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LATTICE POINTS ON ELLIPSOIDS

by

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1 - INTRODUCTION

The theories of integral quadratic forms and modular forms are intimately connected. This connection has not been fully exploited for ternary forms due to a barrier in the estimation of the Fourier coefficients of cusp forms of weight half an odd integer. Recently Iwaniec [6] has shown how to break this barrior for such holomorphic cusp forms of weight 5/2 or greater. In [2] it is shown that his method extends in a straightforward way to include certain non-holomorphic Maass forms as well as holomorphic cusp forms of weight 3/2. There applications are given of the non-holomorphic case to certain geometric distribution questions related to indefinite ternary forms. In this note I will briefly indicate how the case of weight 3/2 applies to the classical problem of determining the asymptotic behavior of the representation number of an arbitrary integral positive definite ternary quadratic form. This representation number is the number of lattice points on an appropriate ellipsoid in Euclidean 3-space.

2 - REPRESENTATION OF INTEGERS BY POSITIVE QUADRATIC FORMS

Let $Q[X]$ be a non-degenerate integral quadratic form in $m \geqslant 3$ variables represented by an mxm non-singular symmetric semi-integral matrix Q. Two such forms Q_1 and Q_2 belong to the same class (in the wide sense) if there is a $U \in GL_m(\mathbb{Z})$ such that $Q[U] \overset{df}{=} {}^tUQ_1U = Q_2$ and to the same genus if this holds for some $U_p \in GL_m(\mathbb{Z}_p)$ for all primes p, including $p = \infty$. The genus of a given Q, denoted by genQ, is known to consist of finitely many classes. For indefinite forms under rather general conditions this class number is one, while for positive forms it is in general large (see [1] p. 202 and p. 364). This circumstance makes the study of representations by individual positive forms more difficult in general than for indefinite forms.

Let now Q be positive definite and define the arithmetic functions for $n \in \mathbb{Z}^+$

$$r_Q(n) = \# \{\alpha \in \mathbb{Z}^m \ ; \ Q[\alpha] = n\} \text{ and}$$

$$r^*_Q(n) = \# \{\alpha \in \mathbb{Z}^m \text{ primitive} \ ; \ Q[\alpha] = n\}.$$

Clearly $r_Q(n) = \sum_{d^2 | n} r^*_Q(n/d^2)$ so also

$$r^*_Q(n) = \sum_{d^2 | n} \mu(d) r_Q(n/d^2).$$

Both functions are class invariants so we may define the "average" genus representation number

(1) $$r(n,\text{gen}Q) = \Big(\sum_{Q' \in \text{gen}Q} r_{Q'}(n) E(Q')^{-1}\Big) \Big(\sum_{Q' \in \text{gen}Q} E(Q')^{-1}\Big)^{-1}$$

and similarly $r^*(n,\text{gen}Q)$, where

$$E(Q) = \# \{U \in GL_m(\mathbb{Z}) \ ; \ Q[U] = Q\}$$

is the (finite) number of units of Q. Siegel's formula evaluates $r(n,\text{gen}Q)$ explicitly as the n^{th} Fourier coefficient of an Eisenstein series. The set of n with $r(n,\text{gen}Q) > 0$ or $r^*(n,\text{gen}Q) > 0$ are determined locally by the solvability of certain congruences. For $m \geq 5$ it follows that

(2) $$r(n,\text{gen}Q) \gg n^{m/2-1}$$

as $n \to \infty$ through n such that $r(n,\text{gen}Q) > 0$. For $m = 4$ this holds provided $r^*(n,\text{gen}Q) > 0$. For $m = 3$ the best statement is given in [15] but requires the notion of a spinor genus, which is an intermediate classification between class and genus. We will not develop this notion here but will be content to impose the additional condition that n not lie in finitely many quadratic sequences $\{t_i \ell^2; \ell \in \mathbb{Z}^+\}$ where t_i are explicitly determined divisors of det(2Q) (see [19]). For n with $r^*(n,\mathrm{gen}Q) > 0$ it is known that ([15] p. 295) for all $\varepsilon > 0$.

(3) $$r(n,\mathrm{gen}Q) \underset{\varepsilon}{\gg} n^{1/2-\varepsilon}$$

as $n \to \infty$, where Siegel's theorem is being employed, making the implied constant ineffective. We shall denote by Ω_Q the set of n indicated for which (2) or (3) holds.

In general, $r(n,\mathrm{gen}Q)-r_Q(n)$ is the n^{th} Fourier coefficient of a holomorphic cusp form for $\Gamma_0(N)$ of weight m/2 with a certain real character χ_Q, where

$$N = \min\{\ell \in \mathbb{Z}^+ ; (\ell/2)(2Q)^{-1} \text{ is semi-integral}\}$$

is the level of Q. The bound corresponding to Weil's estimate for Kloosterman sums (which is much easier to obtain for m odd) gives the asymptotic formula

$$r_Q(n) = r(n,\mathrm{gen}Q) + O(n^{(m-1)/4+\varepsilon}).$$

Thus, for $m \geqslant 4$ by (2) every sufficiently large $n \in \Omega_Q$ is represented by Q. For m = 3 the error term exceeds the main term by n^{ε}. This breakdown of the "analytic method" was one motivation for Linnik's ergodic method, which was partially successful in handling the ternary case. However, for a general ternary always an extra unnatural condition or an unproved hypothesis is assumed (see [5],[7],[10],[11],[19], and cited literature).

By Iwaniec's method we may now return successfully to the analytic method for positive ternary forms.

THEOREM 1. - *For* $n = t\ell^2 \in \Omega_Q$ *with* t *square-free and* $(\ell,N) = 1$ *we have, as* $n \to \infty$,

$$r_Q(n) - r(n,\mathrm{gen}Q) \underset{\varepsilon,Q}{\ll} t^{3/14} n^{1/4+\varepsilon}$$

for any positive definite integral ternary quadratic form Q *and any* $\varepsilon > 0$.

COROLLARY. *Notation as above, for any* $\varepsilon > 0$

$$r_Q(n) \underset{\varepsilon,Q}{\gg} n^{1/2-\varepsilon}$$

as $n \to \infty$, *provided* $n \in \Omega_Q$ *and* $(\ell, N) = 1$. *The implied constant is ineffective.*

Thus, in particular, every sufficiently large square-free integer which is represented by some form in the genus of Q is represented by Q. This was conjectured by Ross and Pall ([13] p. 60.) in 1946.

3. AN ESTIMATE FOR FOURIER COEFFICIENTS

The essential new ingredient in the proof of Theorem 1 is a deep estimate of Iwaniec for a certain sum of Kloosterman sums (Theorem 3 in [6]). In this section the resulting estimate for half-integral weight Fourier coefficients from which Theorem 1 follows will be stated.

Let, for $k = 1/2 + L$, $L \in \mathbb{Z}^+$, and χ a real character mod$N$, $N \equiv 0 \pmod 4$, $S_k(N,\chi)$ be the space of holomorphic cusp forms of weight k with character χ for $\Gamma = \Gamma_0(N)$ (see e.g. [16]). $S_k(N,\chi)$ is a finite dimensional Hilbert space with the inner product

$$\langle f,g \rangle = \int_{\Gamma/H} f(z)\overline{g(z)}\, y^{k-2}\, dxdy.$$

If $k = 3/2$ let U be the subspace of $S_{3/2}(N,\chi)$ which is spanned by theta functions of the form $\sum_{n \geqslant 1} \psi(n) n e(tn^2 z)$ where t is square-free, ψ is an odd Dirichlet character, and $e(z) = \exp(2\pi i z)$. Let $U^\perp$ be the orthogonal complement of U in $S_{3/2}(N,\chi)$ with respect to $\langle .,. \rangle$.

THEOREM 2.- *Suppose* $f(z) = \sum_{n \geqslant 1} a(n) e(nz) \in S_k(N,\chi)$ *with* $f \in U^\perp$ *if* $k = 3/2$. *Let* $\varepsilon > 0$ *be given. Then, for* $n = t\ell^2$ *with* t *square-free,*

$$a(n) \ll_{f,\varepsilon} t^{3/14}\, n^{(k-1)/2+\varepsilon}$$

as $n \to \infty$, provided $(\ell, N) = 1$.

Theorem 1 follows from Theorem 2 by the result of Schulze-Pillot ([15], Satz 4) that

$$\sum_{n \geqslant 1} (r_Q(n) - r(n, \mathrm{spn} Q))\, e(nz) \in U^\perp$$

where $r(n,\mathrm{spn}Q)$ is defined similarly as in (1) with respect to the spinor genus. We use that $r(n,\mathrm{gen}Q) = r(n,\mathrm{spn}Q)$ for $n \in \Omega_Q$.

Theorem 2 follows from Theorem 5 in [2] after reduction to square-free n by the Shimura lift. More precisely, define as in [16] for square-free t the t-Shimura lift F_t of f by $F_t(z) = \sum_{n\geq 1} A_t(n)e(nz)$ where

$$(4) \qquad \sum_{n\geq 1} A_t(n)n^{-s} = L(s-k+3/2,\chi_t)\sum_{n\geq 1} a(tn^2)n^{-s},$$

with $\chi_t(n) = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2}\left(\frac{t}{n}\right)$. Under our conditions it is known that $F_t \in S_{2k-1}(N/2)$ (see [15] p. 285 for a discussion and references). We may also suppose that F is an eigenfunction of all Hecke operators T_{p^2} for $p\nmid N$ prime. Then by [16] cor. 1.8. for $(\ell,N) = 1$,

$$(5) \qquad A_t(\ell) = a(t)A_1(\ell).$$

Thus, by (4) and (5) for $(\ell,N) = 1$:

$$a(t\ell^2) = \sum_{d\mid\ell} \chi_t(d)\mu(d)A_t(\ell/d)$$

$$= a(t)\sum_{d\mid\ell} \chi_t(d)\mu(d)A_1(\ell/d)$$

$$\ll |a(t)|\ell^{k-1+\varepsilon} \quad \text{by Deligne's estimate}$$

$$\ll t^{k/2-2/7+\varepsilon}\ell^{k-1+\varepsilon} \quad \text{by Theorem 5 in [2].}$$

Theorem 2 now follows.

In conclusion we remark that by using theta functions with spherical harmonic coefficients we may employ Theorem 2 to prove that the lattice points $\alpha \in \mathbb{Z}^3$ with $Q[\alpha] = n$ become uniformly distributed on the ellipsoid $\{Q[X] = n\}$ as $n \to \infty$, provided $n \in \Omega_Q$ and $(\ell,n) = 1$. In connection with this we refer to [3],[7],[8], [10], and [12].

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