, s \in k[t] \) with
\[
M = \binom{n \ r \ s}{m} \in \text{SL}_2(k[t]).
\]
Thus \( Q|M = (a', *, *) \). Using \( T_f \) from (3.4) with an appropriate \( f \) found by the division algorithm, we can force \( |b'| < |a'| \).

ii) Let \( n \in k[t] \) be a polynomial of minimal degree represented by \( Q \). The representation is proper. By i) we may assume that \( Q = (n, b, c) \), where \( |b| < |n| \). Since \( c \) is properly represented by \( Q \) we must have \( |b| < |n| \leq |c| \), which implies that \( 2|n| \leq |nc| = |D| \).

iii) Let \( p_1 \in k[t] \) be the product of all primes dividing \( a, c \) and \( f \). Let \( p_2 \) be the product of all primes dividing \( a \) and \( f \) but not \( c \). Let \( p_3 \) be the product of all primes dividing \( c \) and \( f \) but not \( a \). Let \( p_4 \) be the product of all remaining primes dividing \( f \). Then
\[
n = Q(p_2, p_3, p_4) = a(p_2)^2 + 2b(p_2)(p_3p_4) + c(p_3p_4)^2
\]
is prime to \( f \) and \( \gcd(p_2, p_3p_4) = 1 \). \( \Box \)

Forms with a negative discriminant. The following definition streamlines the formulation of theorems and is suggestive, if unconventional.

Definition 1. Say that \( n \in k[t] \) is positive (and write \( n > 0 \)) if the leading coefficient of \( n^* = (-1)^{|n|}n \) is in \( k^2 \) and negative \( (n < 0) \) if \( -n \) is positive. Here \( |n| = \deg n \).

The product of two positive or two negative polynomials is positive and the product of one positive and one negative polynomial is negative.

Definition 2. When \( D \in k[t] \) is negative and square-free with \( |D| \) odd, let \( Q_D \) denote the set of those \( Q = (a, b, c) \) with discriminant \( D \), for which \( a > 0 \).

Observe that \( Q_D \) is non-empty, since it contains the principal form \((1, 0, -D)\). The following lemma shows that the action \( Q \mapsto Q|M \) splits \( Q_D \) into a set of equivalence classes and that the forms in \( Q_D \) are similar to positive definite integral forms.

Lemma 5. Suppose that \( D \in k[t] \) is negative and square-free with \( |D| \) odd. If \( Q = (a, b, c) \in Q_D \), then \( Q|M \in Q_D \) for \( M \in \text{SL}_2(k[t]) \). In particular, any \( n \) properly represented by \( Q \) is positive.

Proof. From \( D = b^2 - ac \) and \( a > 0 \), we will show that \( c > 0 \). Clearly \( c \neq 0 \). Next use that the leading coefficient of \( D \) is in \( k^2 \) with \( |D| \) odd to check the different combinations of parities of the degrees of \( a \) and \( c \). If they are both even or both odd the leading terms of \( b^2 \) and \( ac \) must cancel. Otherwise the leading term of \(-ac\) must be in \( k^2 \).

The first statement for any \( M \) now follows from Lemma 2 and the formulas (3.4). The second statement follows from the first and i) of Lemma 4. \( \Box \)

Definition 3. For \( D \in k[t] \) negative and square-free with \( |D| \) odd, let \( C_D \) be the set of all \( \text{SL}_2(k[t]) \)-classes of forms \( Q \in Q_D \).

We have the following analogue of Lagrangian reduction.

Proposition 1. Each class \( C \in C_D \) contains a unique form \( Q = (a, b, c) \) with
\[
|b| < |a| < \frac{1}{2}|D|,
\]
where \( a^* \) is monic.
Proposition 2. Let $\mathcal{C}_n$ from (3.4) to show that the product of classes is well-defined. Together with this and Proposition 2, arguments of Gauss [12, Art. 237–239] showing that

Together with this and Proposition 2, arguments of Gauss [12, Art. 237–239] showing that

By completing the square,

$$ad' = (am + bn)^2 - Dn^2;$$

where $Q' = Q|M$ with $M = \left( \begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix} \right) \in SL_2(k[t])$. Therefore

$$|D| > |a| + |a'| = |(am + bn)^2 - Dn^2|.$$ 

This implies that $n = 0$ and so from the first equation of (3.2) we have that $am^2 = a'$. By symmetry we deduce that $m = s = 1$ and $a = a'$. The second equation of (3.2) yields $b' = b + ra$, which implies that $r = 0$ since

$$|b' - b| < |a'| = |a|.$$ 

Thus $Q = Q'$.

Class group and ambiguous classes. A straightforward modification of Dirichlet and Dedekind's [8] version of Gaussian composition can be defined for classes in $\mathcal{C}_D$, making it into an abelian group. We do, however, have to account for our condition on $(a, b, c) \in \mathcal{Q}_D$ that $a > 0$, but this leads to no problem since if $a > 0$ and $a' > 0$ then $aa' > 0$. Say two forms $Q = (a, b, c), Q' = (a', b', c') \in \mathcal{Q}_D$ are concordant if

$$\gcd(a, a', b + b') = 1.$$ 

By Lemma 4, any pair of classes will contain a pair of concordant forms. The following result can be proven by adapting §145–§148 of [8] (see also [34, pp.129–132]). Use is made of Lemma 1 to show that the product of classes is well-defined.

Proposition 2. Let $D(t) \in k[t]$ be negative, square-free with $|D|$ odd and $C, C' \in \mathcal{C}_D$.

i) If $(a, b, c) \in C$ and $(a', b', c') \in C'$ are concordant, there exist $e, f \in k[t]$ such that $(a, e, a'f) \in C$ and $(a', e, af) \in C'$.

ii) The class $C'' \in \mathcal{C}_D$ of $(aa', e, f)$ is well-defined and makes $\mathcal{C}_D$ into an abelian group through $C \otimes C' = C''$, where the identity is the class containing $(1, 0, -D)$ and the inverse $C^{-1}$ of the class $C$ is the class of $(a, -b, c)$.

Letting $Q = (a, e, a'f), Q' = (a', e, af)$ and $Q'' = (aa', e, f)$ we have that $Q''$ is directly composed of $Q$ and $Q'$ in the sense of Gauss [12, Art. 235]:

$$Q(x, y)Q'(x, y) = Q''(X, Y),$$

where $(X, Y) = (xx' - fyy', axy' + a'x'y + 2eyy').$

Together with this and Proposition 2, arguments of Gauss [12, Art. 237–239] showing that composition of classes is well-defined adapt to give the following.

Proposition 3. Suppose that $Q \in C$ and $Q' \in C'$ where $C, C' \in \mathcal{C}_D$. If

$$Q''(X, Y) = Q(x, y)Q'(x', y') \quad \text{with}$$

$$(X, Y) = (a_1xx' + b_1xy' + c_1yx' + d_1yy', a_2xx' + b_2xy' + c_2yx' + d_2yy'),$$

where $a_j, b_j, c_j, d_j \in k[t]$ for $j = 1, 2$ are such that

$$a_1b_2 - a_2b_1 = Q(1, 0) \quad \text{and} \quad a_1c_2 - a_2c_1 = Q'(1, 0),$$

then the class $C''$ of $Q''$ equals $C \otimes C'$.

An important consequence of Proposition 2 is the following result.
Proposition 4. Suppose that $Q = (a, b, c) \in C$ and $Q' = (a', b', c') \in C'$, where $C, C' \in \mathcal{C}_D$ are not necessarily distinct.

i) If $a$ and $a'$ are coprime, then any form in $C \otimes C'$ will properly represent $aa'$.

ii) If $a$ is prime to $D$, then any form in $C \otimes C$ will properly represent $a^2$.

Proof. For i), observe that $Q$ and $Q'$ are then concordant. For ii), if $a$ is prime to $D$ then it is prime to $b$ and so $Q$ and $Q'$ are concordant.

The following easily proven result shows that for our purposes we may assume if convenient that a negative $D$ is monic.

Lemma 6. For any fixed $\alpha \in k^*$, the map $(a, b, c) \mapsto (a, \alpha b, \alpha^2 c)$ defines a group isomorphism from $\mathcal{C}_D$ to $\mathcal{C}_D^\alpha$.

By an ambiguous form is meant a form of type $(a, 0, c)$. Note: over $k[t]$ we need not consider non-diagonal ambiguous forms, which are necessary over $\mathbb{Z}$. A class $C \in \mathcal{C}_D$ is called ambiguous if it has order 1 or 2.

Lemma 7. A class of $\mathcal{C}_D$ is ambiguous if and only if it contains an ambiguous form.

Proof. Clearly the class of $(a, 0, c)$ has order 1 or 2. For the converse, we adapt an argument of Georg Cantor [5] from the classical case. Suppose that the square of the class of $Q = (a, b, c)$ is the identity. What is the same, $(a, b, c)$ is equivalent to $(a, -b, c)$. By Lemma 1 there are $m, n, r, s \in k[t]$ such that $ms - nr = 1$ and where

\begin{equation}
\begin{aligned}
a &= am^2 + 2bmn + cn^2, \\
am &= as, \\
ar &= 2bm + cn.
\end{aligned}
\end{equation}

It follows that $m = s$ so from $ms - nr = 1$ we see that $Q' = (r, m, n)$ has discriminant 1. By Proposition 4, $Q'$ is equivalent to $(\alpha, 0, -\alpha^{-1})$ for some $\alpha \in k^*$, which is equivalent to

\begin{equation}
(0, -1, 0) = (\alpha, 0, -\alpha^{-1}) \left( \frac{1}{\alpha} \right)^{1/2}.
\end{equation}

Hence there will exist $N = (\frac{a_1}{a_3} \frac{a_2}{a_4}) \in \text{SL}_2(k[t])$ with $(0, -1, 0)|N = (r, m, n)$. Thus

\begin{equation}
\begin{aligned}
r &= -2a_1a_3 \\
m &= -a_1a_4 - a_2a_3 \\
n &= -2a_2a_4.
\end{aligned}
\end{equation}

Finally, using these identities with the middle equation of (3.2) we get

\begin{equation}
(a, b, c)|N^t = (*, -2bm - cn - ar, *),
\end{equation}

so the last equation of (3.9) shows that $(a, b, c)|N^t = (*, 0, *)$ is ambiguous.

For $n \in k[t]$, let $\omega(n)$ be the number of distinct primes dividing $n$.

Proposition 5. The subgroup of $\mathcal{C}_D$ of elements of order at most two has order $2^{\omega(D) - 1}$.

Proof. By Proposition 1 and Lemma 7, an ambiguous class of $\mathcal{C}_D$ contains a unique ambiguous form $(a, 0, c)$ where $|a| < |c|$ and $a^*$ is monic. The number of such forms is $2^{\omega(D) - 1}$.

Principal genus theorem. In this subsection I will prove the $k[t]$ version of the principal genus theorem. We will use a Hilbert-type Steinberg symbol to define “genus characters.” A reference with proofs of those properties we give is [22, p.98]. The field extension of $k$ determined by $F_p = k[t]/(p)$ for a prime $p \in k[t]$, has degree $|p|$ and can be identified with $k(\alpha)$, where $\alpha \in \bar{k}$ has $p(\alpha) = 0$, where $\bar{k}$ is the algebraic closure of $k$. Define for $a, b \in k(t)^*$

\begin{equation}
(a, b)_p = (-1)^{v_p(a)v_p(b)}a^{v_p(b)}b^{v_p(a)} \in F_p^*/F_p^{*2},
\end{equation}

where $v_p(a)$ is determined by $a = p^{v_p(a)}u$ with $u$ prime to $p$. Also, set

\begin{equation}
(a, b)_\infty = (-1)^{|a||b|}a^{|b|}b^{|a|} \in k^*/k^{*2}.
\end{equation}
For any \( p \) (inc. \( p = \infty \)) and \( a, b \in k(t)^* \) the following hold

\[
\begin{align*}
(a, b)_p &= (b, a)_p \\
(ab, c)_p &= (a, c)_p(b, c)_p \\
(a, 1 - a)_p &= 1, \; a \neq 1 \\
(a, b^2)_p &= 1.
\end{align*}
\]

We also have the product formula

\[
(a, b)_\infty \prod_p N_{F_p/k}(a, b)_p = 1,
\]

where \( N_{F_p/k} : F_p \to k \) is the norm.

For any \( p \) including \( p = \infty \) define the genus “character” \( \chi_p : Q_D \to F_p^*/F_p^{*2} \) by

\[
\chi_p(Q) = (n, D)_p,
\]

where \( n \) is any polynomial prime to \( p \) that is properly represented by \( Q \).

**Lemma 8.** The value of \( \chi_p(Q) \) is well-defined; it is independent of the \( n \) chosen. In addition, \( \chi_p(Q) = 1 \) for any \( p \nmid d \), including \( p = \infty \).

**Proof.** We have the identity\(^2\)

\[
Q(x, y)Q(x', y') = (axx' + b(xy' + x'y) + cyy')^2 - D(t)(xy' - x'y)^2.
\]

Suppose that \( u \) and \( u' \) are prime to \( D \) and properly represented by \( Q \). For \( p \mid D \) it follows from (3.18), (3.15) and (3.13) that \( 1 = (uu', D)_p = (n, D)_p(u', D)_p \). Suppose that \( p \mid n \).

By Lemmas 4 i) and 5, we have that \( Q \) is properly equivalent to some \( (n, b, c) \in Q_D \), so \( D = b^2 - nc \) and \( D \) is a square modulo \( p \). Thus \( (n, D)_p = 1 \). For \( p = \infty \), since \( n > 0 \), we have that \( (n, D)_\infty = 1 \) from (3.11). For any other \( p \), that \( (n, D)_p = 1 \) is immediate from the definition. \( \square \)

Each \( \chi_p \) is well defined on the class of \( Q \). It induces a homomorphism from \( C_D \) to \( F_p^*/F_p^{*2} \) by (3.13) and i) of Proposition 4 together with Lemma 4. The intersection of the kernels of \( \chi_p \) for \( p \mid D \) will be called the principal genus and the cosets of the principal genus the genera.

Let \( G_D = G_D(k) \) denote the group of genera. The principal genus is that genus containing \((1, 0, -D)\). Thus we have

**Proposition 6.** The genera \( G_D \) comprise an abelian group in which every non-identity element is of order two. Each genus \( G \) is characterized by the values \( \chi_p(Q) \) for \( p \mid d \), where \( Q \) is any form in \( G \).

It follows from (3.16) that the characters are not independent but satisfy

\[
\prod_{p \mid D} N_{F_p/k}(\chi_p(Q)) = 1.
\]

\(^2\)This is not a direct composition; it fails to satisfy the last conditions given in Proposition 3.
Remark. The product formula (3.16) is equivalent to Weil reciprocity for $k(t)$. For $a = p$ and $b = q$ with $p, q$ distinct primes in $k[t]$, it’s truth is obvious when it is written in the form

$$(-1)^{b||q} \prod_{\beta; q(\beta)=0} p(\beta) = \prod_{\alpha; p(\alpha)=0} q(\alpha).$$

Weil reciprocity can be used to prove quadratic reciprocity (1.1) when $k$ is finite (see [30]).

Say two forms $Q, Q’ \in Q_D$ are rationally equivalent if (3.1) holds for some $M \in \text{GL}_2(k(t))$. Such an $M$ must have $\det M = \pm 1$.

Proposition 7. Two forms $Q, Q’ \in Q_D$ are in the same genus if and only if they are rationally equivalent.

The proof of the corresponding result over $\mathbb{Z}$ given in [3] is based on the Hasse-Minkowski theorem in the important special case given by Legendre. Most proofs of Legendre’s theorem are based on some sort of reduction method and rely on the fact that there are only finitely many integers of a bounded absolute value. In order to prove Proposition 7 we must use a method that is different. The result we need is a determination of the Witt ring of $k(t)$ [36], obtained in the form we need by Milnor [21] (c.f.[17]). A useful exposition of this result can be found in [18]. I will give here a consequence that is easy to apply.

For a field $F$ let $W(F)$ be its Witt ring. See [18, Chap. II] for its definition and basic properties (assuming that the characteristic of $F$ is not 2). We define a set of maps between Witt rings as follows. For each prime $p \in k[t]$ let

$$\psi(p) : W(k(t)) \to W(F_p)$$

be defined through

$$\psi(p)(\langle a \rangle) = \begin{cases} \langle \bar{u} \rangle, & \text{if } \ell \text{ is odd} \\ 0, & \text{otherwise,} \end{cases}$$

where $a = p^\ell u$ with $v_p(u) = 0$.

Also, define $\psi(\infty) : W(k(t)) \to W(k)$ by

$$\psi(\infty)(\langle a \rangle) = \begin{cases} \langle \bar{a}_0 \rangle, & \text{if } \ell \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

when $a(t^{-1}) = t^{-\ell}(a_0 + a_1 t + \ldots)$.

Then the following result is a consequence of [18, Milnor’s Thm. p. 306]:

Proposition 8. The map $\psi = \psi(\infty) \times \prod_p \psi(p)$ induces an isomorphism

$$W(k(t)) \to W(k) \times \prod_{p < \infty} W(F_p).$$

Proposition 7 follows from the next proposition and the definition of genera from the previous subsection.

Proposition 9. Let $D \in k[t]$ be square-free and negative with $|D|$ odd. Suppose that the forms $Q = (a, b, c)$ and $Q’ = (a’, b’, c’)$ are in $Q_D$ with $a, a’$ prime to $D$. Then $Q$ and $Q’$ are equivalent over $k(t)$ if and only if

$$(a, D)_p = (a’, D)_p$$

for all $p|D$. 
Proof. Each \( Q = (a, b, c) \in \mathcal{Q}_d \) is equivalent over \( k(t) \) to \( \langle a, -\frac{D}{a} \rangle := (a, 0, -\frac{D}{a}) \). We will use Proposition 8 to show that \( \langle a, -aD \rangle \) and \( \langle a', -a'D \rangle \) are Witt-equivalent if and only if \( (a, D)_p = (a', D)_p \) for all \( p \). This is enough since as forms over \( k(t) \) they have the same dimension, namely 2 (see e.g. [18, Prop.1.4, p.29]).

For \( p < \infty \) we have

\[
(3.21) \quad \psi(p)\langle a, -aD \rangle = \begin{cases} \langle -\bar{a}a \rangle, & D = pu \\ 0, & \text{otherwise} \end{cases} \quad (a, D)_p = \begin{cases} \bar{a}, & p|D \\ 1, & \text{otherwise}. \end{cases}
\]

In both statements we use that \( D \) is a square modulo any \( p \) with \( p|a \); in the first \( \psi(p)\langle a, -aD \rangle \) is isotropic and in the second \( (a, D)_p = 1 \). Note that \( u \) in (3.21) only depends on \( D \). Also, we have

\[
(3.22) \quad \psi(\infty)\langle a, -aD \rangle = \langle 1 \rangle, \quad (a, D)_\infty = 1.
\]

Here the first statement follows using \( a \succ 0 \) and checking each case when \( |a| \) is odd or even, referring to (3.20). That \( (a, D)_\infty = 1 \) was shown in Lemma 8. Comparison of both sides of (3.21) and (3.22) for each prime \( p \) and \( \infty \), together with Proposition 8, completes the proof. \( \square \)

Now we apply Proposition 7 to prove the following analogue of Gauss’s principal genus theorem.

**Proposition 10.** The principal genus coincides with \( \mathcal{C}_D^2 \) and hence \( \mathcal{G}_D = \mathcal{C}_D/\mathcal{C}_D^2 \).

**Proof.** First suppose that the class of \( Q \) is the square of the class of \( Q' = (a', b', c') \), where \( a' \) is chosen to be prime to \( D \). By ii) of Proposition 4 we have that \( Q \) properly represents \( a'^2 \) and so \( \chi_p(Q) = 1 \) for all \( p|D \), hence \( Q \) is in the principal genus.

Conversely, suppose that \( Q = (a, b, c) \) is in the principal genus. By Proposition 7 we know that \( Q \) is rationally equivalent to \( (1, 0, -D) \), so there are \( q_1, q_2, r \in k[t] \) with \( r \neq 0 \) such that

\[
a(\frac{q_1}{q_2})^2 + 2b(\frac{q_1}{q_2})(\frac{q_2}{p}) + c(\frac{q_2}{p})^2 = 1.
\]

Hence \( Q \) represents \( r^2 \), but perhaps improperly. In any case, by Lemma 4 there are \( s, b', c' \in k[t] \) with \( s \succ 0 \) so that \( Q \) is properly equivalent to \( (s^2, b', c') \) where \( D = b'^2 - s^2c' \). If \( b' \) and \( s \) share a prime factor \( q \), then \( q^2 \) divides both \( s^2 \) and \( b'^2 \) so must also divide \( D \). But \( D \) is square-free. Thus \( s \) is prime to \( b' \). By Proposition 2, the form \( Q' = (s, b', c's) \in \mathcal{Q}_D \) will have the property that the square of its class is the class of \( Q \). \( \square \)

**The class group \( \mathcal{C}_D \) and the Mordell-Weil group \( \mathcal{J}_D \).** Let \( D \in k[t] \) be square-free and negative with odd degree \( |D| = 2g + 1 \). In this section I will show that \( \mathcal{C}_D \) is isomorphic to \( \mathcal{J}_D \) and so, in particular, is finitely generated as a consequence of the Mordell Weil theorem.

Let \( \mathcal{H} \) be a complete smooth curve with function field \( k(t, s) \), where

\[
s^2 = D(t).
\]

The function \( t \) on \( \mathcal{H} \) has a double pole that we denote \( \infty \). Let \( Q = (a, b, c) \) represent a class \( C \in \mathcal{C}_D \). Then since \( a \succ 0 \) we can write

\[
(3.23) \quad a(t) = \alpha^2 \prod_{i=1}^{m} (t_i - t)^{n_i}.
\]
with \( \alpha \in k^* \) and distinct \( t_i \in \bar{k} \) and \( n_i \in \mathbb{Z}^+ \). Set \( s_i = b(t_i) \). Let \( \text{div}(1,0,-D) = 0 \) and otherwise associate to \( Q \) the degree zero divisor on \( \mathcal{H} \) given by

\[
\text{div}(Q) = \sum_{i=1}^{m} n_i(t_i, s_i) - N\infty,
\]

where \( N = \sum_{i=1}^{m} n_i \). It is defined over \( k \) and is semi-reduced in that if \((t, s)\) occurs in \( \text{div}(Q) \), then \((t, -s)\) does not, unless \( s = 0 \), in which case the multiplicity of \((0, t)\) is one.

As before, let \( J_d \) be the group of \( k \)-rational divisor classes of degree 0 on \( \mathcal{H} \). We want to prove the following.

**Proposition 11.** The map \( Q \mapsto \text{div}(Q) \) induces an isomorphism of groups from \( C_D \) onto \( J_D \).

**Lemma 9.** The map \( Q \mapsto \text{div}(Q) \) induces a map \( \phi \) from \( C_D \) to \( J_D \).

**Proof.** We must show that \( Q \sim Q' \) implies that \( \text{div}(Q) \) is equivalent to \( \text{div}(Q') \). Let \( a(t) \) be as in (3.23). If \( Q' \) results from \( Q \) by a translation \( Q' = Q|T_n \) with \( n \in k[t] \), we see that \( \text{div}(Q) = \text{div}(Q') \) since from (3.4),

\[
b(t_i) + n(t_i)a(t_i) = b(t_i).
\]

By Lemma 2, we are reduced to showing the needed equivalence when \( Q' = Q|S = (c, -b, a) \). As in the proof of Lemma 5, we know that \( c > 0 \), so we have

\[
c(t) = \gamma^2 \prod_{j=1}^{m'} (t'_j - t)^{n'_j}
\]

with \( \gamma \in k^* \) and \( N' = \sum_{j=1}^{m'} n'_j \), where \( n'_j \in \mathbb{Z}^+ \). Thus by (3.24) we need to show that

\[
\text{div}(a, b, c) = \sum_{i=1}^{m} n_i(t_i, b(t_i)) - N\infty \quad \text{and} \quad \text{div}(c, -b, a) = \sum_{j=1}^{m'} n'_j(t'_j, -b(t'_j)) - N'\infty
\]

are equivalent. Define the rational function \( f \) on \( \mathcal{H} \) by

\[
f = \frac{c(t)}{s - b(t)} = \frac{-s + b(t)}{a(t)}.
\]

Now we will show that

\[
(f) = (\text{div}(c, -b, a) - \text{div}(a, b, c))
\]

where, as usual, \((f)\) is the divisor of \( f \). This will prove the claim.

Suppose first that \( b(t_i) \neq 0 \) and \( b(t'_j) \neq 0 \). Clearly \((t_i, b(t_i)) \neq (t'_j, -b(t'_j))\). The factors \( s - b(t) \) and \( c(t) \) have zeros of order \( n_i \) at \((t_i, b(t_i))\) and \( n'_j \) at \((t'_j, -b(t'_j))\), respectively. Of the latter, the zeros at \((t'_j, b(t'_j))\) are cancelled by the factor \( s - b(t) \). A similar argument works for any points \((t_i, 0)\) or \((t'_j, 0)\) that occur. Since \( d \) is square-free, \((t_i, 0) \neq (t'_j, 0)\).

Finally, the behavior at infinity also matches since \( t \) has a double pole at \( \infty \).

\[\square\]

**Lemma 10.** The map \( \phi : C_D \to J_D \) is a group homomorphism.

**Proof.** By Lemma 4 and Proposition 2, any two classes \( C, C' \in C_D \) contain forms \( Q = (a, b, ca) \) and \( Q' = (a', b, ac) \), resp., where \( a, a' \) are coprime and prime to \( D \), and the composition of \( C \) and \( C' \) contains the form \( Q'' = (aa', b, *) \). Thus \( \text{div}(Q) + \text{div}(Q') = \text{div}(Q'') \). \[\square\]
Proposition 11 now follows from the next lemma.

**Lemma 11.** The map $\phi : C_D \to J_D$ is bijective.

**Proof.** By Proposition 1, every class $C \in C_D$ contains a unique form $Q = (a, b, c)$ where $|b| < |a| \leq g$, with $(-1)^{|a|}a$ monic. By the Riemann-Roch theorem (see e.g. [19]) and a straightforward argument (c.f. [4, pp. 96-97]), every divisor class has a unique representative of the form $\div(Q)$ for such a $Q$. Here $b$ is uniquely determined by the condition $|b| < |a|$. □

**Proposition 12.** For $k$ a number field, $D \in k[t]$ square-free and negative of odd degree, the group $C_D$ is finitely generated.

**Proof.** This follows from Proposition 11 and the Mordell-Weil Theorem [35], [20]. □

**Remark.** As already mentioned, that a result like Proposition 11 should hold can be inferred from the paper of D. Cantor [4] (see also [26]). However, the isomorphism we give does not seem to be well-known.

**Proof of Theorem 1.** Suppose that $J_q$ is finite. By Proposition 11 we know that $C_q$ is finite. Since $q$ is a prime, it follows from Proposition 5 that $C_q$ has no classes of positive even order and so any class $C \in C_q$ is in $C_q^2$. Let $p$ be any prime different from $q$. If $\left(\frac{q}{p}\right) = 1$ then $q = b^2 + pc$ for some $b, c$, so $Q = (p^*, b, \pm c) \in Q_q$. Thus since $C \in C_q^2$, we have that $\chi_q(Q) = (p^*, q)_q = 1$ and $\left(\frac{p}{q}\right) = 1$.

By Proposition 11 again, if $J_q$ is infinite so is $C_q$ and by Proposition 12 it must contain a class $C$ not in $C_q^2$. Let $Q(x, y) = (a, b, c) \in C$, so $b^2 - ac = q$. By Proposition 1 and the proof of Lemma 5, we may assume that $|b| < |a| < |c|$ with $c^*$ monic. The polynomial

$$F(x, t) = a(t)x^2 + 2b(t)x + c(t)$$

is irreducible in $k[y, t]$. To see this, observe that $\gcd(a, b, c) = 1$ and, as a quadratic in $x$, the discriminant of $F$ is $4q(t)$, which is not a square. By Hilbert’s irreducibility theorem [14], [20] there are infinitely many $x \in k$ such that $F(x, t) = Q(x, 1)$ is irreducible in $k[t]$. For any such $x$, it follows from Lemma 4 that $C$ contains the form $Q' = (p^*, b', c')$, where

$$p(t) = (-1)^{|c|}q(x, 1) \neq q(t)$$

is prime. Thus $q = b'^2 - p^*c'$ and so $\left(\frac{q}{p}\right) = 1$. Since $C \not\subseteq C_q^2$, by the principal genus theorem Proposition 10, we must have that $\chi_q(Q') = (p^*, q)_q \neq 1$. Thus $\left(\frac{p}{q}\right) \neq 1$. □

**Remark.** It is easy to modify the above argument to produce infinitely many primes $p$ of either odd or even degree with $\left(\frac{p}{q}\right) \neq 1$.

**Proof of Theorem 2.** By Proposition 11, we require a characterization of those classes in $C_D$ of order at most 2 that are contained in $C_D$. Claim: the number of such classes is given by $2^{e_2(J)}$, where as before we write $J = J_D$. To see this, let $m$ be the number of primary cyclic subgroups of order 2. The number of classes $C$ for which $C^4 = 1$ is

$$2^m 4^{e_2(J)} = 2^{m+e_2(J)} 2^{e_2(J)} = 2^{e_2(J)} 2^{e_2(J)} = \# \text{ (classes of order at most 2)} \times 2^{e_2(J)}.$$

For each class in $C_2 \subseteq C_D$ of order at most 2 contained in $C_D$, choose a fixed $C_1 \in C_D$ with $C_1^2 = C_2$. By multiplication of each $C_1$ by all classes of order at most 2, we get all classes $C$ with $C^4 = 1$. The claim follows by a comparison with (3.25).
Proposition 10 implies that $2^{c_2(J)}$ is the number of classes of order at most 2 that are contained in the principal genus of $C_D$. By (the proof of) Proposition 5, these classes are in one-to-one correspondence with the ambiguous forms $(D_1, 0, -D_2)$ that are killed by all genus characters, where $D = D_1D_2$ is as in the statement of the theorem. Using the definition of the genus characters given in (3.17), these forms correspond to the decompositions described in Theorem 2.

□

4. Binary cubic forms over $k[t]$

Preliminaries. Many of the general identities and transformation properties of binary cubic forms over $\mathbb{Z}$ can be readily adapted to forms with coefficients in $k[t]$. Some references for the classical theory are [15] and [31]. For $a, b, c, d \in k[t]$ let

$$F(u, v) = au^3 + 3bu^2v + 3cuv^2 + dv^3 = (a, b, c, d)(u, v).$$

Its discriminant is

$$\Delta = \Delta_F = a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2 = (ad - bc)^2 - 4(ac - b^2)(bd - c^2).$$

The Hessian of $F$ is

$$(4.1) \quad H_F(u, v) = (ac - b^2)u^2 + (ad - bc)uv + (bd - c^2)v^2 = (ac - b^2, \frac{1}{2}(ad - bc), bd - c^2)(u, v)$$

and the cubic covariant is

$$J_F(u, v) = (2b^3 - 3abc + a^2d, b^2c - 2ac^2 + abd, -bc^2 + 2b^2d - acd, -2c^3 + 3bcd - ad^2).$$

The discriminant of $H_D$ is $D = \frac{1}{4}\Delta_F$. The group $\text{SL}_2(k[t])$ acts as expected on $F, H_F, J_F$: thus for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $F|_M(x, y) = F(mx + ny, rx + sy)$. The discriminant $\Delta_F$ is invariant and the forms $H_F$ and $J_F$ are covariants of $F$, meaning that $H_F|_M = H_F|M$ and $J_F|_M = J_F|M$. The principal form of discriminant $\Delta = 4D$

$$(4.2) \quad F_0 = (0, 1, 0, D)$$

has $-H_0 := -H_{F_0} = (1, 0, -D)$ and $\frac{1}{2}J_0 := \frac{1}{2}J_{F_0} = (1, 0, D, 0)$.

The syzygy and its converse. After Eisenstein [9] and Cayley [6], a cubic form and its covariants satisfy the syzygy

$$J_F^2 = -4H_F^3 + \Delta F^2,$$

which can be checked directly. Setting $X = -H_F, Y = \frac{1}{2}J_F$ and $Z = F$ we have

$$(4.3) \quad Y^2 = X^3 + DZ^2,$$

where $D = \frac{1}{4}\Delta$ is the discriminant of $H_F$. The following converse result is crucial. Over $\mathbb{Z}$ the corresponding result is due to Arndt [1] and Mordell [23], [25, p.216].

Proposition 13. Suppose that $D \in k[t]$ is square-free. If $X, Y, Z \in k[t]$ satisfy $\gcd(X, Z) = \gcd(Y, Z) = 1$ and

$$Y^2 = X^3 + DZ^2,$$

then there exists a binary cubic form $F = (Z, b, c, d)$ with $b, c, d \in k[t]$ and discriminant $4D$ for which $-H_F = (X, *, *)$ and $\frac{1}{2}J_F = (Y, *, *, *)$. 
Proof. First note that $\gcd(X, Y) = 1$ since $D$ is square-free. Thus both $X$ and $Y$ are prime to $D$. From (4.3) we see that $D$ is a square modulo $X$ so there exists a binary form $(X, B, C)$ with $B^2 - XC = D$. We have that $\gcd(Z, X) = 1$ and from (4.3) we may suppose that

$$ZB \equiv -Y \pmod{X^3}. \tag{4.4}$$

The class of $(X, B, C)$ is uniquely determined once we force $B$ to satisfy condition (4.4). Clearly $X \neq 0$. After Arndt [1], set

$$b = \frac{1}{X}(Y + BZ), \quad c = \frac{1}{X^2}(DZ + 2YB + ZB^2), \quad d = \frac{1}{X^2}(YD + 3BDZ + 3YB^2 + ZB^3). \tag{4.5}$$

A calculation shows that the discriminant of $F = (Z, b, c, d)$ is $4D$, that $-H_F = (X, B, C)$ and that $\frac{1}{2}J_F = (Y, *, *, *)$.

We need to show that $b, c, d \in k[t]$. Clearly $b \in k[t]$.

To see that $c \in k[t]$, consider

$$U \times V = (DZ + 2YB + ZB^2)(DZ - 2YB + ZB^2) = Z^2(D + B^2)^2 - 4Y^2B^2 \equiv Z^2(D + B^2)^2 - 4DZ^2B^2 \pmod{X^3} \text{ by (4.3)} = Z^2(B^2 - D)^2 \equiv 0 \pmod{X^2}.$$

Now $U$ and $V$ cannot share any primes with $X$ since $U - V = 4YB$. That $c \in k[t]$ will follow if we can show that $X|ZU$. From $X|(ZB + Y)$ we have that

$$ZU \equiv Z(DZ + YB) \equiv Y(Y + ZB) \equiv 0 \pmod{X}.$$

Finally, from (4.5) we must prove that

$$X^3d = YD + 3BDZ + 3YB^2 + ZB^3 \equiv 0 \pmod{X^3}.$$

Using (4.4) we have that

$$Z^2X^3d \equiv Z^2YD - 3YDZ^2 + (3Y^3 - Y^3) \equiv 2Y(Y^2 - DX^2) \equiv 0 \pmod{X^3},$$

as desired. \hfill \Box

**Proposition 14.** Suppose that $D \in k[t]$ is square-free and negative with odd degree. Let $F = (a, b, c, d)$ with $a, b, c, d \in k[t]$ have discriminant $4D$. Then $Q = -H_F = (A, B, C) \in Q_D$ and its class has order 1 or 3 in $C_D$. The only other cubic form with Hessian $-Q$ is $-F$.

**Proof.** To show that $Q \in Q_D$, first note that the discriminant of $Q$ is $D$. We also must have $A > 0$. To see this we can apply (4.3). First, if $|A|$ is even we need the leading coefficient $A_0$ of $A$ to be in $k^2$. Since $DZ^2$ has odd degree we have $A_0^3 = Y_0^2$ where $Y_0$ is the leading coefficient of $Y$, so $A_0 = (\frac{Y_0}{A_0})^2$. If $|A|$ is odd the leading coefficients of $A^3$ and $DZ^2$ must cancel so again $A_0 \in k^2$.

Next we show that the class of $Q = (A, B, C)$ has order 1 or 3. We have the following identity (from [7]) that can be verified directly:

$$Q \otimes Q = Q^{-1},$$

where $A = b^2 - ac, B = bc - ad$ and $C = c^2 - bd$. Thus by Proposition 3 and ii) of Proposition 2 we have $Q \otimes Q = Q^{-1}$, so that the class of $Q$ has order 1 or 3 in $C_D$.\hfill \Box
The proof of the final statement is also adapted from [7, 2nd letter], but the argument required some reworking to make it more understandable. Suppose that $F' = (a', b', c', d')$ has $-H_{F'} = (A, B, C)$. By (4.6) we have

\[(A, -B, C)(x, y) = (A, -B, C)(x', y')\]

where

\[(4.7) \quad x = buu' + cvv' + ca'v + d'v'\]
\[y = auu' + buv' + bv' + cvv'\]
\[x' = b'uu' + c'uv' + c'u'v + d'v'\]
\[y' = a'uu' + b'uv' + b'u'v + c'v'v'.\]

Also by assumption

\[(4.8) \quad A = b^2 - ac = b'^2 - a'c' \quad B = bc - ad = b'c' - a'd' \quad C = c^2 - bd = c'^2 - b'd'.\]

First we will show that

\[(4.9) \quad (x', y')^t = M(x, y)^t \quad \text{for} \quad M = \left( \begin{array}{cc} m & r \\ n & s \end{array} \right) \in SL_2(k(t)).\]

After verifying this, we will show that in fact we may take $m, r, n, s \in k[t]$. Given this, the uniqueness of $F'$ up to sign follows from Lemma 3, which implies that $Q$ has only trivial automorphs.

A calculation shows that if

\[(4.10) \quad m = \frac{bh' - ac'}{A} \quad r = \frac{-cb' + ba'}{A} \quad n = \frac{ba' - ab'}{A} \quad s = \frac{-ca' + bb'}{A}\]

then $\det M = 1$. Here we use the first formula of (4.8) and that $A \neq 0$. Another calculation using (4.10) gives that

\[bm + ar = b' \quad bn + as = a'\]
\[cm + br = c' \quad cn + bs = b'.\]

To verify (4.9) it is now enough to show that $dm + cr = d'$ and $dn + cs = c'$. By (4.10) and (4.8)

\[d(bh' - ac') + c(-cb' + ba') = -C' + c'B = b'^2d' - a'c'd' = d'A,\]

which gives $dm + cr = d'$. That $dn + cs = c'$ follows similarly.

Finally, we must show that $m, r, n, s \in k[t]$. In case $B = 0$ we have from (4.7) and (4.9) that

\[cm + br = c', \quad dm + cr = d' \quad \text{and} \quad cn + bs = b', \quad dn + cs = c',\]

hence that

\[m = \frac{cc' - bd'}{c} \quad r = \frac{-cd' + ce'}{c} \quad n = \frac{cb' - bd'}{c} \quad s = \frac{-db' + ce'}.\]

Since $\gcd(A, C) = 1$, from this and (4.10) we must have that $m, r, n, s \in k[t]$. If $B \neq 0$ we argue similarly by giving a third set of fractions from (4.7) and (4.9) for $m, r, n, s$, now with denominator $B$, starting with

\[bm + ar = b', \quad dm + cr = d' \quad \text{and} \quad bn + as = a', \quad dn + cs = c'.\]

\[\square\]

Propositions 14 and 12 immediately imply the following result, since the number of classes in $C_D$ of order at most three is finite.

**Proposition 15.** The class number of binary cubics over $k[t]$ with discriminant $4D$ is finite, when $D \in k[t]$ is negative, square-free and of odd degree.
Elliptic curves over $k(t)$. Consider the elliptic curve with $j$-invariant zero over $k(t)$ defined by
\begin{equation}
\mathcal{E}_D : y^2 = x^3 + D(t)
\end{equation}
where $D \in k[t]$ is square-free. Recall that $D' = -27D(-3t)$, which is also square-free and that if $D$ is negative with odd degree so is $D'$. By the Mordell-Weil theorem over function fields (see [33, p.230]) the rational points on $\mathcal{E}_D$ and $\mathcal{E}_{D'}$ form finitely generated abelian groups.

We have the pair of dual 3-isogenies $\psi : \mathcal{E}_D \to \mathcal{E}_{D'}$ and $\psi' : \mathcal{E}_{D'} \to \mathcal{E}_D$ given by
\begin{equation}
\psi : (x, y) \mapsto \left(\left(\frac{x^3 + 4D}{x^2}\right)^\sigma, \left(\frac{y(x^3 - 8D)}{x^3}\right)^\sigma\right)
\end{equation}
and
\begin{equation}
\psi' : (x', y') \mapsto \left(\left(\frac{x'^3 + 4D'}{9x'^2}\right)^{\sigma^{-1}}, \left(\frac{y'(x'^3 - 8D')}{27x'^2}\right)^{\sigma^{-1}}\right),
\end{equation}
where $\sigma : k(t) \to k(t)$ is the automorphism defined by $t \mapsto -3t$. Then $\psi' \circ \psi$ and $\psi \circ \psi'$ give the tripling maps on $\mathcal{E}_D$ and $\mathcal{E}_{D'}$, respectively.

Suppose that $D$ is negative and of odd degree. Given a (finite) rational point $P = (x, y)$ on $\mathcal{E}_D$ we can write $(x, y) = \left(\frac{X}{V^2}, \frac{Y}{V}\right)$ where $V \in k[t]$ is monic and $\text{gcd}(X, V) = \text{gcd}(Y, V) = 1$. Taking $Z = V^3$, we associate to $P$ the class $C_P$ of any form $Q = -H_F$ that, together with $F$ and $J_F$, gives rise to $P$ via Proposition 13. Note that the class of $Q$ is uniquely determined by (4.4). Thus by Proposition 14, $C_P$ has order 1 or 3 in $\mathcal{C}_D$. Similarly we associate to any point $P' = (x', y')$ on $\mathcal{E}_{D'}$ the class $C_{P'}$, which has order 1 or 3 in $\mathcal{C}_{D'}$.

**Lemma 12.** The class $C_P \in \mathcal{C}_D$ has order 1 if and only if $P$ is the image of some $Q' \in \mathcal{E}_{D'}$ under $\psi'$. The class $C_{P'} \in \mathcal{C}_{D'}$ has order 1 if and only if $P'$ is the image of some $Q \in \mathcal{E}_D$ under $\psi$.

**Proof.** Note that $F(u, v) = V^3$ if and only if $F(\frac{y'}{y}, \frac{v}{y}) = 1$. Recall the principal form $F_0$ from (4.2). It is readily checked that there is a bijection between the set
\begin{equation}
\{(x, y) \in k(t)^2; F_0(x, y) = 3x^2y + Dy^3 = 1\}
\end{equation}
and the finite rational points $(x', y')$ on $\mathcal{E}_{D'}$ given by
\begin{equation}
(x', y') = \left(\left(\frac{x^3}{y^3}\right)^\sigma, \left(\frac{y^3}{x^3}\right)^\sigma\right) \quad \text{and} \quad (x, y) = \left(\left(\frac{y}{x^3}\right)^{\sigma^{-1}}, \left(\frac{x}{y^3}\right)^{\sigma^{-1}}\right).
\end{equation}
A computation using (4.12) and (4.2) shows that for $(x, y)$ with $3x^2y + Dy^3 = 1$ and $(x', y')$ from (4.14) we have
\begin{equation}
\psi'(x', y') = (-H_0(x, y), \frac{1}{2}J_0(x, y)).
\end{equation}
The first statement follows since we have from Proposition 14 that $-H_F$ is principal if and only if $F$ is principal.

The proof of the second statement is similar, using that for $F'_0 = (0, 1, 0, D')$ there is a bijection between the set
\begin{equation}
\{(x', y') \in k(t)^2; F'_0(x', y') = 3x'^2y' + D'y'^3 = 1\}
\end{equation}
and the finite rational points $(x, y)$ on $\mathcal{E}_D$ from (4.11) given by
\begin{equation}
(x, y) = \left(\left(\frac{y'}{x'^3}\right)^{\sigma^{-1}}, \left(\frac{x'}{y'^3}\right)^{\sigma^{-1}}\right) \quad \text{and} \quad (x', y') = \left(\left(\frac{x}{y^3}\right)^\sigma, \left(\frac{y^3}{x}\right)^\sigma\right)
\end{equation}
Now use that
\begin{equation}
\psi(x, y) = (-H'_0(x', y'), \frac{1}{2}J'_0(x', y'))
\end{equation}
where $-H'_0 = -H_{F'_0} = (1, 0, -D')$ and $\frac{1}{2}J'_0 = \frac{1}{2}J_{F'_0} = (1, 0, D', 0)$ and apply Proposition 14 as before. \qed
Proof of Theorem 3. To prove the first statement, suppose that $P = (x, y) \in \mathcal{E}_D$. Then
$$2P = (x', y') = \left(\frac{9x^4}{8y^3} - 2x, \frac{8y^4 - 36x^3y^2 + 27x^6}{8y^3}\right).$$
Write $(x, y) = \left(\frac{X}{V^3}, \frac{Y}{V^2}\right)$ where $X, Y, V \in \mathbb{k}[t]$ with $\gcd(X, V) = \gcd(Y, V) = 1$. Then we have
\begin{align}
Y^2 &= X^3 + DV^6 \quad \text{and} \\
V^2 &= 9X^4 - 18X^2Y + 27Y^2 \quad \text{and} \\
V^3 &= 8Y^4 - 36X^3Y^2 + 27X^6.
\end{align}
Since $D$ is square-free we must have $\gcd(X, Y) = 1$. There are two cases to consider.

i) If $|Y| > 0$ then the denominator of $x'$ has larger degree than that of $x$ and so the first statement of Theorem 3 follows.

ii) If $Y \in k^*$ then we may write $(x', y') = \left(\frac{X'}{V'}, \frac{Y'}{V'}\right)$ where $X', Y' \in \mathbb{k}[t]$. Here $\gcd(X', V) = \gcd(Y', V) = 1$, since by (4.15) and (4.16) we have for any prime $P|V$
$$4Y^2X' = X(9X^3 - 8Y^2) \equiv XY^2 \pmod{P}$$
and
$$8Y^3Y' = 8Y^4 - 36X^3Y^2 + 27X^6 = 27D^2V^{12} - 18DV^6Y^2 - Y^4 \equiv -Y^4 \pmod{P},$$
where $Y \not\equiv 0$ and $P \nmid X$. Since $|D| > 0$ we have from (4.15) that $|X| > 0$, hence from the first equation of (4.17) that $|Y'| > 0$. Repeat the argument from the beginning with $P = (x', y') \in \mathcal{E}_D$ in place of $(x, y)$. Now i) applies and again the first statement of Theorem 3 follows.

Now we prove the (contrapositive of the) second statement of Theorem 3. By the first statement and Mordell-Weil for $\mathcal{E}_D$, if $\mathcal{E}_D$ has a finite rational point, it is of infinite order and we may assume that it is not the triple of any other point. Let $P$ be such a point. If the $C_P \in \mathcal{C}_D$ that corresponds to $P$ as above Lemma 12 is not trivial we are done, since $P$ is then of order 3. Otherwise, by Lemma 12, $P$ is the image under $\psi'$ of some point $P' \in \mathcal{E}_{D'}$. In that case the class $C_{P'} \in \mathcal{C}_{D'}$ corresponding to $P'$ is of order 3, since otherwise $P' = \psi(Q)$ for some $Q \in \mathcal{E}_D$ and then $P = \psi'(\psi(Q)) = 3Q$, and $P$ is a triple. The second statement now follows since $\mathcal{J}_D$ is isomorphic to $\mathcal{C}_D$ and $\mathcal{J}_{D'}$ is isomorphic to $\mathcal{C}_{D'}$ by Proposition 11. \hfill \square

References


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