THE INDEX OF A MODULAR DIFFERENTIAL OPERATOR

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In memory of Raja Varadarajan (1937-2019)

Abstract. The space of all weakly holomorphic modular forms and the space of all holomorphic period functions of a fixed weight for the modular group are realized as locally convex topological vector spaces that are topologically dual to each other. This framework is used to study the kernel and range of a linear differential operator that preserves modularity and to define and describe its adjoint. The main results are an index formula for such a differential operator that is holomorphic at infinity and the identification of the co-kernel of the operator as a cohomology group of the modular group acting on the kernel.

1. Introduction

The theory of meromorphic differential equations was one of Varadarajan's many mathematical interests [42], [43]. In this paper I will present some new results about ordinary differential operators that act on modular forms. The main results are a formula for the index of certain of these operators that remains invariant under perturbations and the identification of the co-kernel of such an operator as a cohomology group of the modular group acting on the kernel.

Suppose that $k \in 2\mathbb{Z}$ and that f is a holomorphic function on the upper half-plane $\mathcal{H}$ that satisfies for all $z \in \mathcal{H}$ the functional equations

$$f(z) - z^{-k} f(-\tfrac{1}{z}) = 0 \quad \text{and} \quad f(z) = f(z+1). \tag{1.1}$$

Since f is periodic and holomorphic on $\mathcal{H}$ it has a convergent Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a(n) q^n. \tag{1.2}$$

Assume that f is meromorphic at $i\infty$, meaning that this series satisfies

$$\mathrm{ord}_\infty f := \min\{n;\ a(n) \neq 0\} > -\infty.$$

Here $q = e(z) := e^{2\pi i z}$. Denote by $\mathcal{M}_k$ the space of all such f, which are called weakly holomorphic modular forms of weight k for the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.

On the other hand, denote by $\mathcal{W}_k$ the space of all $\psi : \mathcal{H} \to \mathbb{C}$ holomorphic that satisfy the two functional equations

$$\psi(z) + z^{-k}\psi(-\tfrac{1}{z}) = 0 \quad \text{and} \quad \psi(z) + z^{-k}\psi(1 - \tfrac{1}{z}) + (z-1)^{-k}\psi(\tfrac{-1}{z-1}) = 0. \tag{1.3}$$

Such a ψ is an example of a period function of weight k that is holomorphic on $\mathcal{H}$. For our purposes we will refer to ψ as a holomorphic period function of weight k.

For any f, g holomorphic on $\mathcal{H}$ define

$$\langle f, g\rangle = \int_i^\rho f(z) g(z) dz, \tag{1.4}$$

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where the integral is over any smooth curve from i to $\rho = e(1/6) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ in $\mathcal{H}$. Clearly $\langle f, g\rangle$ is well-defined.

Both $\mathcal{M}_k$ and $\mathcal{W}_k$ are infinite dimensional vector spaces over $\mathbb{C}$. We may consider $\mathcal{W}_{2-k}$ to be a subspace of the (algebraic) dual space of $\mathcal{M}_k$ by restricting $\langle\cdot,\cdot\rangle$ to $\mathcal{M}_k \times \mathcal{W}_{2-k}$. We show in Corollary 1 below that by means of $\langle\cdot,\cdot\rangle$ the space $\mathcal{W}_{2-k}$ separates points in $\mathcal{M}_k$ and that $\mathcal{M}_k$ separates points in $\mathcal{W}_{2-k}$. It follows (see e.g. [38, Example 1, p. 127]) that $\mathcal{M}_k$ becomes a locally convex topological vector space with topology generated by the semi-norms

$$\rho_\psi(f) = |\langle f, \psi\rangle|$$

for $\psi \in \mathcal{W}_{2-k}$, and that $\mathcal{W}_{2-k}$ may be identified with the topological dual of $\mathcal{M}_k$. Similar considerations apply to $\mathcal{W}_{2-k}$, whose topological dual can be identified with $\mathcal{M}_k$.

In this paper continuity of an operator is always taken to be in terms of these topologies. A standard reference for the theory of linear operators on topological vector spaces is [39]. In particular, any linear and continuous operator

$$T : \mathcal{M}_{2-k} \to \mathcal{M}_k$$

has a unique linear adjoint $T' : \mathcal{W}_{2-k} \to \mathcal{W}_k$ defined through

$$\langle Tf, \psi\rangle = \langle f, T'\psi\rangle.$$

Also, T' is continuous and $(T')' = T$. If $\alpha(T) := \dim\ker T$ and $\alpha(T') := \dim\ker T'$ are finite, it is of interest to determine them. While they will change under certain perturbations of T, one may hope that their difference

$$\operatorname{ind} T = \alpha(T) - \alpha(T'), \tag{1.5}$$

which we take to define the index of T, remains invariant. Clearly $\operatorname{ind} T' = -\operatorname{ind} T$.

When $k > 0$ consider a linear differential operator of order $k-1$ acting on any holomorphic f on $\mathcal{H}$ by

$$Tf = h_0 f^{(k-1)} + h_1 f^{(k-2)} + \cdots + h_{k-2} f' + h_{k-1} f, \tag{1.6}$$

where each h_j is holomorphic on $\mathcal{H}$, periodic and meromorphic at $i\infty$. Assume that h_0 is a non-zero constant. Here and throughout the paper we write $f' = \frac{1}{2\pi i}\frac{df}{dz}$ and $f^{(2)} = f''$ etc. Say that T in (1.6) is *modular* if

$$T\big(z^{k-2} f(-\tfrac{1}{z})\big) = z^{-k}(Tf)(-\tfrac{1}{z}) \tag{1.7}$$

and that T is *holomorphic* if in addition $\operatorname{ord}_\infty h_j \geq 0$ for each j. It is easy to see that such a modular T defines a linear operator from $\mathcal{M}_{2-k}$ to $\mathcal{M}_k$ and we show in Lemma 4 that T is continuous. It is apparent that $\alpha(T)$ is the dimension of the space of modular solutions to the differential equation $Tf = 0$. By the standard theory of linear ODE's we know that

$$\alpha(T) = \dim\ker T \leq k-1.$$

To interpret $\alpha(T')$ we employ the classical Lagrange adjoint of T, which acts on any f holomorphic on $\mathcal{H}$ by

$$T^* f = -h_0 f^{(k-1)} + (h_1 f)^{(k-2)} - \cdots - (h_{k-2} f)' + h_{k-1} f \tag{1.8}$$

(see e.g. [22]). Now T^* is modular when T from (1.6) is modular; this is shown in Lemma 3. In Lemma 4 we prove that the adjoint $T' : \mathcal{W}_{2-k} \to \mathcal{W}_k$ is given by $T' = T^*$, when T^* is restricted to $\mathcal{W}_{2-k}$. Therefore $\alpha(T')$ is the dimension of the space of all holomorphic period functions $\psi \in \mathcal{W}_{2-k}$ that satisfy $T^*\psi = 0$, and again $\alpha(T') \leq k-1$.

For a fixed k the quantities $\alpha(T)$ and $\alpha(T')$ are certainly sensitive to T. We will illustrate this in an example below. Our first main result shows that the index is an invariant, at least when T is holomorphic. Define

$$\ell_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor & \text{otherwise.} \end{cases} \tag{1.9}$$

Theorem 1. *For T a holomorphic modular differential operator of the form (1.6) we have*

$$\operatorname{ind} T = -2\ell_k - 1.$$

A consequence of Theorem 1 is that for any holomorphic modular T of the form (1.6) there will exist a non-zero holomorphic period function ψ of weight $2-k$ such that $T\psi = 0$, provided that $k > 2$.

Example 1.1. For any $m = 0, 1, 2, \ldots$ Bol's identity [7, p. 28] states that

$$\big(z^{m-1} f(-\tfrac{1}{z})\big)^{(m)} = z^{-m-1} f^{(m)}(-\tfrac{1}{z}). \tag{1.10}$$

For a direct proof see Lemma 5 of [1]. In particular, T defined by $Tf = f^{(k-1)}$ is modular and holomorphic. Since there are no modular polynomials of non-zero weight, we have that $\alpha(T) = 0$ unless $k = 2$, when $\alpha(T) = 1$. On the other hand, since $T^* f = -f^{(k-1)}$, we know that $\alpha(T')$ is the dimension of the space of $\psi \in \mathcal{W}_{2-k}$ that are polynomials of degree at most $k-2$. These ψ are called period polynomials. By Theorem 1 we see that $\alpha(T') = 2\ell_k + 1$ when $k > 2$, while $\alpha(T') = 0$ when $k = 2$. This is well known. See [11] for another proof due, essentially, to Poincaré [35] and for further references.

By (1.10) it is straightforward to verify that if T is defined by

$$Tf = f^{(k-1)} + g f' + (\tfrac{1}{2} g' + h) f, \tag{1.11}$$

where $g \in \mathcal{M}_{2k-4}$ and $h \in \mathcal{M}_{2k-2}$, then T is modular. The Lagrange adjoint of this T is determined by

$$T^* f = -f^{(k-1)} - g f' + (-\tfrac{1}{2} g' + h) f. \tag{1.12}$$

To illustrate how $\alpha(T)$ and $\alpha(T')$ may vary with T we need some classical modular forms. The Eisenstein series may be defined for any $k \geq 0$ by

$$E_k(z) = 1 + \tfrac{2}{\zeta(1-k)} \sum_{n \geq 1} \sigma_{k-1}(n) q^n. \tag{1.13}$$

Here σ is the sum of divisors function. When $k \neq 2$ it is well-known that $E_k \in \mathcal{M}_k$. Also

$$\Delta(z) = \tfrac{1}{12^3}\big(E_4^3(z) - E_6^2(z)\big) = q \prod_{m \geq 1} (1 - q^m)^{24} \tag{1.14}$$

is in $\mathcal{M}_{12}$. We have Ramanujan's formulas

$$E_2' = \tfrac{1}{12}(E_2^2 - E_4), \quad E_4' = \tfrac{1}{3}(E_2 E_4 - E_6), \quad E_6' = \tfrac{1}{2}(E_2 E_6 - E_4^2). \tag{1.15}$$

For proofs of these standard facts see [45].

Example 1.2. Now consider (1.11) when $k=4$, $g=c_1E_4$ and $h=c_2E_6$:

$$Tf = f^{(3)} + c_1E_4f' + (\tfrac{c_1}{2}E_4' + c_2E_6)f.$$

For $c_1=c_2=0$ we saw in Example 1.1 that $\alpha(T)=0$ and $\alpha(T')=1$. Here $\ker T'$ is spanned by $\psi(z)=1-z^2$.

On the other hand, when $c_1=-\frac{31}{36}$ and $c_2=\frac{5}{36}$ we have that $Tf=0$ for

$$f=\frac{E_4E_6}{\Delta}\in\mathcal{M}_{-2}.$$

This can be checked by using (1.15). Therefore $\alpha(T)\geq 1$. In fact $\alpha(T)=1$ since a computation shows that $\alpha(T')=\dim\ker T'\leq 2$. See around (3.20) in Example 3.4 for details.

It is also natural to consider the ranges and cokernels of the maps $T:\mathcal{M}_{2-k}\to\mathcal{M}_k$ and $T':\mathcal{W}_{2-k}\to\mathcal{W}_k$ for a holomorphic modular T. For any modular T let $\mathcal{P}_T$ be the space of **all** solutions f of the equation $Tf=0$. Now $\mathcal{P}_T$ and $\mathcal{P}_{T^*}$ are naturally acted on by Γ and $\dim\mathcal{P}_T=\dim\mathcal{P}_{T^*}=k-1$. A crucial role in understanding the relationship between $\mathcal{P}_T$ and $\mathcal{P}_{T^*}$ is played by the bilinear concomitant, which for our T is defined for f and g holomorphic on $\mathcal{H}$ by

$$[f,g]_T = gf^{(k-2)} + \big(h_1g-g'\big)f^{(k-3)}+\cdots+\big(h_{k-2}g-(h_{k-3}g)'+\cdots+g^{(k-2)}\big)f.$$

An identity of Lagrange (see (4.3) below) gives that

$$g\,T\,f - f\,T^*g = \tfrac{1}{2\pi i}\tfrac{d}{dz}[f,g]_T.$$

It follows that $[f,g]_T$ is a scalar when $f\in\mathcal{P}_T$ and $g\in\mathcal{P}_{T^*}$. In fact, as will be shown below, $\mathcal{P}_T$ and $\mathcal{P}_{T^*}$ are dual with respect to $[f,g]_T$. If $\{f_0,\dots,f_{k-2}\}$ and $\{g_0,\dots,g_{k-2}\}$ are dual bases with $[f_m,g_n]_T=\delta_{m,n}$, consider the kernel function

$$k_T(z,\tau)=\sum_{n=0}^{k-2} f_n(z)g_n(\tau).$$

By the classical method of variation of parameters the inhomogeneous equation $TF=f$, when $f\in\mathcal{M}_k$, is solved by the *generalized Eichler integral*

$$F(z)=\int_{z_0}^{z} k_T(z,\tau)f(\tau)\,d\tau,$$

where the integral is over any smooth curve from z_0 to z in $\mathcal{H}$. For $A=\pm\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ let

$$\sigma_A(z)=\int_{z_0}^{A^{-1}z_0} k_T(z,\tau)f(\tau)\,d\tau. \tag{1.16}$$

We will see that $A\to\sigma_A$ determines a cohomology class in $H^1(\Gamma,\mathcal{P}_T)$, whose definition is recalled in §5. For $Tf=f^{(k-1)}$ we have that

$$k_T(z,\tau)=\tfrac{1}{(k-2)!}(z-\tau)^{k-2},$$

so the generalized Eichler integral becomes the usual Eichler integral of f. The generalized version together with Theorem 1 yield a proof of the following result.

Theorem 2. *Suppose that T is a holomorphic modular differential operator of the form (1.6). Then the map $f\to\sigma_A$ gives an isomorphism from $\mathcal{M}_k/T\mathcal{M}_{2-k}$ to $H^1(\Gamma,\mathcal{P}_T)$. Their dimension is $2\ell_k+1+\alpha(T)$.*

When $Tf = f^{(k-1)}$ the result of Theorem 2 is close to, but not equivalent to, the specialization of Eichler's main result in [13] to the modular group. See also his exposition [14]. Eichler's version of $\mathcal{M}_k/T\mathcal{M}_{2-k}$ is in terms of meromorphic modular forms and he must restrict to those that have well-defined Eichler integrals. Eisenstein series are not admitted and his isomorphism is with the parabolic cohomology group. Multi-valued Eichler integrals do not occur for weakly holomorphic forms.

Eichler applied his theory to give trace formulas for Hecke operators and most further developments on the number-theoretic side have been formulated in terms of cusp forms, mainly due to their connection with L-functions. Some classical references are [37], [32] and [28]. More recently, aspects of the Eichler–Shimura theory have been generalized to mock-modular forms [8]. Earlier, Eichler's theory was taken in a different direction by Ahlfors [2] and Bers [6] to study Kleinian groups. See also [29] and the more recent survey [30] and its references to this large body of research.

The study of indices of operators is of course also extensive; one need only mention the famous results of Atiyah and Singer beginning with [4], that determine the indices of certain elliptic operators on a manifold. Rather general linear differential operators in one variable with appropriate domains may be treated using the theory of unbounded linear operators in a Banach space. For more on this see [18] and its references.

For our modular differential operators, whose domain consists of analytic functions, it seems better to use topological vector spaces and semi-norms. This set-up also clarifies the role period functions play in the theory of these differential operators. If T were a (certainly unbounded) Fredholm operator on a Banach space the index of T could be equivalently defined to be $\alpha(T) - \beta(T)$, where

$$\beta(T) = \operatorname{codim} T, \tag{1.17}$$

rather than $\alpha(T) - \alpha(T')$, as we have done. By Theorems 1 and 2 it is true that $\beta(T) = \alpha(T')$ for T a holomorphic modular differential operator[1], but already in the simplest case where $Tf = f'$ we have that $\beta(T') = \infty$ while $\alpha(T) = 1$. See Example 3.5 below. Therefore the definition we give of the index seems appropriate for modular differential operators T since $\operatorname{ind} T' = -\operatorname{ind} T$.

Linear differential equations whose solution spaces are stable under a modular action have been much studied in recent years. References include the influential paper of Kaneko and Zagier [25]. Some other recent works are [19], [23], [24]. One natural problem that is treated in these papers is to find explicit solutions of specific equations that are modular or quasi-modular for the modular group or for a subgroup of finite index.

In the next section some standard facts from the theory of meromorphic modular forms for the full modular group are reviewed. This is followed by §3, which contains a needed result (Proposition 1) about holomorphic period functions. Since this result is of independent interest we illustrate it with several examples. In §4 the theory of modular differential operators and their Lagrange adjoints is developed. We prove Theorem 1 in §5 by utilizing a formula of Weil [44] for the dimension of a certain parabolic cohomology group. In §6 we apply generalized Eichler integrals together with Theorem 1 and a formula of Curran [9] for the dimension of the full cohomology group to prove Theorem 2.

[1] After seeing a preprint of this paper, Toshiteru Kinjo has given a direct proof that $\alpha(T) - \beta(T) = -2\ell_k - 1$. I thank Masanobu Kaneko for showing this to me.

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2. Weakly holomorphic modular forms

In the study of modular forms it is convenient to employ slash operators. For $k \in 2\mathbb{Z}$ and $A = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G = \mathrm{PSL}(2,\mathbb{R})$ define the weight k slash operator of A on f by

$$(f|_k A)(z) = (cz+d)^{-k} f(Az),$$

where $Az = \frac{az+b}{cz+d}$. Note that $f|_k(AB) = (f|_k A)|_k B$ for all $A, B \in G$ and we may extend the slash operator to the group ring $\mathbb{Z}[G]$. For simplicity we write as the default

$$f|_{2-k} A = f|A.$$

The modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) \subset G$ is a Fuchsian group. As such it has the presentation

$$\Gamma = \langle T, S, U \mid TSU^2 = S^2 = U^3 = 1 \rangle \tag{2.1}$$

where[2] $T = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \pm\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus Γ is generated by any two of T, S, U and may also be presented as the free product of the cyclic groups (S) and (U) generated by S and U, respectively.

Let $\mathcal{F}$ be the usual fundamental domain for Γ given by

$$\mathcal{F} = \{z \in \mathcal{H}; 0 \le \mathrm{Re}\, z \le \tfrac{1}{2}, |z| \ge 1\} \cup \{z \in \mathcal{H}; -\tfrac{1}{2} < \mathrm{Re}\, z < 0, |z| > 1\}. \tag{2.2}$$

A meromorphic modular form of weight $k \in 2\mathbb{Z}$ for Γ is a meromorphic function on $\mathcal{H}$ that satisfies

$$f|_k(1-A) = 0$$

for all $A \in \Gamma$ and is meromorphic at $i\infty$, meaning that its Fourier expansion (1.2) converges for $\mathrm{Re}\, z$ sufficiently large and that $\mathrm{ord}_\infty f > -\infty$. The divisor of a non-zero meromorphic modular form f, denoted (f), is defined multiplicatively in the usual way by its zeros and poles in $\mathcal{F} \cup \{i\infty\}$, except that at i or ρ we define the exponent by multiplying the usual order of f by $\frac{1}{2}$ or $\frac{1}{3}$, respectively. Denote by $|\mathfrak{d}|$ the degree of a divisor $\mathfrak{d}$, defined to be the sums of its exponents. The valence formula states that for any non-zero meromorphic modular form f

$$|(f)| = \frac{k}{12}. \tag{2.3}$$

One obtains (2.3) by integrating the Γ-invariant differential $(\frac{f'(z)}{f(z)} - \frac{k}{4\pi \mathrm{Im}\, z})dz$ around the boundary of $\mathcal{F}$, with the usual detours. For any divisor $\mathfrak{d}$ on $\mathcal{F} \cup \{i\infty\}$ let $M_k(\mathfrak{d})$ denote the space of meromorphic modular forms f of weight k for Γ that satisfy $\mathfrak{d}|(f)$. A version of the Riemann-Roch theorem that is useful in the context of modular forms was given by Petersson [34], who refers to his earlier work [33]. The theorem gives an invariant formula for the difference between the dimensions of two "dual" spaces of modular forms. For Γ and any $k \in 2\mathbb{Z}$ it takes the following elegant form. Note that $\ell_{2-k} = -\ell_k - 1$.

Theorem. [34, Satz 1.12 p. 20] *If the divisor $\mathfrak{d}$ on $\mathcal{F}$ has integral exponents then*

$$\dim M_k(\mathfrak{d}) - \dim M_{2-k}(\mathfrak{d}') = -|\mathfrak{d}| + \ell_k + 1, \tag{2.4}$$

where $\mathfrak{d}' = (i\infty)/\mathfrak{d}$ and ℓ_k is from (1.9).

[2]There is little danger that this T will be confused with a differential operator.

For $n \in \mathbb{Z}$ let $\mathcal{M}_{k,n} = M_k((i\infty)^n)$. In the usual notation

$$M_k = \mathcal{M}_{k,0} \quad \text{and} \quad S_k = \mathcal{M}_{k,1}.$$

By (2.4) and (2.3) we obtain the dimension formula

$$\dim \mathcal{M}_{k,n} = 1 - n + \ell_k \tag{2.5}$$

for even k and $n \leq k+1$.

Now $\mathcal{M}_k$ has a simple algebraic structure and (2.5) can also be established constructively (still using (2.3)). Since Δ from (1.14) does not vanish on $\mathcal{H}$ the j-function

$$j = \tfrac{E_4^3}{\Delta} = \tfrac{E_6^2}{\Delta} + 12^3$$

is in $\mathcal{M}_0$. It has integral Fourier coefficients

$$j(z) = q^{-1} + 744 + 196884q + \cdots .$$

Let $k' \in \{0, 4, 6, 8, 10, 14\}$ be uniquely determined by the equation $k = 12\ell_k + k'$ and set

$$f_k = \Delta^{\ell_k} E_{k'} \in \mathcal{M}_{k,\ell_k}. \tag{2.6}$$

Observe that these $E_{k'}$, together with E_2, are precisely those Eisenstein series that have integral Fourier coefficients. By (2.3) we have that

$$\operatorname{ord}_\infty f \leq \ell_k. \tag{2.7}$$

for non-zero $f \in \mathcal{M}_k$. We can now easily establish that

$$\mathcal{M}_k = f_k\, \mathbb{C}[j].$$

In fact, $\mathcal{M}_k$ has a natural basis $\mathcal{B} = \{f_{k,m}\}_{m \geq -\ell_k}$ with $f_{k,m}$ given by

$$f_{k,m}(z) = f_k P_{k,m}(j) = q^{-m} + \sum_{n > \ell_k} a_k(m,n)\, q^n, \tag{2.8}$$

where $P_{k,m}$ is a unique monic polynomial of degree $\ell_k + m$ called a Faber polynomial. Clearly $\{f_{k,-m}\}_{n \leq m \leq \ell_k}$ give a basis for $\mathcal{M}_{k,n}$, where $k > 2$ and $n \leq 1$, so (2.5) follows.

The Fourier coefficients of $f_{k,m}$ are integers that satisfy the duality relation

$$a_k(m,n) = -a_{2-k}(n,m). \tag{2.9}$$

This is a consequence of the generating function (see [3], [12])

$$\frac{f_k(\tau) f_{2-k}(z)}{j(z) - j(\tau)} = \sum_{m \geq -\ell_k} f_{k,m}(\tau) q^m, \tag{2.10}$$

where f_k is defined in (2.6).

3. Holomorphic period functions

Denote by $\mathcal{W}_{2-k}$ the space of all holomorphic functions ψ on $\mathcal{H}$ that satisfy

$$\psi|(1+S) = \psi|(1+U+U^2) = 0. \tag{3.1}$$

Recall our convention that $\psi|A = \psi|_{2-k}A$. As mentioned above, we refer to $\psi \in \mathcal{W}_{2-k}$ as a holomorphic period function of weight $2-k$.

Any $\psi \in \mathcal{W}_{2-k}$ satisfies the three term functional equation of a period function:

$$\psi|(1 - T - V) = 0, \tag{3.2}$$

where $V = T^\top$. This follows from the identity

$$1 - (T+V) = 1 + S - (1 + U + U^2)S.$$

Conversely, if ψ satisfies (3.2) then

$$F = \psi|(1+S) = \psi|(1+U+U^2)$$

must satisfy $F|(1-A) = 0$ for all $A \in \Gamma$. This argument is from [31, p. 249], where general period functions are studied.

If F is a periodic holomorphic function on $\mathcal{H}$ then

$$\psi = F|(1-S) \in \mathcal{W}_{2-k}. \tag{3.3}$$

To see this, observe that the first relation in (3.1) is trivial and the second follows by using

$$T^{-1} = SU^2 \quad \text{and} \quad U - S = TS - S = (T-1)S.$$

We next show the converse; given $\psi \in \mathcal{W}_{2-k}$ there exists F that is meromorphic at $i\infty$ where

$$\mathrm{ord}_\infty F \geq -\ell_k,$$

for which (3.3) holds. It follows from (2.7) applied with weight $2-k$ that such an F must be unique, so we denote it $\hat{\psi}$.

Proposition 1. *For any $k \in 2\mathbb{Z}$ let $\psi \in \mathcal{W}_{2-k}$. Then there exists a unique*

$$\hat{\psi}(z) = \sum_{n \geq -\ell_k} b(n) q^n$$

convergent on $\mathcal{H}$ such that $\psi(z) = \hat{\psi}|(1-S)$. The Fourier coefficients of $\hat{\psi}$ are given by

$$b(n) = \langle f_{k,n}, \psi \rangle, \tag{3.4}$$

where $f_{k,n}$ is defined in (2.8). In general, if $f(z) = \sum_m a(m) q^m \in \mathcal{M}_k$ then

$$\langle f, \psi \rangle = \sum_{n+m=0} a(m) b(n), \tag{3.5}$$

where the sum is finite. We also have

$$\langle f, \psi \rangle = \int_{\rho^2}^{\rho} f(z) \hat{\psi}(z) dz. \tag{3.6}$$

Proof. Denote by C the directed circular path from i to ρ on the edge of $\overline{\mathcal{F}}$, where $\mathcal{F}$ was given in (2.2). Since the Faber expansion (2.10) converges uniformly on compact subsets in τ provided that $\mathrm{Im}(z)$ is sufficiently large, the integral

$$F(z) = f_{2-k}(z) \int_C \frac{f_k(\tau)\psi(\tau)}{j(z) - j(\tau)} d\tau \tag{3.7}$$

will have the desired Fourier expansion for $\mathrm{Im}(z)$ sufficiently large. We must show that F has an analytic continuation and that it satisfies

$$\psi = F|(1-S). \tag{3.8}$$

We can do this by showing that we may continue F to the neighborhood of any point of $\mathcal{H}$. By the monodromy theorem we will get a unique continuation since $\mathcal{H}$ is simply connected.

For fixed z with $E_{14}(z) = E_4^2(z)E_6(z) \neq 0$ the integrand of (3.7) has a simple pole at $\tau = z$. Now for f_k defined in (2.6) we have

$$f_{2-k}(z) f_k(z) = E_{14}(z)/\Delta(z) = -j'(z).$$

Hence

$$\mathrm{Res}_{\tau=z}\frac{f_{2-k}(z)f_k(\tau)}{j(z)-j(\tau)}=\lim_{\tau\to z}\frac{(\tau-z)f_{2-k}(z)f_k(z)}{j(z)-j(\tau)}=\frac{1}{2\pi i}. \tag{3.9}$$

The zeros of E_{14} are exactly the elliptic points.

The images of C under Γ decompose $\mathcal{H}$ into infinitely many connected components. Let Ω_1 be any component and let Ω_2 be an adjacent component. Let $F_j(z)$ be the analytic function in Ω_j defined by the integral in (3.7). Choose any non-elliptic point z_1 on an edge between Ω_1 and Ω_2. Make a small semi-circular deformation[3] of C so that the new integral continues $F_1(z)$ to a neighborhood N of z_1. For each z in $N\cap\Omega_2$ the value of the analytic continuation of F_1 is given by

$$F_1(z)=F_2(z)\pm\psi(z), \tag{3.10}$$

coming from the pole of the integrand at $\tau=z$. Here the sign is the same for all such z and depends only on the orientation of the edge containing z_0. Since ψ is holomorphic in $\mathcal{H}$, this provides an analytic continuation of F to a neighborhood of every non-elliptic point of $\mathcal{H}$. We want to show that this continuation is unique. We do this by continuing the original F to *every* point of $\mathcal{H}$.

So far we have not used that ψ is a holomorphic period function. By the first formula in (3.1) we see that we may write

$$F(z)=\tfrac12 f_{2-k}(z)\int_{\rho^2}^{\rho}\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau, \tag{3.11}$$

where now we take a straight line segment between ρ^2 and ρ, whose images under Γ thus avoid images of i. By the same procedure as before, we may now continue F to every image of i. To treat the images of ρ, observe that by the second formula in (3.1) we have

$$3\int_i^{\rho}\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau=\Big(2\int_i^{\rho}-\int_{1+i}^{\rho}-\int_{\frac12+\frac{i}{2}}^{\rho}\Big)\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau.$$

Hence

$$F(z)=\tfrac13 f_{2-k}(z)\Big(\int_i^{1+i}\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau+\int_i^{\frac12+\frac{i}{2}}\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau\Big),$$

where now the contours are chosen as line segments in each integral to avoid images of ρ under Γ. In this way we may also continue F to each image of ρ. Thus we have continued F to all of $\mathcal{H}$. In particular we know that its Fourier series converges in $\mathcal{H}$.

The functional equation now also follows. For z in an open disc contained in $\mathcal{F}\subset\Omega_1$, we have by (3.10) that

$$F(z)=(F|S)(z)+\psi(z),$$

since for such z we have that $F|S=F_2$ by (3.7) and the choice of sign of $\psi(z)$ is easily verified. Thus (3.8) holds throughout $\mathcal{H}$ by analytic continuation and we have that

$$\hat{\psi}(z)=f_{2-k}(z)\int_i^{\rho}\frac{f_k(\tau)\psi(\tau)}{j(z)-j(\tau)}d\tau. \tag{3.12}$$

The last two statements of Proposition 1 now follow easily. □

[3]The deformation can be either away from or toward the origin depending on the orientation of the arc on which z_1 lies.

Remarks.

(1) Proposition 1 was first proven shortly after [11] was written and was motivated by conversations with Özlem Imamoḡlu and Árpad Tóth. It remained unpublished until now.
(2) The bilinear form $\langle f, \psi \rangle$ when written in the form (3.6), where ψ is a period polynomial, is essentially one used by Eichler [13]. When $k = 2$ this form occurs in a paper of Hecke [20].

The following corollary of Proposition 1 justifies the claims made in the introduction about $\mathcal{M}_k$ and $\mathcal{W}_{2-k}$ that follow from their being a dual pair with respect to the bilinear form $\langle \cdot, \cdot \rangle$.

Corollary 1. *Fix $k \in 2\mathbb{Z}$. Then*

i) Fix $\psi \in \mathcal{W}_{2-k}$. If $\langle f, \psi \rangle = 0$ for all $f \in \mathcal{M}_k$ then $\psi = 0$.

ii) Fix $f \in \mathcal{M}_k$. If $\langle f, \psi \rangle = 0$ for all $\psi \in \mathcal{W}_{2-k}$ then $f = 0$.

Proof. We will use the basis $\{f_{k,m}\}_{m \geq -\ell_k}$ for $\mathcal{M}_k$, where

$$f_{k,m}(z) = q^{-m} + \sum_{n > \ell_k} a_k(m, n)\, q^n.$$

For i), if $\langle f, \psi \rangle = 0$ for all $f \in \mathcal{M}_k$ then $\langle f_{k,m}, \psi \rangle = 0$ for all $m \geq -\ell_k$ so from (3.4) we have that $b(m) = 0$ for all m, hence $\psi = 0$.

For ii), suppose that $\langle f, \psi \rangle = 0$ for all $\psi \in \mathcal{W}_{2-k}$. Write

$$f = \sum_{m \geq -\ell_k} c(m) f_{k,m},$$

a finite sum. Then choose $\psi = \hat{\psi}|(1 - S)$ where $\hat{\psi}(q) = \sum_m \overline{c(m)} e^{2\pi i m z}$. Using (3.4) again,

$$0 = \sum c(m) \langle f_{k,m}, \psi \rangle = \sum c(m) b(m) = \sum |c(m)|^2,$$

so for all m we have $c(m) = 0$. $\square$

Example 3.1. Period polynomials are of basic arithmetic interest. Suppose that

$$L(s) = \sum_{n \geq 1} a(n) n^{-s}$$

converges for $\mathrm{Re}(s)$ sufficiently large and that its completion $\Lambda(s) = (2\pi)^{-s} \Gamma(s) L(s)$ is entire and satisfies the functional equation $\Lambda(k - s) = (-1)^{\frac{k}{2}} \Lambda(s)$. Then

$$\psi(z) = (-1)^{\frac{k}{2}} \sum_{m=0}^{k-2} \tfrac{(2\pi i)^m}{m!} L(k - m - 1) z^m \in \mathcal{W}_k. \tag{3.13}$$

In fact, Hecke's converse theorem [21] together with a standard calculation of the Eichler integral gives that

$$\hat{\psi}(z) = \sum_{n \geq 1} a(n) n^{1-k} q^n.$$

Thus by Proposition 1 we get the pretty formula

$$n^{1-k} a(n) = (-1)^{\frac{k}{2}} \sum_{m=0}^{k-2} \tfrac{(2\pi i)^m L(k-m-1)}{m!} \int_i^{\rho} z^m f_{k,n}(z) dz.$$

Example 3.2. In case $k = 0$ it is usual to write

$$f_{0,m} = P_{0,m}(j) = j_m,$$

where the first few Faber polynomials $P_{0,m}$ are given by

$$P_{0,0}(x) = 1,\ P_{0,1}(x) = x - 744,\ P_{0,2}(x) = x^2 - 1488x + 159768,$$
$$P_{0,3}(x) = x^3 - 2232x^2 + 1069956x - 36866976.$$

The real vector space $\mathcal{M}_0^{\mathbb{R}}$ consisting of $f \in \mathcal{M}_0$ with real Fourier coefficients is an inner product space with the inner product defined by

$$(f, g) = \tfrac{6}{\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta.$$

A calculation using (1.13) and (1.14) gives the relation $\frac{\Delta'}{\Delta} = E_2$, which implies Hurwitz's result

$$z^{-2}E_2(-\tfrac{1}{z}) = E_2(z) - \tfrac{6i}{\pi z}. \tag{3.14}$$

Note that this shows that $E_2 = \hat{\psi}$, where $\psi(z) = \frac{6i}{\pi z} \in \mathcal{W}_2$ is the simplest holomorphic period function that is not a polynomial. Since $f(e^{i\theta}), g(e^{i\theta})$ are real if $f, g \in \mathcal{M}_0^{\mathbb{R}}$, we have $(f, g) = \langle fg, \psi\rangle$. Thus if $f(z)g(z) = \sum_n c(n)q^n$ then by (3.5) we reproduce the known formula of Atkin (see [25])

$$(f, g) = \sum_{n+m=0} c(n)\sigma'(n),$$

where $\sigma'(1) = 1$ and $\sigma'(n) = -24\sigma(n)$ for $n > 1$. The monic polynomials A_m in the orthogonal basis $\{A_m(j)\}_{m\geq 0}$, formed by orthogonalizing $\{j_m\}_{m\geq 0}$ with respect to $(\cdot, \cdot)$, have numerous interesting arithmetic properties. They are called Atkin polynomials. The first few are given by

$$A_0(j) = 1,\quad A_1(j) = j - 720,\quad A_2(j) = j^2 - 1640j + 269280.$$

A recent paper [15] has related the Atkin polynomials to associated Jacobi polynomials.

Example 3.3. Indefinite binary quadratic forms yield a general construction of holomorphic period functions that are rational functions of z. For d a positive discriminant let

$$Q(x, y) = ax^2 + bxy + cy^2 = [a, b, c]$$

be an integral binary quadratic form of discriminant $b^2 - 4ac = d$ and (Q) the class of forms (under the usual action of Γ) that contains Q. Then for any $k \in 2\mathbb{Z}$

$$\psi_{2-k,Q}(z) = \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \operatorname{sgn}(c)(az^2 + bz + c)^{\frac{k}{2}-1} \in \mathcal{W}_{2-k}.$$

When $k > 2$ the associated holomorphic period functions are polynomials that were introduced and studied in [28]. For $k < 2$ they are rational period functions and were introduced by Knopp [27] and have been much studied since. For some references see [11]. It was shown in [11, Theorem 3] that for any $k \in 2\mathbb{Z}$ the Fourier coefficients of $\hat{\psi}_{2-k,Q}$ are given by certain invariant cycle integrals of the $f_{k,n}$:

$$\hat{\psi}_{2-k,Q}(z) = \sum_{m\geq -\ell_k} r_Q(f_{k,m})q^m \quad \text{where} \quad r_Q(f) = \int_{\tau}^{A\tau} f(z)Q(z,1)^{\frac{k}{2}-1}dz. \tag{3.15}$$

Here the integral is over any smooth curve from some $\tau \in \mathcal{H}$ to $A\tau$, where $A \in \Gamma$ is a certain hyperbolic element determined by Q. Theorem 1 actually implies the formulas of

(3.15). To see this, apply Proposition 1 to $\psi_{2-k,Q}$, making use of the well-known fact that we may assume that A is a word in T and V, then transform the integral in (3.12) to the cycle integral. Details are readily supplied. The proof is somewhat similar to that of Theorem 7 in [28, p. 233].

Example 3.4. There exist explicit holomorphic period functions that are also modular forms. For any $k \in \mathbb{R}$ define, using the principal branch of the logarithm,

$$\Delta_k := \exp(\tfrac{k}{12} \log \Delta). \tag{3.16}$$

When $k \in 2\mathbb{Z}$ satisfies $k \equiv \pm 2 \pmod{12}$ then $\psi = \Delta_k \in \mathcal{W}_k$. This is an easy consequence of the fact that

$$\Delta_k|_k A = \chi_k(A)\Delta_k \quad \text{with } A = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tag{3.17}$$

defines a character χ_k of Γ with $\chi_k(S) = i^k$ and $\chi_k(U) = \rho^k = e(\frac{k}{6})$. These period functions are modular forms for the commutator subgroup Γ' of Γ.

For example, if $\psi = \Delta_{-2}$ then $\psi \in \mathcal{W}_{-2}$. By changing variables $x \mapsto J(\tau) = \frac{1}{1728} j(\tau)$ we get by (3.12)

$$\Delta_2(z)\hat{\psi}(z) = \frac{E_{10}(z)}{\Delta_{10}(z)} \int_i^\rho \frac{E_4(\tau)\Delta(\tau)^{-\frac{1}{6}}}{j(z) - j(\tau)} d\tau = t^{\frac{1}{3}}(t-1)^{\frac{1}{2}} \int_0^1 x^{-\frac{1}{3}}(x-1)^{-\frac{1}{2}}(x-t)^{-1}\, dx, \tag{3.18}$$

where $t = J(z)$. The right-hand side of (3.18) is, after a linear change of variables, a Jacobi function of the second kind (see e.g. [41]). For $\hat{\psi}_1 := c\hat{\psi}$, where

$$c = \sqrt{\tfrac{\pi}{3}}\, \Gamma\left(\tfrac{7}{6}\right) \Gamma\left(\tfrac{2}{3}\right)^{-1},$$

we get after a calculation the formula

$$\begin{aligned} \hat{\psi}_1(z) &= \tfrac{E_6(z)}{E_4(z)^2}\, {}_2F_1\left(\tfrac{2}{3}, 1; \tfrac{7}{6}; \tfrac{1}{J(z)}\right) \\ &= 1 + \tfrac{24}{7}\, q + \tfrac{1080}{91}\, q^2 + \tfrac{8160}{247}\, q^3 + \tfrac{741576}{8645}\, q^4 + \tfrac{7776432}{38285}\, q^5 + O(q^6). \end{aligned} \tag{3.19}$$

It can be checked that for T from the second part of Example 1.2

$$T^*\psi_1 = -\psi_1^{(3)} + \tfrac{31}{36} E_4 \psi_1' + (\tfrac{31}{72} E_4' - \tfrac{5}{36} E_6)\psi_1 = 0. \tag{3.20}$$

Therefore $\psi_1 \in \ker T'$, given that $T' = T^*$. The claim made there that $\dim \ker T' \leq 2$ can be checked numerically since if $\psi \in \ker T'$ then $T^*\hat{\psi} \in M_4$. The resulting recursion formula shows that there are at most two linearly independent solutions to the equation $T^*\psi = 0$, where we use that M_4 is spanned by E_4. For example, $T^*\hat{\psi}_1 = \frac{36}{5} E_4$. In fact, $\ker T'$ is spanned by ψ_1 and ψ_2 where

$$\hat{\psi}_2(z) = q + \tfrac{8640}{221}\, q^2 + \tfrac{75074580}{96577}\, q^3 + \tfrac{146742747136}{14003665}\, q^4 + \tfrac{47154337662474}{434113615}\, q^5 + O\left(q^6\right)$$

satisfies $T^*\hat{\psi}_2 = 0$.

Example 3.5. As was mentioned at the end of the Introduction around (1.17), for $Tf = f'$ we have that

$$\beta(T') = \operatorname{codim} T' = \dim \mathcal{W}_2 / T'\mathcal{W}_0 = \infty.$$

We claim that for any $\psi \in \mathcal{W}_2$ with $\psi = \hat{\psi}|(1-S)$, where

$$\hat{\psi}(z) = \sum_{n \geq 0} b(n) q^n$$

has the property that $b(0) \neq 0$, there is no $\phi \in \mathcal{W}_0$ such that $T'\phi = -\phi' = \psi$. For ϕ we would have $\phi = \hat{\phi}|_0(1-S)$ with $\hat{\phi}(z) = \sum_{n\geq -1} c(n)q^n$. Then $\hat{\psi} = -(\hat{\phi})' + f$ for some $f \in \mathcal{M}_2$, and such $\hat{\psi}$ has zero constant term. Finally, we may choose the $b(n)$ in $\hat{\psi}$ arbitrarily.

4. Modular differential operators and their Lagrange adjoints

We need some results about differential operators, in particular properties of the Lagrange adjoint and the bilinear concomitant. Let D be the linear differential operator of order $m \geq 0$ defined by

$$Df = h_0 f^{(m)} + h_1 f^{(m-1)} + \cdots + h_m f, \tag{4.1}$$

where now we only assume that each h_j is holomorphic on $\mathcal{H}$ and that $h_0 \neq 0$. Then the Lagrange adjoint of D is defined by

$$D^* f = (-1)^m (h_0 f)^{(m)} + (-1)^{m-1}(h_1 f)^{(m-1)} + \cdots + h_m f.$$

A basic result due to Lagrange is that (see [36, p. 38])

$$(D^*)^* = D.$$

The bilinear concomitant associated to D is defined by $[f,g]_D = 0$ if D has order zero and otherwise

$$\begin{aligned}[f,g]_D = {} & g f^{(m-1)} \\ & + \big(h_1 g - (h_0 g)'\big) f^{(m-2)} + \\ & \quad \vdots \\ & + \big(h_{m-1} g - (h_{m-2} g)' + \cdots + (-1)^{m-1}(h_0 g)^{(m-1)}\big) f.\end{aligned} \tag{4.2}$$

We have Lagrange's identity (see e.g. [22]):

$$g\, D\, f - f\, D^* g = \tfrac{1}{2\pi i}\tfrac{d}{dz}[f,g]_D. \tag{4.3}$$

This implies that $[f,g]_D$ is a scalar if $Df = 0$ and $D^*g = 0$ and that $[f,g]_D = \mp[g,f]_D$ when $D^* = \pm D$. The following additional facts were given by Frobenius [17] (see also [36]):

Lemma 1. *Let D, E be linear differential operators. Then*
a) $(D+E)^* = D^* + E^*$,
b) $(DE)^* = E^* D^*$,
c) $[f,g]_{D+E} = [f,g]_D + [f,g]_E$,
d) $[f,g]_{DE} = [Ef,g]_D + [f, D^*g]_E$.

Definition. *The differential operator D from (4.1) is modular of order m and weight k if*

$$D(f|_k A) = (Df)|_{k+2m} A \tag{4.4}$$

for all $A \in \Gamma$ and each h_j, which is necessarily periodic, is meromorphic at $i\infty$. We also suppose that h_0 is a non-zero constant. Such an operator is said to be holomorphic if $\mathrm{ord}_\infty h_j \geq 0$ *for each j.*

To compare with our earlier convention, a modular differential operator of the form (1.6) is modular of order $k-1$ and weight $2-k$.

It is very convenient to express these modular differential operators in terms of a special modular derivative, usually called the Serre derivative. The Serre derivative of weight k for any real k is defined by

$$D_k f := f' - \tfrac{k}{12}E_2 f, \tag{4.5}$$

where E_2 was given above in (1.13). Using (3.14) it is easy to see that for $k \in 2\mathbb{Z}$ the Serre derivative D_k is modular of order 1 and weight k. Also, for $k_1, k_2 \in \mathbb{R}$ it obeys the product rule

$$D_{k_1+k_2}(fg) = g(D_{k_1}f) + f(D_{k_2}g). \tag{4.6}$$

Define the higher derivatives by

$$D_k^m := D_{k+2m-2}\cdots D_{k+2}D_k \tag{4.7}$$

where $m \in \mathbb{Z}^+$. Clearly D_k^m is modular of order m and weight k.

Lemma 2. *A differential operator D of the form (4.1), where h_0 is a non-zero constant, is modular of order $m \geq 0$ and weight k if and only if*

$$D = f_0 D_k^m + f_1 D_k^{m-1} + \cdots + f_{m-1}D_k^1 + f_m \tag{4.8}$$

where $f_j \in \mathcal{M}_{2j}$ for $j = 0, \ldots, m$ with f_0 a non-zero constant. In this case D is holomorphic if and only if $f_j \in M_{2j}$ for each such j.

Proof. The "if" part of the first statement is clear. For the converse, observe that if D defined by

$$Df = h_0 f^{(m)} + h_1\, f^{(m-1)} + \cdots + h_{m-1}\, f' + h_m f$$

is modular of order m and weight k, then by (4.4)

$$D\big(z^{-k}f(-\tfrac{1}{z})\big) = z^{-k-2m}(Df)(-\tfrac{1}{z}). \tag{4.9}$$

Choose $f_0 = h_0$. Now define f_1 by

$$(D - f_0 D_k^m)f = f_1 f^{(m-1)} + \text{ lower order terms.} \tag{4.10}$$

We claim that $f_1 \in \mathcal{M}_2$. By (4.9) we must have

$$\begin{aligned} f_1(z)\big(z^{-k}f(-\tfrac{1}{z})\big)^{(m-1)} &= z^{-k-m}f_1(-\tfrac{1}{z})z^{-m-2}f^{(m-1)}(-\tfrac{1}{z}) + \text{ lower order terms} \\ &= z^{-k-m}f_1(-\tfrac{1}{z})\big(z^{m-2}f(-\tfrac{1}{z})\big)^{(m-1)} + \text{ lower order terms,} \end{aligned}$$

by (1.10) and so

$$f_1(z)\big(z^{-k}f(-\tfrac{1}{z})\big)^{(m-1)} = z^{-2}f_1(-\tfrac{1}{z})\big(z^{-k}f(-\tfrac{1}{z})\big)^{(m-1)} + \text{ lower order terms.}$$

Since f_1 is periodic and meromorphic at $i\infty$, it follows that $f_1 \in \mathcal{M}_2$. Of course it is possible that $f_1 = 0$, which in fact must hold if each h_j is holomorphic at $i\infty$ since then $f_1 \in M_2 = \{0\}$. By continuing this process we are led ultimately to the desired representation.

The final statement follows easily. □

For results related to Lemma 2 in cases $m = 2$ and $m = 3$ see [23] and [24].

Lemma 3. *Suppose that*

$$T = f_0 D_{2-k}^{k-1} + \cdots + f_{k-1} \tag{4.11}$$

is modular of order $k-1$ and weight $2-k$.

i) The Lagrange adjoint T^* of T is determined by

$$T^* f = -f_0 D_{2-k}^{k-1}(f) + D_{4-k}^{k-2}(f_1 f) + \cdots - D_{k-2}^1(f_{k-2}f) + f_{k-1}f.$$

ii) T^* is modular of order $k-1$ and weight $2-k$.

Proof. The proof of i) uses a) and b) of Lemma 1. From (4.5) we have $D_k^* = -D_{-k}$. Using b) of Lemma 1 we get

$$(D_{k+2m-2}\cdots D_{k+2}D_k)^* = (-1)^m(D_{-k}D_{-k-2}\cdots D_{-k-2m+2})$$

and for $f_m \in \mathcal{M}_{2m}$ with $m = 0,\dots,k-1$ we deduce that

$$(f_m D_{2-k}^{k-1-m})^* = (-1)^m D_{2-k+2m}^{k-1-m} f_m.$$

Thus by this and a) of Lemma 1 the proof of i) is finished.

Now ii) follows easily from the formula in i) and (4.6). □

Example 4.1. Consider (1.11) when $k = 4$:

$$Tf = f^{(3)} + gf' + (\tfrac{1}{2}g' + h)f,$$

where $g \in \mathcal{M}_4$ and $h \in \mathcal{M}_6$. We have

$$T = D_{-2}^3 + f_2 D_{-2} + f_3,$$

where $f_2 = \frac{1}{36}E_4 + g \in \mathcal{M}_4$ and $f_3 = h - \frac{1}{216}E_6 + \frac{1}{2}D_4 g \in \mathcal{M}_6$. Also,

$$T^*f = -f^{(3)} - gf' + (-\tfrac{1}{2}g' + h)f = -D_{-2}^3 f - D_{-2}(f_2 f) + f_3 f.$$

We now can justify the other basic statements made about T and T' in the introduction.

Note. From now on we always assume without saying so that T denotes a modular differential operator of order $k-1$ and weight $2-k$, where k is a positive even integer.

Lemma 4. *The operator T is a continuous linear map from $\mathcal{M}_{2-k}$ to $\mathcal{M}_k$. The adjoint $T' : \mathcal{W}_{2-k} \to \mathcal{W}_k$ is given by $T' = T^*$.*

Proof. First note that T maps $\mathcal{W}_{2-k}$ to $\mathcal{W}_k$. Thus by ii) of Lemma 3 the Lagrange adjoint T^* maps $\mathcal{W}_{2-k}$ to $\mathcal{W}_k$. If $f \in \mathcal{M}_{2-k}$ and $\psi \in \mathcal{W}_k$ then by Proposition 1 and (4.3)

$$\langle Tf, \psi\rangle = \int_{\rho^2}^{\rho} \hat{\psi}(z)Tf(z)\,dz = \int_{\rho^2}^{\rho} f(z)T^*\hat{\psi}(z)\,dz = \langle f, T^*\psi\rangle, \tag{4.12}$$

since $[f,\hat{\psi}]_T$ is periodic. By (4.12) we have that

$$|\langle Tf,\psi\rangle| = |\langle f,\phi\rangle|$$

where $\phi = T^*\psi$ so from e.g. [38, Thm V.2] it follows that T is continuous. Furthermore, we must have $T^* = T'$. □

By default we write $[f,g] = [f,g]_T$.

Lemma 5. *For f, g holomorphic on $\mathcal{H}$ and $A \in \Gamma$ we have*

$$[f|A,\ g|A] = [f,g]\,|_0 A.$$

Proof. It follows from (4.3) and the fact that both T and T^* are modular that

$$\tfrac{d}{dz}[f|A, g|A] = (g|A)(T\,f)|_k - (f|A)\,(T^*g)|_k A = \tfrac{d}{dz}[f,g]|_2 A.$$

Thus for some $c(A) \in \mathbb{C}$

$$[f|A, g|A] = [f,g]|_0 A + c(A).$$

Now for $A, B \in \Gamma$ we have that $c(AB) = c(A)+c(B)$. Since Γ has no nonzero homomorphisms into $\mathbb{C}$ we must have $c(A) = 0$. □

Example 4.2. For $Tf = f^{(k-1)}$, the bilinear concomitant coincides with a special Rankin-Cohen bracket denoted in this case by $[f,g]_{k-2}$ (see [45, p. 53]):

$$[f,g]_T = \sum_{\substack{r,s\geq 0\\ r+s=k-2}} (-1)^r f^{(r)} g^{(s)}. \tag{4.13}$$

When restricted to polynomials $f(z) = \sum_{m\leq k-2} a_m z^m$ and $g(z) = \sum_{n\leq k-2} b_n z^n$ it reads

$$[f,g] = \sum_{n=0}^{k-2} (-1)^n \tbinom{k-2}{n}^{-1} a_n b_{k-2-n},$$

which is a well-known symmetric, Γ-invariant and non-degenerate pairing (see [28, p. 243]).

Example 4.3. We can apply c) and d) of Lemma 1 to compute $[f,g]_T$ when T is written as in (4.11). For instance, when $T = D_{2-k}^{k-1}$ we derive a formula similar to (4.13):

$$[f,g]_T = \sum_{\substack{r,s\geq 0\\ r+s=k-2}} (-1)^r D_{2-k}^r(f) D_{2-k}^s(g).$$

Let $\mathcal{P}_T$ be the space of all solutions f of the equation $Tf = 0$. It follows from the standard theory of ordinary differential equations (see e.g. [22] or [36]) that $\dim \mathcal{P}_T = \dim \mathcal{P}_{T^*} = k-1$. When restricted to $\mathcal{P}_T \times \mathcal{P}_{T^*}$ the bilinear concomitant is scalar valued. Since $\mathcal{P}_T$ is acted on by the weight $2-k$ slash operator, the *monodromy map* $f \mapsto f|A$ for $f \in \mathcal{P}_T$ and $A \in \Gamma$ defines a representation $\Gamma \to \mathrm{GL}(\mathcal{P}_T)$.

Lemma 6. *The spaces $\mathcal{P}_T$ and $\mathcal{P}_{T^*}$ are dual with repect to the bilinear concomitant $[\cdot,\cdot]_T$. The dual of the monodromy map $f \mapsto f|A$ on $\mathcal{P}_T$ for $A \in \Gamma$ is given on $\mathcal{P}_{T^*}$ by $g \mapsto g|A^{-1}$.*

Proof. This is a "modern" formulation of classical results due to Jacobi, Fuchs and Frobenius. Let

$$W(z) = \begin{vmatrix} f_0 & f_1 & \cdots & f_{k-2} \\ f_0' & f_1' & \cdots & f_{k-2}' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k-2)} & f_2^{(k-2)} & \cdots & f_{k-2}^{(k-2)} \end{vmatrix}$$

be the Wronskian of the basis $\{f_0, \ldots, f_{k-2}\}$ for $\mathcal{P}_T$. Now define for each $m = 1, \ldots, k-1$

$$g_m(z) = \tfrac{W_m(z)}{W(z)}, \tag{4.14}$$

where $W_m(z)$ is the cofactor of the element $f_m^{(k-2)}$. In other words, let $\{g_0, \ldots, g_{k-2}\}$ be the final row of $W(z)^{-1}$. It is a classical result (see e.g. [40, p. 62]) easily proven that $\{g_0, \ldots, g_{k-2}\}$ is a basis for $\mathcal{P}_{T^*}$. A computation shows it to be a dual basis for $\{f_0, \ldots, f_{k-2}\}$ so that $[f_m, g_n] = \delta_{m,n}$, thereby making $\mathcal{P}_{T^*}$ dual to $\mathcal{P}_T$.

The second statement of the Lemma was pointed out by Fuchs and Frobenius.[4] It now follows immediately from Lemma 5 applied to the scalar $[f,g]$:

$$[f|A, g] = [f, g|A^{-1}].$$

□

[4] See the footnote on p.408 of the paper [17]. For another proof see [40, p.65].

For a subgroup $\Gamma_1 \subset \Gamma$ let

$$\mathcal{P}_T^{\Gamma_1} = \{f \in \mathcal{P}_T; f|A = f \text{ for all } A \in \Gamma_1\}.$$

Lemma 7. *We have that*

$$\dim (\mathcal{P}'_T)^\Gamma = \dim \mathcal{P}^\Gamma_{T^*},$$

where $\mathcal{P}'_T$ is the dual of $\mathcal{P}_T$. If T is holomorphic then any $f \in \mathcal{P}_T^{(T)}$ must be meromorphic at $i\infty$. In particular,

$$\mathcal{P}^\Gamma_T \subset \mathcal{M}_{2-k}.$$

Proof. The first statement follows from Lemma 6.

If T is holomorphic then the equation $Tf = 0$ takes the following form in the variable q:

$$\tfrac{d^{k-1}}{dq^{k-1}} f + g_1 q^{-1} \tfrac{d^{k-2}}{dq^{k-2}} f + \cdots + q^{1-k} g_{k-1} f = 0,$$

where each g_m is holomorphic at $q = 0$. This equation is regular in $|q| < 1$ except that it has a regular singularity at $q = 0$. By a fundamental theorem of Fuchs (see [22, p. 365]) every solution has the form

$$f(z) = p(z) q^\alpha \sum_{n\geq 0} a(n) q^n,$$

where $p \in \mathbb{C}[z]$ has degree at most $k-2$ and $\alpha \in \mathbb{C}$. The rest of the lemma now follows. $\square$

For Δ_k from (3.16) we have $D_k\,\Delta_k = 0$ and so deduce from (4.6) that

$$D_{k_1+k_2}(\Delta_{k_1} f) = \Delta_{k_1} D_{k_2}(f).$$

From this we have

$$D^m_{k_1+k_2}(\Delta_{k_1} f) = \Delta_{k_1} D^m_{k_2} f. \tag{4.15}$$

This formula may be used to shift the weight of a modular differential operator.

Example 4.4. Consider the operator

$$T = D^3_{-2} - \kappa E_4 D_{-2} + \tfrac{1}{6}\kappa E_6,$$

where κ is a constant. The equation $Tf = 0$ is by (4.15) equivalent to

$$D_0^3 \phi - \kappa E_4 (D_0 \phi) + \tfrac{1}{6} \kappa E_6\, \phi = 0$$

for $\phi = \Delta_2 f$. Changing variables $t = J(z)$ we get for $w(t) = \phi(z)$ the third-order equation

$$t^2(1-t) w''' - \tfrac{1}{2}(7t-4) t\, w'' - \tfrac{1}{9}\big((14 - 9\kappa)t - 2\big) w' + \tfrac{1}{6}\kappa w = 0. \tag{4.16}$$

Here we have used that $\frac{(J')^2}{J(1-J)} = -E_4$ and $\frac{(J')^3}{J^2(1-J)} = E_6$.

When $\kappa = \frac{1}{3} n(1+3n)$ a calculation shows that (4.16) is satisfied by the generalized hypergeometric series

$${}_3F_2(-n, \tfrac{1}{3}+n, \tfrac{1}{6}; \tfrac{1}{3}, \tfrac{2}{3} |\, t).$$

For each $n = 0, 1, 2, \ldots$ this series terminates and becomes a polynomial of degree n in t. Thus we have

$$\psi_n(z) := \Delta_{-2}(z) P_n\big(J(z)\big) \in \mathcal{W}_{-2} \cap \mathcal{P}_T,$$

where P_n is this polynomial normalized in some way by multiplying it by a non-zero constant. Actually we may take

$$P_n(t) = \begin{cases} \Big(P_{\frac{n}{2}}^{(-\frac{1}{2},-\frac{1}{3})}(1-2t)\Big)^2, & n \text{ even} \\ (1-t)\Big(P_{\frac{n-1}{2}}^{(\frac{1}{2},-\frac{1}{3})}(1-2t)\Big)^2, & n \text{ odd,} \end{cases}$$

where $P_n^{(\alpha,\beta)}(t)$ is the usual Jacobi polynomial. By Proposition 1 we get the following generalization of (3.18), which is the $n=0$ case:

$$\Delta_2(z)\hat{\psi}_n(z) = c_n t^{\frac{1}{3}}(t-1)^{\frac{1}{2}} \int_0^1 P_n(x)x^{-\frac{1}{3}}(x-1)^{-\frac{1}{2}}(x-t)^{-1}\,dx, \tag{4.17}$$

where $t = J(z)$ and c_n is a constant. We can show that $\hat{\psi}_n \in \mathcal{P}_T$ since the integral on the right hand side of (4.17) satisfies (4.16) for each n, when $\kappa = \frac{1}{3}n(3n+1)$.

Example 4.5. Using (4.15) repeatedly, a calculation shows that for $T = D_{2-k}^{k-1}$ a basis for $\mathcal{P}_T$ is given by $\{v_1, \ldots, v_{k-2}\}$, where

$$v_j(z) = \Delta_{2-k}(z)u^j(z)$$

and

$$u(z) = 2\pi i \int_{i\infty}^{z} \Delta_2(\tau)\,d\tau = q^{\frac{1}{6}}(6 - \tfrac{24}{7}q + \tfrac{12}{13}q^2 \cdots). \tag{4.18}$$

It is well-known that $z \mapsto \big(\wp(u(z)), \wp'(u(z))\big)$ maps $\Gamma'\backslash\mathcal{H}$ to the elliptic curve given by

$$y^2 = 4x^3 - \tfrac{4}{27},$$

where Γ' is the commutator subgroup of Γ. Using this correspondence we may interpret our results in terms of operators on elliptic functions having extra symmetries inherited from this CM curve. Roughly speaking, the iterated Serre operator of order m in z corresponds to $\frac{d^m}{du^m}$.

5. Parabolic cohomology

In this section we prove Theorem 1. For this we apply a formula of Weil [44] for the dimension of the parabolic cohomology group of Γ acting on $\mathcal{P}_T$. This formula also depends on the action of Γ on the dual space $\mathcal{P}'_T$.

After Weil, a 1-cocycle on Γ with coefficients in $\mathcal{P}_T$ is a map $\sigma : \Gamma \to \mathcal{P}_T$ with the property that for each $A \in \Gamma$

$$f \mapsto f|A + \sigma_A \tag{5.1}$$

gives an affine automorphism of $\mathcal{P}_T$. Equivalently, for all $A, B \in \Gamma$

$$\sigma_{AB} = \sigma_A|B + \sigma_B. \tag{5.2}$$

A cocycle σ is a coboundary if the associated affine map (5.1) is conjugate under translation by some *fixed* $h \in \mathcal{P}_T$ to the linear map $f \mapsto f|A$. This is equivalent to the condition that

$$\sigma_A = h|(1-A)$$

for all $A \in \Gamma$. The space of all 1-cocycles is denoted by $Z(\Gamma, \mathcal{P}_T)$, while the subspace of coboundaries is denoted by $B(\Gamma, \mathcal{P}_T)$. The first cohomology group is defined to be

$$H^1(\Gamma, \mathcal{P}_T) = Z(\Gamma, \mathcal{P}_T)/B(\Gamma, \mathcal{P}_T).$$

Since Γ is a Fuchsian group with presentation (2.1), after Weil again we say $\sigma \in Z(\Gamma, \mathcal{P}_T)$ is a parabolic[5] cocycle provided that there exist $h_1, h_2, h_3 \in \mathcal{P}_T$ depending on σ such that

$$\sigma_T = h_1|(1-T) \quad \sigma_S = h_2|(1-S) \quad \text{and} \quad \sigma_U = h_3|(1-U).$$

[5] We emphasize that the parabolic condition requires the presentation of Γ as a Fuchsian group.

Since S and U have finite order, h_2 and h_3 exist for all σ. Also a coboundary is automatically parabolic. Let $Z_{\mathrm{par}}(\Gamma, \mathcal{P}_T)$ denote the space of parabolic cocycles and

$$H^1_{\mathrm{par}}(\Gamma, P_T) = Z_{\mathrm{par}}(\Gamma, \mathcal{P}_T)/B(\Gamma, \mathcal{P}_T)$$

be the parabolic first cohomology group. Clearly $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_T)$ is a subgroup of $H^1(\Gamma, \mathcal{P}_T)$.

Lemma 8. *We have*

$$\dim H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_T) = k - 1 + \dim \mathcal{P}_T^{\Gamma} + \dim \mathcal{P}_{T^*}^{\Gamma} - \dim \mathcal{P}_T^{(S)} - \dim \mathcal{P}_T^{(U)} - \dim \mathcal{P}_T^{(T)}. \tag{5.3}$$

Proof. From Weil [44, p. 156] we have

$$\dim H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_T) = k - 1 + \dim \mathcal{P}_T^{\Gamma} + \dim (\mathcal{P}'_T)^{\Gamma} - \dim \mathcal{P}_T^{(S)} - \dim \mathcal{P}_T^{(U)} - \dim \mathcal{P}_T^{(T)}.$$

Clearly (5.3) follows from this and the first statement of Lemma 7. $\square$

Let $\mathcal{P}_T^{\dagger} = \mathcal{W}_{2-k} \cap \mathcal{P}_T$ be the kernel of T acting on $\mathcal{W}_{2-k}$ and set

$$\mathcal{P}_T^{\ddagger} = \{\psi \in \mathcal{W}_{2-k}; T\hat{\psi} = 0\}.$$

Since $\psi = \hat{\psi}|(1 - S)$, it follows that $\mathcal{P}_T^{\ddagger} \subset \mathcal{P}_T^{\dagger}$.

Lemma 9. $H^1_{\mathrm{par}}(\Gamma, \mathcal{P}_T)$ *is isomorphic to* $\mathcal{P}_T^{\dagger}/\mathcal{P}_T^{\ddagger}$. *Also,*

$$\dim \mathcal{P}_T^{\ddagger} = \dim \mathcal{P}_T^{(T)} - \dim \mathcal{P}_T^{\Gamma}.$$

Proof. For the first statement, map $\psi \in \mathcal{P}_T^{\dagger}$ to $\sigma \in Z_{\mathrm{par}}(\Gamma, \mathcal{P}_T)$ defined by $\sigma_A = \hat{\psi}|(1 - A)$. This σ is a coboundary if and only if $\psi \in \mathcal{P}_T^{\ddagger}$. Observe that a parabolic cohomology class for Γ is always represented by a cocycle σ with $\sigma_T = 0$ and so is completely determined by σ_S.

For the second statement, map $F \in \mathcal{P}_T^{(T)}$ to $\psi = F|(1 - S) \in \mathcal{P}_T^{\ddagger}$ and note that the kernel of this surjective homomorphism is $\mathcal{P}_T^{\Gamma}$.

$\square$

We need to determine $\dim \mathcal{P}_T^{(S)}$ and $\dim \mathcal{P}_T^{(U)}$.

Lemma 10. *We have*

$$k - 1 - \dim \mathcal{P}_T^{(S)} - \dim \mathcal{P}_T^{(U)} = 2\ell_k + 1.$$

Proof. This follows easily by induction on k once we show the following formulas:

$$\dim \mathcal{P}_T^{(S)} = 2\lfloor \tfrac{k-2}{4} \rfloor + 1$$

$$\dim \mathcal{P}_T^{(U)} = 2\lfloor \tfrac{k-2}{6} \rfloor + 1.$$

These are proven by computing the indicial equations in the z variable of $Tf = 0$ around and $z = i$ and $z = \rho$. Thus around $z = i$ we have the expansion

$$f(z) = (z - i)^r \sum_{m \geq 0} a(n)(z - i)^n.$$

The indicial equation for $Tf = 0$ is

$$r(r - 1)(r - 2) \cdots (r - (k - 2)) = 0. \tag{5.4}$$

To be fixed under S is equivalent to having $r \equiv \frac{k}{2} - 1 \pmod 2$. The number of such r is $2\lfloor \frac{k-2}{4} \rfloor + 1$, which gives $\dim \mathcal{P}_T^{(S)}$. Around $z = \rho$ we have the expansion

$$f(z) = (z - \rho)^r \sum_{m \geq 0} a(n)(z - \rho)^n,$$

for which the indicial equation is again (5.4). To be fixed under U is equivalent to having $r \equiv 2k-1 \pmod 3$. The number of such r is $2\lfloor\frac{k-2}{6}\rfloor+1$, which gives $\dim \mathcal{P}_T^{(U)}$.

□

Proof of Theorem 1. It follows from Lemma 4 that

$$\dim\ker T' = \dim \mathcal{P}^{\dagger}_{T^*}. \tag{5.5}$$

Also, since T is assumed to be holomorphic, by Lemma 7 we know that

$$\dim\ker T = \dim \mathcal{P}^{\Gamma}_{T}. \tag{5.6}$$

By Lemmas 8 and 9 we have

$$\begin{aligned}\dim \mathcal{P}^{\dagger}_{T^*} &= \dim H^1_{\text{par}}(\Gamma, \mathcal{P}_{T^*}) + \dim \mathcal{P}^{(T)}_{T^*} - \dim \mathcal{P}^{\Gamma}_{T^*}\\ &= k-1+\dim \mathcal{P}^{\Gamma}_{T} - \dim \mathcal{P}^{(S)}_{T^*} - \dim \mathcal{P}^{(U)}_{T^*}.\end{aligned}$$

By (5.5) and Lemma 10 we have that

$$\dim\ker T' = 2\ell_k + 1 + \dim \mathcal{P}^{\Gamma}_{T}. \tag{5.7}$$

Thus by (5.6)

$$\dim\ker T - \dim\ker T' = -2\ell_k - 1,$$

as desired. □

6. Generalized Eichler integrals

We now prove Theorem 2. Again let $\{f_0, \ldots, f_{k-2}\}$ be a fixed basis for $\mathcal{P}_T$. In addition to its role in the proof of Lemma 7, another application of the dual basis $\{g_0, \ldots, g_{k-2}\}$ for $\mathcal{P}_{T^*}$ defined in (4.14) is an integral formula for a particular solution to the inhomogeneous equation $TF = f$. Define the kernel function

$$k_T(z,\tau) = \sum_{n=0}^{k-2} f_n(z) g_n(\tau).$$

Note that k_T is independent of the choice of dual bases since a change of basis of $\mathcal{P}_T$ is matched by its inverse transpose to get the dual basis for $\mathcal{P}_{T^*}$. Therefore by Lemma 5

$$k_T(Az, A\tau) = (cz+d)^{2-k}(c\tau+d)^{2-k}\, k_T(z,\tau) \tag{6.1}$$

for $A = \pm\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$.

By the classical method of variation of parameters (see e.g. [5]) we have

$$F(z) = \int_{z_0}^{z} k_T(z,\tau) f(\tau)\, d\tau, \tag{6.2}$$

where the integral is over any smooth curve from z_0 to z, all in $\mathcal{H}$. In case $f \in \mathcal{M}_k$ the integral in (6.2) is a generalized Eichler integral of f. Making a change of variables and using (6.1) we get

$$\sigma_A(z) = F|(1-A)(z) = \int_{z_0}^{A^{-1}z_0} k_T(z,\tau) f(\tau)\, d\tau.$$

Example 6.1. For $T = D^{k-1}_{2-k}$ a calculation shows that

$$k_T(z,\tau) = \Delta_{2-k}(z)\Delta_{2-k}(\tau)\big(u(z)-u(\tau)\big)^{k-2},$$

where u was given in (4.18).

Proof of Theorem 2. There is also a formula for the dimension of the full first cohomology group. Since Γ is the free product of (S) and (U) it follows from [9], Lemma 7 and (5.7) that

$$\dim H^1(\Gamma, \mathcal{P}_T) = 2\ell_k + 1 + \dim \mathcal{P}_T^\Gamma. \tag{6.3}$$

Therefore by (5.7)

$$\dim H^1(\Gamma, \mathcal{P}_T) = \dim \ker T'. \tag{6.4}$$

It is easy to see that

$$f \mapsto \sigma_A$$

defines a homomorphism from $\mathcal{M}_k$ to $H^1(\Gamma, \mathcal{P}_T)$. It is well-defined since a different choice of $z_0 \in \mathcal{H}$ leads to an equivalent cocycle. Also its kernel is precisely $T\mathcal{M}_{2-k}$. Thus

$$\dim(\mathcal{M}_k/T\mathcal{M}_{2-k}) \leq \dim \ker T'. \tag{6.5}$$

We want to show that the homomorphism is surjective.

There is a natural map from $\ker T'$ to the dual of $\mathcal{M}_k/T\mathcal{M}_{2-k}$ defined by sending $\psi \in \ker T'$ to the functional λ determined by

$$\lambda(g + T\mathcal{M}_{2-k}) = \langle g, \psi \rangle,$$

where either $g \in \mathcal{M}_k$ or $g + T\mathcal{M}_{2-k} \in \mathcal{M}_k/T\mathcal{M}_{2-k}$. This map is well-defined, since for $f \in \mathcal{M}_{2-k}$ we have $\langle Tf, \psi \rangle = \langle f, T'\psi \rangle = 0$ by Lemma 4. It is also injective by i) of Corollary 1. Hence

$$\dim(\mathcal{M}_k/T\mathcal{M}_{2-k}) \geq \dim \ker T',$$

which by (6.5), (6.4) and (6.3) completes the proof. $\square$

Remarks.

(1) It would be interesting to have a direct proof of Theorem 2. The main difficulty in any case is to show that the map $f \mapsto \sigma_A$ is surjective. We use Theorem 1 and proceed by comparing dimensions in a manner that is close in spirit to Eichler's original proof of his result. See [26] and [30] for other approaches when $Tf = f^{(k-1)}$. We remark that our proof of Theorem 2 may be generalized to differential operators that intertwine other Fuchsian groups.

(2) For simplicity we have only considered even weights k in our main theorems. Generalizations that include odd weights are certainly possible.

(3) There are Fuchsian equations that are uniformized by the modular group and that correspond to differential operators on $\mathcal{H}$ whose coefficients have poles at elliptic fixed points. It is a natural problem to extend the framework and results of this paper to cover these more general modular operators.

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