A DIOPHANTINE PROBLEM ABOUT KUMMER SURFACES

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ABSTRACT. Upper and lower bounds are given for the number of rational points of bounded height on a double cover of projective space ramified over a Kummer surface.

1. INTRODUCTION

Let $F(x) = F(x_0, \ldots, x_n)$ with $n \geq 2$ be an integral form with $\deg F \geq 2$ and set

$$N_F(T) = \#\{x \in \mathbb{Z}^{n+1} | F(x) = z^2 \text{ for some } z \in \mathbb{Z},\ \gcd(x_0, \ldots, x_n) = 1 \ \&\ \|x\| \leq T\}, \tag{1.1}$$

where $\|x\| = \max_j(|x_j|)$. The behavior of $N_F(T)$ for large T is of basic Diophantine interest. When $\deg F$ is even, $N_F(T)$ counts rational points of bounded height on a double cover of $\mathbb{P}^n_{\mathbb{Q}}$ ramified over the hypersurface given by $F(x) = 0$.

Assume that $\deg F$ is even and that $z^2 - F(x)$ is irreducible over $\mathbb{C}$. It follows from Theorem 3 on p.178 of [13] that for any $\epsilon > 0$

$$N_F(T) \ll T^{n+\frac{1}{2}+\epsilon}. \tag{1.2}$$

As discussed after Theorem 3 in [13], it is reasonable to expect that

$$N_F(T) \ll T^{n+\epsilon}. \tag{1.3}$$

Broberg [3] improved 5/2 to 9/4 in (1.2) when $n = 2$. For $n \geq 3$ various improvements and generalizations of (1.2) are given in [11], [7] and [2], assuming that $F(x) = 0$ is nonsingular. Certain non-homogeneous F are treated in [7].

In this note I will consider the problem of estimating $N_F(T)$ from above *and below* when $n = 3$ for a special class of quartic F, namely those for which $F(x) = 0$ define certain Kummer surfaces. These surfaces have singularities (nodes).

For our purpose we will define a Kummer surface in terms of an integral sextic polynomial $P(t)$. For fixed $a, b, c, d, e, f, g \in \mathbb{Z}$ with $a \neq 0$ let

$$P(t) = at^6 + bt^5 + ct^4 + dt^3 + et^2 + ft + g.$$

Suppose that the discriminant of P is not zero. Define the symmetric matrices

$$S_0 = \begin{pmatrix} a & \frac{b}{2} & 0 & 0 \\ \frac{b}{2} & c & \frac{d}{2} & 0 \\ 0 & \frac{d}{2} & e & \frac{f}{2} \\ 0 & 0 & \frac{f}{2} & g \end{pmatrix} \tag{1.4}$$

and

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

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For $x = (x_0, x_1, x_2, x_3)$ define the matrix

$$S_x = x_0 S_0 + x_1 S_1 + x_2 S_2 + x_3 S_3.$$

For a row vector v let $S(v) = vSv^t$ denote the quadratic form associated to a symmetric matrix S. It is easy to check that *for any* x we have the identity

$$x_0 P(t) = S_x(t^3, t^2, t, 1).$$

Define the associated quartic form F by

$$F(x) := 16 \det S_x. \tag{1.6}$$

Over $\mathbb{C}$ the surface given by $F(x) = 0$ is a Kummer surface, a special determinantal quartic surface that is singular with sixteen nodes, including the points $(t^3, t^2, t, 1)$ where t is a root of $P(t) = 0$. The Jacobian variety of the genus two hyperelliptic curve $y^2 = P(t)$ is a double cover of the Kummer surface ramified over these nodes. For details on the geometry of Kummer surfaces see e.g. [10] and [5]. Some arithmetic aspects of Kummer surfaces are considered in [4]. The construction of a Kummer surface using the S_j from (1.4) and (1.5) occurs in a slightly different form in [1, p.69]. See also [4, p.42].

Our main result is the following.

Theorem. *Suppose that $P(t) = at^6 + bt^5 + ct^4 + dt^3 + et^2 - 2t$ with integral a, b, c, d, e has non-zero discriminant and $a \neq 0$. Let F be defined in (1.6) and $N_F(T)$ in (1.1). Then for any $\epsilon > 0$*

$$T^2 \ll N_F(T) \ll T^{3+\epsilon}, \tag{1.7}$$

where the first implied constant depends only on P and the second depends only on P and ϵ.

Our approach to these estimates relies on the special form of the Kummer surfaces we consider. In particular, for the upper bound we use that in P we assume that $g = 0$. For the lower bound we use that $g = 0$ and $f = -2$. The upper bound coincides with that given in (1.3). An example of an equation to which Theorem 1 applies, when $P(t) = t^6 - 2t$, is

$$z^2 = x_3^2(x_1^2 + 8x_0x_2) + x_3(-16x_0^3 - 2x_1x_2^2) - 4x_0x_1^3 - 8x_0^2x_1x_2 + x_2^4.$$

Numerical calculations in this case show that we seem to have $N_F(T) \gg T^{3-\epsilon}$. It would be of interest to find the correct order of magnitude of $N_F(T)$ for some P.

Remark. Most research on $N_F(T)$ in (1.1) has concentrated on giving upper bounds for $N_F(T)$ for quite general F, where $F(x) = 0$ is usually assumed to be nonsingular. The proofs often make use of intricate estimates of character and exponential sums (for example see [7]). In contrast, the proof of the upper bound of (1.7) is rather straightforward. Although it is likely not sharp, the lower bound of (1.7) is probably more interesting and certainly deeper. Its proof uses a remarkable and not well-known identity of Schottky to explicitly produce solutions to $F(x) = z^2$. Along somewhat similar lines, invariant theory was recently applied to asymptotically count integer points on quadratic twists of certain elliptic curves and give a class number formula for binary quartic forms [6]. It is reasonable to hope that some other classical identities of algebraic geometry and syzygies of invariant theory, some of which are beautifully presented in [5], could have still undiscovered applications to the problem of finding lower bounds for counting functions like $N_F(T)$.

2. Proof of the theorem

Upper bound. The mechanism behind the proof of the upper bound in (1.7) is that a quadratic Diophantine equation in two variables has "few" solutions. The argument relies on the fact that for $P(t)$ of the assumed form (so that in particular $g = 0$), the associated F has the property that it is quadratic in one of its variables. It will become clear that similar arguments can be applied to other F with this property.

For a general $P(t)$ we have the explicit formula

$$\begin{aligned}F(x) = x_0^4(&16aceg - 4acf^2 - 4ad^2g - 4b^2eg + b^2f^2)\\ &- 2x_0^3(-8acgx_1 + 2adfx_1 - 4adgx_2 - 8aegx_3 + 2af^2x_3 + 2b^2gx_1 + bdfx_2 + 2bdgx_3)\\ &+ x_0^2(-4aex_1^2 + 4afx_1x_2 + 16agx_1x_3 - 4agx_2^2 - 4bex_1x_2 - 2bfx_1x_3\\ &+ 2bfx_2^2 + 4bgx_2x_3 - 4cex_2^2 - 4cfx_2x_3 - 4cgx_3^2 + d^2x_2^2)\\ &- 2x_0(2ax_1^3 + 2bx_1^2x_2 + 2cx_1x_2^2 + dx_1x_2x_3 + dx_2^3 + 2ex_2^2x_3 + 2fx_2x_3^2 + 2gx_3^3)\\ &+ (x_2^2 - x_1x_3)^2.\end{aligned}$$

For $P(t) = at^6 + bt^5 + ct^4 + dt^3 + et^2 - 2t$ we have that F has an expansion that is quadratic in x_3:

$$\begin{aligned}F(x) =& x_3^2\left(x_1^2 + 8x_2x_0\right) + x_3(-16ax_0^3 + 4bx_0^2x_1 + 8cx_0^2x_2 - 2dx_0x_1x_2 - 4ex_0x_2^2 - 2x_1x_2^2)\\ &+ 4b^2x_0^4 - 16acx_0^4 + 8adx_0^3x_1 - 4aex_0^2x_1^2 - 4ax_0x_1^3 + 4bdx_0^3x_2 - 8ax_0^2x_1x_2\\ &- 4bex_0^2x_1x_2 - 4bx_0x_1^2x_2 - 4bx_0^2x_2^2 + d^2x_0^2x_2^2 - 4cex_0^2x_2^2 - 4cx_0x_1x_2^2 - 2dx_0x_2^3 + x_2^4.\end{aligned} \tag{2.1}$$

Thus given a solution x of $z^2 = F(x)$, upon completing the square we will get a solution (y, z) of

$$y^2 - (x_1^2 + 8x_2x_0)z^2 = k(x_0, x_1, x_2) \tag{2.2}$$

where

$$k(x_0, x_1, x_2) = 8x_0x_2^5 - 64a^2x_0^5 + \cdots$$

is a homogeneous integral form of degree 6 that is not identically zero, and where

$$y = (x_1^2 + 8x_2x_0)x_3 + (8ax_0^3 - 2bx_0^2x_1 - 4cx_0^2x_2 + dx_0x_1x_2 + 2ex_0x_2^2 + x_1x_2^2). \tag{2.3}$$

The number of x_0, x_1, x_2 with $|x_0|, |x_1|, |x_2| \le T$ where either

$$k(x_0, x_1, x_2) = 0 \qquad \text{or} \qquad x_1^2 + 8x_2x_0 = 0$$

is $\ll T^2$. For such x_0, x_1, x_2, by (2.2) and (2.3) the total number of solutions of $F(x) = z^2$ with $|x_3| \le T$ is $\ll T^3$.

For any other x_0, x_1, x_2 with $|x_0|, |x_1|, |x_2| \le T$ we can apply the well-known estimate

$$d(k) \ll k^\epsilon$$

for the divisor function and [9, Lemma 1], which follows from [8, Lemma 5], to conclude that the total number of solutions of $F(x) = z^2$ with $|x_1|, |x_2|, |x_3|, |x_0| \le T$ is $\ll T^{3+\epsilon}$.

Lower bound. The tool used to obtain the lower bound of (1.7) is an explicit parameterization of solutions given by an identity of Schottky. This identity has a form that is similar to many of those coming from syzygies connecting covariants and invariants of forms. However, Schottky's identity has a different origin and does not appear to come from invariant theory.

The Jacobian of S_0, S_1, S_2, S_3 as given in (1.4) and (1.5) is

$$J(x) = J_{S_0,S_1,S_2,S_3}(x) = \det \begin{pmatrix} \partial_1 S_0 & \partial_2 S_0 & \partial_3 S_0 & \partial_4 S_0 \\ \partial_1 S_1 & \partial_2 S_1 & \partial_3 S_1 & \partial_4 S_1 \\ \partial_1 S_2 & \partial_2 S_2 & \partial_3 S_2 & \partial_4 S_2 \\ \partial_1 S_3 & \partial_2 S_3 & \partial_3 S_3 & \partial_4 S_3 \end{pmatrix} = 2gx_3^3x_0 - 2ax_3x_0^3 + \dots .$$

In case $f = -2$ and $g = 0$ this is given in full by

$$\begin{aligned} (2.4)\quad J(x) = 2(&-ax_3x_0^3 + 3ax_0^2x_1x_2 - 2ax_0x_1^3 - bx_3x_0^2x_1 + bx_0^2x_2^2 + bx_0x_1^2x_2 - bx_1^4 - cx_3x_0x_1^2 \\ &2cx_0x_1x_2^2 - cx_1^3x_2 - dx_3x_1^3 + dx_0x_2^3 + ex_3x_0x_2^2 - 2ex_3x_1^2x_2 + ex_1x_2^3 - 2x_3^2x_0x_2 \\ &2x_3^2x_1^2 + 2x_3x_1x_2^2 - 2x_2^4). \end{aligned}$$

The surface defined by $J(x) = 0$ is a Weddle surface. A variant of the following identity connecting the Weddle and Kummer surfaces, which can be checked directly, is apparently due to Schottky [12, p.241.]. He obtained it via theta functions and used it to show that the Kummer and Weddle surfaces are birationally equivalent over $\mathbb{C}$. It is stated (in a somewhat different form) in [1, p.152, Ex 8].

Proposition 1. *For F in (1.6) (and in (2.1)) when $P(t) = at^6 + bt^5 + ct^4 + dt^3 + et^2 - 2t$, we have identically*

$$F\big(-S_3(x), -2S_2(x), 2S_1(x), S_0(x)\big) = J^2(x), \tag{2.5}$$

where $J(x)$ is given in (2.4).

Note the order of the parameterizing quadrics S_j. It is not obvious (to me) how to modify (2.5) so that it holds for a general $P(t)$ or even if that is possible without changing its basic form.

Let $\mathcal{S}$ be the set of six points $\alpha_j \in \mathbb{P}^3_{\mathbb{C}}$ represented by $(t_j^3, t_j^2, t_j, 1)$, where $P(t_j) = 0$ for $j = 1, \dots, 6$. Recall from the discussion around (1.6) that $S_i(\alpha_j) = 0$ for each i, j. In order to apply Proposition 1 to prove the lower bound of (1.7), we must first examine the map

$$\alpha \mapsto \big(-S_3(\alpha), -2S_2(\alpha), 2S_1(\alpha), S_0(\alpha)\big) \tag{2.6}$$

from $\mathbb{P}^3_{\mathbb{C}} \setminus \mathcal{S}$ to $\mathbb{P}^3_{\mathbb{C}}$. Let V be the space spanned by $\{S_0, S_1, S_2, S_3\}$, which is clearly four dimensional. We need to control the degree of the map (2.6). Suppose that $\beta_1, \beta_2, \beta_3 \in \mathbb{P}^3_{\mathbb{C}} \setminus \mathcal{S}$ are distinct and all have the same image in $\mathbb{P}^3_{\mathbb{C}}$ under the map (2.6). Then three independent $S, S', S'' \in V$ will vanish at the nine distinct points $\{\alpha_1, \dots, \alpha_6, \beta_1, \beta_2, \beta_3\}$. This is impossible by Bezout's theorem and shows that there are at most two points in $\mathbb{P}^3_{\mathbb{C}} \setminus \mathcal{S}$ with the same image in $\mathbb{P}^3_{\mathbb{C}}$ under the map (2.6).

Therefore by Proposition 1, the lower bound of (1.7) will follow from

$$\#\{x \in \mathbb{Z}^4;\ \gcd(x_1, x_2, x_3, x_4) = 1, \quad |S_j(x)| \le T, \quad j = 1, 2, 3, 4\} \gg T^2.$$

This estimate is easily established since there is a ball in $\mathbb{R}^4$ centered at the origin of positive radius, all of whose points x satisfy $|S_j(x)| \le 1$ for $j = 1, 2, 3, 4$. Thus a standard lattice point count gives the result. $\square$

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