nonempty subsets E_1 *and* E_2 *of* $[0,1]$ *such that* $K(x,y)=0$ *for* $(x,y)\in\left(E_1\times E_2'\right)\cup\left(E_2\times E_1'\right)$ *(where* E' *denotes the complement of* E*).*

The assumption in Theorem 1 that E_1 and E_2 have nonempty interiors implies that there are continuous functions ψ_i $(i=1,2)$ with $\psi_i(x)=0$ for $x\notin E_i$ and $\int_0^1\psi_i(x)\,dx=1$. Then

$$K(x,y)=\left(\frac{r^{-1}g(x)-f(x)}{r^{-1}-r}\right)\psi_1(y)f(y)^{-1}+\left(\frac{f(x)-rg(x)}{r^{-1}-r}\right)\psi_2(y)f(y)^{-1}$$

has the required properties. The example in the selected solution is of this type, and the calculation given there illustrates the general proof.

Solved also by J. H. Lindsey II and the proposer.

Sums of Complex Numbers of Unit Magnitude

10446 [1995, 359]. *Proposed by Hubert Kiechle, Technische Universität, Munich, Germany.* Let $T=\{z:|z|=1\}$ be the unit circle in the complex plane, and let w be a given nonzero complex number.
(a) If $|w|\le 2$, show that there are unique $z_1,z_2\in T$ such that $w=z_1+z_2$.
(b) If $|w|>2$, show that w can be written as a sum of $\lceil|w|\rceil$ elements of T.
(c) Under what conditions will w be a unique sum of n elements of T?

Solution by A. N. 't Woord, University of Technology, Eindhoven, The Netherlands. **(a)** For $z_1,z_2\in T$, $z_1+z_2=w$ if and only if z_1 and z_2 are the two intersection points of T and the circle $\{z:|w-z|=1\}$ (which coincide when $|w|=2$).
(b) Let $z=w/|w|\in T$ and $n=\lceil|w|\rceil$. Then $|w-(n-2)z|=|w|-\lceil|w|\rceil+2\le 2$, so according to part **(a)** we can find $z_1,z_2\in T$ such that $w=(n-2)z+z_1+z_2$.
(c) For $z_i\in T$ we have $|z_1+\cdots+z_n|\le|z_1|+\cdots+|z_n|=n$ with equality if and only if $\arg(z_1)=\cdots=\arg(z_n)$. So if $|w|>n$, w is not a sum of n elements of T, and if $|w|=n$, w is a unique sum of n elements of T, namely $z_1=z_2=\cdots=z_n=w/|w|$.

Now suppose $|w|<n$ and $n\ge 3$. Let $c\in T$ be such that $|w-c|<n-1$. Using part **(a)** and induction on n, we may assume that $w-c=z_1+\cdots+z_{n-1}$ for certain $z_i\in T$. Since c can be chosen in infinitely many ways, it follows that w is not a unique sum of n elements of T.

Since the situation for $n=1$ is clear, these results prove that w is a unique sum of n elements of T if and only if either $|w|=n$ or $n=2$ and $|w|\le 2$.

Solved also by G. Bach (Germany), R. Barbara (Lebanon), M. Benedicty, K. L. Bernstein, J. C. Binz (Switzerland), R. J. Chapman (U. K.), R. B. Eggleton, S. Eisman, F. J. Flanigan, K. Foltz & N. Rosenberg, D. L. Grant (Canada), H. W. Guggenheimer, G. A. Heuer, F. B. Holt, N. Komanda, J. H. Lindsey II, M. D. Meyerson, C. A. Minh & T. V. Ta, V. Pambuccian, R. Patenaude, W. Paulsen, A. Pedersen (Denmark), J. Sadek & M. Sand, J. H. Steelman, R. Stong, A. A. Tarabay (Lebanon), R. W. W. Taylor, E. Woerner & P. Sun, Anchorage Math Solutions Group, NSA Problems Group, and the proposer.

A Nonnegativity Restriction

10492 [1995, 930]. *Proposed by William Duke, Mathematical Sciences Research Institute, Berkeley, CA.* Let n be a positive integer. Show that the only integral polynomials of degree less than n that are real and nonnegative at all n-th roots of unity and have constant term 1 are of the form

$$1+x^d+x^{2d}+\cdots+x^{n-d}$$

with $d|n$, or

$$1-x^d+x^{2d}-\cdots-x^{n-d}$$

with $2d|n$.

Solution by Robin J. Chapman, University of Exeter, Exeter, U. K. Set $\zeta = e^{2\pi i/n}$, a primitive n-th root of unity. If $d|n$, then

$$1 + x^d + x^{2d} + \cdots + x^{n-d} = \frac{1-x^n}{1-x^d}$$

vanishes for n-th roots of unity that are not d-th roots of unity and equals n/d when x is a d-th root of unity. If $2d|n$, then

$$1 - x^d + x^{2d} - \cdots - x^{n-d} = \frac{1-x^n}{1+x^d}$$

vanishes when x is an n-th root of unity and $x^d \neq -1$ and equals n/d when $x^d = -1$.

Conversely, suppose $f(x) = \sum_{j=0}^{n-1} a_j x^j$ has integer coefficients, $a_0 = 1$, and $f(\zeta^k) \geq 0$ for each k. Let $b_k = f(\zeta^k)$ and consider the subscripts on the a_j and b_k as elements of $\mathbb{Z}_n$, the set of integers modulo n. An easy computation (essentially the Fourier inversion formula) gives

$$na_j = \sum_{k=0}^{n-1} b_k \zeta^{-jk}. \tag{1}$$

In particular, setting $j = 0$ shows that $n = na_0 = \sum_{k=0}^{n-1} b_k$. From the triangle inequality applied to (1) and from the assumption that $b_k \geq 0$, it follows that $n\left|a_j\right| \leq n$ for all j. Hence each $a_j = -1$, 0, or 1. Unless all nonzero terms on the right of (1) have the same argument, $n\left|a_j\right| < n$ and so $a_j = 0$. Let $S = \{k : b_k \neq 0\}$. Since the a_j and b_k are real, ζ^{jk} is real and hence equals ± 1 for $k \in S$. Now $a_j = 1$ if and only if $\zeta^{-jk} = 1$ for all $k \in S$. Similarly, $a_j = -1$ if and only if $\zeta^{-jk} = -1$ for all $k \in S$. For a given k, $\{j : \zeta^{-jk} = 1\}$ forms a subgroup of $\mathbb{Z}_n$. As the intersection of subgroups is a subgroup, the set of j with $a_j = 1$ is a subgroup of $\mathbb{Z}_n$, and so it equals $\{0, d, 2d, \ldots, n-d\}$ for some $d|n$.

The proof is complete if no a_j equals -1. If there are some j with $a_j = -1$, then the same argument shows that $\{j : a_j \neq 0\}$ forms a subgroup G of $\mathbb{Z}_n$, say $\{0, d, 2d, \ldots, n-d\}$. Also the set $H = \{j : a_j = 1\}$ forms a proper subgroup of G. Now $\zeta^{-dk} = \pm 1$ for all $k \in S$, and so $\zeta^{-2dk} = 1$ for all such k. Hence $2d \in H$, and so $H = \{0, 2d, 4d, \ldots, n-2d\}$. Finally, since H is a subgroup of $\mathbb{Z}_n$, we have $2d|n$.

Solved also by J. E. Dawson (Australia), O. P. Lossers (The Netherlands), GCHQ Problems Group (U. K.), and the proposer.

A Complex Combinatorial Identity

10511 [1996, 267]. *Proposed by Bhaskar Bagchi and Gadadhar Misra, Indian Statistical Institute, Bangalore, India.* Let k and l be nonnegative integers, and let I be an index set of size $k + l$. Show that, for any $k + l$ distinct real numbers x_i, $i \in I$, we have

$$\sum_{A,B} \Big(\prod_{h\in B} x_h^p\Big)\Big(\prod_{i\in A, j\in B} (x_i - x_j)^{-1}\Big) = \begin{cases} 0 & \text{if } p = 0, 1, \ldots, k-1, \\ (-1)^{kl} & \text{if } p = k; \end{cases}$$

where the sum is over all ordered partitions (A, B) of I into two sets A and B of sizes k and l, respectively.

Solution I by Donald A. Darling, Newport Beach, CA. Extend the problem by letting $x_1, \ldots, x_{k+l}$ be distinct complex numbers, and let $F = F(k, l)$ be the indicated sum. An empty product equals 1, so $F = 1$ when $l = 0$ and $F = \prod x_i^p$ when $k = 0$; thus the formula is incorrect when $kl = 0$ (except when $p = k > 0$).