INTEGRAL TRACES OF SINGULAR VALUES OF WEAK MAASS FORMS

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ABSTRACT. We define traces associated to a weakly holomorphic modular form f of arbitrary negative even integral weight and show that these traces appear as coefficients of certain weakly holomorphic forms of half-integral weight. If the coefficients of f are integral, then these traces are integral as well. We obtain a negative weight analogue of the classical Shintani lift and give an application to a generalization of the Shimura lift.

1. INTRODUCTION

Recently there has been a resurgence of interest in the classical theory of singular moduli, these being the values of the modular j-function at quadratic irrationalities. This resurgence is due largely to the influential papers of Borcherds [1], [2] and Zagier [26]. The present paper arose from a suggestion made at the end of [26] to extend some of the results given there on traces of singular moduli to higher weights. One such generalization has been given recently by Bringmann and Ono [3], who provide an identity for the traces associated to certain Maass forms in terms of the Fourier coefficients of half-integral weight Poincaré series. However, it does not seem to be known when these traces are integral or even rational. Here we will identify the traces associated to a weakly holomorphic form f of negative integral weight with the coefficients of certain weakly holomorphic forms of half-integral weight and show that these coefficients are integral when the coefficients of f are integral. We will use this identification to obtain a negative weight analogue of the classical Shintani lift. We also give an application to Borcherds' generalization of the Shimura lift to weakly holomorphic modular forms.

Recall that a weakly holomorphic modular form of weight k, where $k \in 2\mathbb{Z}$, is a holomorphic function f on the upper half-plane $\mathcal{H}$ that satisfies

$$\big(f|_k\gamma\big)(\tau) = (c\tau+d)^{-k} f\big(\tfrac{a\tau+b}{c\tau+d}\big) = f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{PSL}(2,\mathbb{Z})$ and that has a q-expansion $f(\tau) = \sum_n a(n)q^n$ with $a(n) = 0$ for all but finitely many $n < 0$; here, as usual, $q = e(\tau) = e^{2\pi i\tau}$. Let $M_k^!$ denote the vector space of all weakly holomorphic modular forms of weight k. Similarly, for $k = s + 1/2$ with $s \in \mathbb{Z}$ let $M_k^!$ denote the space of holomorphic functions on $\mathcal{H}$ that transform like θ^{2k} under $\Gamma_0(4)$, have at most poles in the cusps, and have a q-expansion supported on integers n with $(-1)^s n \equiv 0, 1 \pmod 4$. Here, as usual, $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$. For any k let $M_k \subset M_k^!$ denote the subspace of holomorphic forms and $S_k \subset M_k$ the subspace of cusp forms.

In this paper d is always an integer with $d \equiv 0, 1 \pmod 4$ and D is always a fundamental discriminant (possibly 1). Suppose that $dD < 0$ and that F is a Γ-invariant function on $\mathcal{H}$.

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Define the twisted trace

$$\mathrm{Tr}_{d,D}(F) = \sum_{Q} w_Q^{-1}\, \chi(Q) F(\tau_Q), \tag{1}$$

where the sum is over a complete set of Γ-inequivalent positive definite integral quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ with discriminant $dD = b^2 - 4ac$, and

$$\tau_Q = \frac{-b + \sqrt{dD}}{2a} \in \mathcal{H} \tag{2}$$

is the associated CM point. Here $w_Q = 1$ unless $Q \sim a(x^2 + y^2)$ or $Q \sim a(x^2 + xy + y^2)$, in which case $w_Q = 2$ or 3 respectively, and

$$\chi(Q) = \chi(a,b,c) = \begin{cases} \chi_D(r), & \text{if } (a,b,c,D) = 1 \text{ and } Q \text{ represents } r, \text{ where } (r,D) = 1; \\ 0, & \text{if } (a,b,c,D) > 1, \end{cases} \tag{3}$$

where χ_D is the Kronecker symbol. It is known that χ is well–defined on classes, that χ restricts to a real character (a genus character) on the group of primitive classes, and that all such characters arise this way.

For the usual j-function $j = E_4^3/\Delta \in M_0^!$ with Fourier expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots,$$

it is classical that the value $j(\tau_Q)$ is an algebraic integer in an abelian extension of $\mathbb{Q}(\sqrt{dD})$. Let $j_1 = j - 744$. Zagier showed in [26] that for a fundamental discriminant $D \neq 1$ we have

$$q^{-|D|} + \sum_{d>0} d^{-\frac{1}{2}}\, \mathrm{Tr}_{d,D}(j_1) q^{|d|} \in M_{1/2}^! \qquad \text{if } D < 0,$$
$$q^{-|D|} - D^{-\frac{1}{2}} \sum_{d<0} \mathrm{Tr}_{d,D}(j_1) q^{|d|} \in M_{3/2}^! \qquad \text{if } D > 0,$$

and that both forms have integral Fourier coefficients. For instance, when $D = -3$ and $D = 5$ we have the two weakly holomorphic forms

$$q^{-3} - 248q + 26752q^4 - 85995q^5 + \cdots \in M_{1/2}^! \quad \text{and}$$
$$q^{-5} + 85995q^3 - 565760q^4 + 52756480q^7 + \cdots \in M_{3/2}^!,$$

and $\mathrm{Tr}_{5,-3}(j_1) = \mathrm{Tr}_{-3,5}(j_1) = j(\frac{1+\sqrt{-15}}{2}) - j(\frac{1+\sqrt{-15}}{4}) = -85995\sqrt{5}$.

In this paper we will give such a result when j_1 is replaced by a function f of negative weight. To state it, first define the Maass raising operator ∂_k in $\tau = x + iy$:

$$\partial_k = \mathcal{D} - \frac{k}{4\pi y}, \qquad \text{where} \qquad \mathcal{D} = \frac{1}{2\pi i}\,\frac{d}{d\tau} = q\frac{d}{dq}. \tag{4}$$

Now $\partial_k(f|_k\gamma) = (\partial_k f)|_{k+2}\gamma$ for any $\gamma \in \mathrm{PSL}(2,\mathbb{R})$. Thus, if $f \in M_{2-2s}^!$ for $s \in \mathbb{Z}^+$, the function $\partial^{s-1} f$ is Γ-invariant, where

$$\partial^{s-1} \equiv (-1)^{s-1}\partial_{-2} \circ \partial_{-4} \circ \cdots \circ \partial_{4-2s} \circ \partial_{2-2s}. \tag{5}$$

After Maass we know that $\partial^{s-1} f$ is an eigenfunction of the Laplacian

$$\Delta = -y^{-2}\Big(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\Big)$$

with eigenvalue $s(1-s)$, so $\partial^{s-1}f$ is a weak Maass form (see e.g. [6, p.162] for a precise definition). As was shown in special cases in [26] by a method that readily generalizes, $\partial^{s-1}f$ is a rational function of j and $h = E_2^*E_4E_6/\Delta$, where

$$E_2^*(\tau) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n - \frac{3}{\pi y} \qquad \Big(\sigma(n) = \sum_{m|n} m\Big)$$

is the nonholomorphic weight 2 Eisenstein series. For a CM point like τ_Q given in (2), it was shown by Ramanujan that $h(\tau_Q)$ is algebraic [22, eq. (23) p. 33]. More precisely,

$$h(\tau_Q) \in \mathbb{Q}(j(\tau_Q))$$

(see [18, Thm. A1, p. 114].) Using this, we can deduce the remarkable fact that for any $f \in M^!_{2-2s}$ with $s \geq 1$ and with rational Fourier coefficients, the "singular" value of the weak Maass form $\partial^{s-1}f(\tau_Q)$ is algebraic. We are thus motivated to study $\mathrm{Tr}_{d,D}(\partial^{s-1}f)$ for such f.

For D a fundamental discriminant let $\hat{s} = s$ if $(-1)^sD > 0$ and $\hat{s} = 1-s$ otherwise. It is also convenient to set

$$\mathrm{Tr}^*_{d,D}(f) = (-1)^{\lfloor\frac{\hat{s}-1}{2}\rfloor}\,|d|^{-\frac{\hat{s}}{2}}\,|D|^{\frac{\hat{s}-1}{2}}\;\mathrm{Tr}_{d,D}(\partial^{s-1}f). \tag{6}$$

Suppose that $f \in M^!_{2-2s}$ for $s \geq 2$ has Fourier coefficients $a(n)$. For D fundamental, define the D^{th} Zagier lift of f to be

$$\mathfrak{Z}_Df(\tau) = \sum_{m>0} a(-m)m^{s-\hat{s}}\sum_{n|m}\chi_D(n)n^{\hat{s}-1}q^{-\frac{m^2|D|}{n^2}} + \tfrac{1}{2}L(1-s,\chi_D)a(0) + \sum_{d:\ dD<0}\mathrm{Tr}^*_{d,D}(f)q^{|d|}.$$

The linear map $f \mapsto \mathfrak{Z}_D(f)$ is a negative weight analogue of the Shintani lift on integral weight cusp forms. This follows from our main result, whose proof will be completed in Section 5.

Theorem 1. *Suppose that $f \in M^!_{2-2s}$ for an integer $s \geq 2$. If D is a fundamental discriminant with $(-1)^sD > 0$, we have that $\mathfrak{Z}_Df \in M^!_{3/2-s}$, while if $(-1)^sD < 0$, then $\mathfrak{Z}_Df \in M^!_{s+1/2}$. If f has integral Fourier coefficients, then so does $\mathfrak{Z}_Df$.*

Here we will not treat the case $s = 1$, which requires special considerations and which can be dealt with by the methods of [26]. Furthermore, when $s = 2, 3, 4, 5, 7$, Theorem 1 can also be deduced from results of [26]. The first new example occurs when $s = 6$ and $D = 1$, where we have the pair

$$f(\tau) = \tfrac{E_{14}(\tau)}{\Delta(\tau)^2} = q^{-2} + 24q^{-1} - 196560 - 47709536\,q + \cdots \in M^!_{-10}$$
$$\mathfrak{Z}_1f(\tau) = q^{-4} + 56\,q^{-1} + 390 + 15360\,q^3 + 42264\,q^4 + 615240\,q^7 + \cdots \in M^!_{-9/2}.$$

Here $-\frac{1}{2}\zeta(-5)\cdot 196560 = 390$ and the first few values of $\mathrm{Tr}^*_{d,1}(f)$ are

$$3^{-4}\partial^5 f(\tfrac{1+\sqrt{-3}}{2}) = 15360 \qquad 2^{-7}\partial^5 f(i) = 42264 \qquad 7^{-3}\partial^5 f(\tfrac{1+\sqrt{-7}}{2}) = 615240.$$

Similarly, when $D = -3$ we have

$$\mathfrak{Z}_{-3}f(\tau) = 2^{11}q^{-12} - 8q^{-3} - 15360q - 53319598080q^4 + \cdots \in M^!_{13/2}.$$

The main new difficulty in proving Theorem 1 comes from the existence of cusp forms in $M^!_{2s}$. The method of Poincaré series adapts nicely to handle it. A key dividend of the method is the last statement of Theorem 1, showing that the integrality of coefficients is preserved under the lift.

Remarks:

- It follows from Theorem 1 that if $(-1)^s D > 0$ then the image $\mathfrak{Z}_D(f) \in M^!_{3/2-s}$ is determined by its principal part, hence by the principal part of f. Furthermore, $a(0)$ is divisible by the denominator of each of the L-values $\frac{1}{2}L(1-s,\chi_D)$, provided that the Fourier coefficients of f are integral. Using well-known properties of the generalized Bernoulli numbers, one can reproduce the divisibility properties that follow from work of Siegel [23, pp. 254–256]. On the other hand, if $(-1)^s D < 0$ then $\frac{1}{2}L(1-s,\chi_D) = 0$.
- It can be shown that the Zagier lift is compatible with the Hecke operators. For more details, see the end of Section 5.
- Using a theta lift, Bruinier and Funke [4] have generalized Zagier's result in various other ways, for instance to higher levels, where the existence of cusp forms in the dual weight is also a complication (see also [11]).

As another application of these methods we will give a simple proof of a basic property of the Shimura lift for weakly holomorphic modular forms. For

$$g(\tau) = \sum_n b(n) q^n \in M^!_{s+\frac{1}{2}}$$

with $s \in \mathbb{Z}^+$ and fundamental D with $(-1)^s D > 0$, define the D^{th} Shimura lift of g by

$$\mathscr{S}_D\, g(\tau) = \tfrac{1}{2}L(1-s,\chi_D) b(0) + \sum_{m>0} \Big(\sum_{n|m} \chi_D(n) n^{s-1} b\big(\tfrac{m^2|D|}{n^2}\big) \Big) q^m. \tag{7}$$

When g is holomorphic this is the usual definition. We will repeatedly use the basic fact that $\mathscr{S}_D g \in M_{2s}$ if $g \in M_{s+1/2}$ (see [16]). Recall that a CM point is a point in $\mathcal{H}$ of the form $\frac{-b+\sqrt{b^2-4ac}}{2a}$ for integral a, b, c. The proof of the following result will be completed in Section 6. In the case $D = 1$ it is due to Borcherds and follows from a special case of Theorem 14.3 in [2] (see Example 14.4).

Theorem 2. *For $g \in M^!_{s+\frac{1}{2}}$ with $s \geq 2$ and D a fundamental discriminant with $(-1)^s D > 0$, $\mathscr{S}_D\, g$ is a meromorphic modular form of weight $2s$ for Γ whose possible poles are of order at most s and occur at CM points.*

2. Weakly holomorphic forms

In this section we will define a canonical basis for the space $M^!_k$ for any $k = s + 1/2$ with $s \in \mathbb{Z}$ in which all basis elements have integral Fourier coefficients. Then we will construct forms in $M^!_k$ when $s \geq 2$ using Poincaré series.

We begin by recalling the canonical basis for $M^!_{2s}$ defined in [9] for any $s \in \mathbb{Z}$. Write $2s = 12\ell + k'$ with uniquely determined $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$, so that if $\ell \geq 0$, then ℓ is the dimension of the space S_{2s} of cusp forms of weight $2s$. For every integer $m \geq -\ell$, there exists a unique $f_{2s,m} \in M^!_{2s}$ with a q-expansion of the form

$$f_{2s,m}(\tau) = q^{-m} + \sum_{n>\ell} a_{2s}(m,n) q^n,$$

and together they form a basis for $M^!_{2s}$. The basis element $f_{2s,m}$ can be given explicitly in the form

$$f_{2s,m} = f_{2s} P(j),$$

where $f_{2s} = f_{2s,-\ell} = \Delta^\ell E_{k'}$ and P is a polynomial of degree $m+\ell$. As shown in [9], the basis elements have the following generating function:

$$\sum_{m\geq -\ell} f_{2s,m}(z)q^m = \frac{f_{2s}(z)f_{2-2s}(\tau)}{j(\tau)-j(z)} = -\sum_{m\geq \ell+1} f_{2-2s,m}(\tau)r^m, \tag{8}$$

where $r = e(z)$. It follows from this that the coefficients $a_{2s}(m,n)$ are integral and satisfy the duality relation

$$a_{2s}(m,n) = -a_{2-2s}(n,m). \tag{9}$$

In order to formulate a similar result for $M_k^!$ when $k = s+1/2$ with $s\in\mathbb{Z}$, let ℓ be defined by $2s = 12\ell + k'$ as above. By the Shimura correspondence given in [14] one finds that the maximal order of a nonzero $f \in M_k^!$ at $i\infty$ is

$$A = \begin{cases} 2\ell - (-1)^s, & \text{if } \ell \text{ is odd;} \\ 2\ell, & \text{otherwise.} \end{cases}$$

If $B < A$ is the next admissible exponent we can construct functions in $M_k^!$ of the form

$$f_k(\tau) = q^A + O(q^{B+4}) \qquad \text{and} \qquad f_k^*(\tau) = q^B + O(q^{B+4}).$$

Writing $s = 12a + b$, where $b \in \{6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19\}$, f_k and f_k^* can be given explicitly in the form

$$f_k(\tau) = \Delta(4\tau)^a f_{b+1/2}(\tau) \qquad \text{and} \qquad f_k^*(\tau) = \Delta(4\tau)^a f_{b+1/2}^*(\tau),$$

where the forms $f_{b+1/2}, f_{b+1/2}^* \in M_{b+1/2}$ are given in the appendix and have integral Fourier coefficients. Using them it is easy to construct a unique basis for $M_k^!$ consisting of functions of the form

$$f_{k,m}(\tau) = q^{-m} + \sum_{n>A} a_k(m,n)q^n, \tag{10}$$

where $m \geq -A$ satisfies $(-1)^{s-1}m \equiv 0, 1 \pmod 4$. Here $f_{k,-A} = f_k$ and $f_{k,-B} = f_k^*$. This can be done recursively; $f_{k,m}(\tau)$ is obtained by multiplying $f_{k,m-4}(\tau)$ by $j(4\tau)$ and then subtracting a suitable linear combination of the forms $f_{k,m'}(\tau)$ with $m' < m$. We also have the following generating function, whose proof is similar to Zagier's proof of the $k = 1/2$ case in [26]:

$$\sum_m f_{k,m}(z)q^m = \frac{f_k(z)f_{2-k}^*(\tau) + f_k^*(z)f_{2-k}(\tau)}{j(4\tau)-j(4z)} = -\sum_m f_{2-k,m}(\tau)r^m. \tag{11}$$

By (11) and the fact that f_k and f_k^* have integral Fourier coefficients, we have the following result.

Proposition 1. *The Fourier coefficients $a_k(m,n)$ defined in (10) are integral and satisfy the duality relation*

$$a_k(m,n) = -a_{2-k}(n,m) \tag{12}$$

for all $m, n \in \mathbb{Z}$.

Another way to construct weakly holomorphic forms is by Poincaré series. In this paper we only need them for $k = s + 1/2$ where $s \geq 2$. Set $j(\gamma,\tau) = \theta(\gamma\tau)/\theta(\tau)$ for $\gamma \in \Gamma_0(4)$. For $m \in \mathbb{Z}$ define the Poincaré series

$$P_{k,m}(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} e(m\gamma\tau)j(\gamma,\tau)^{-2k},$$

where Γ_∞ is the subgroup of translations in $\Gamma_0(4)$. For $k \geq 5/2$ this is absolutely convergent and represents a weakly holomorphic form of weight k for $\Gamma_0(4)$, but it is not in $M_k^!$ since its Fourier coefficients are not supported on n with $(-1)^s n \equiv 0, 1 \pmod 4$. When $m = 0$ the Poincaré series is an Eisenstein series that Cohen [7] projected to a form in M_k and whose Fourier coefficients are expressed in terms of the values of Dirichlet L-functions at $1 - s$. When $m > 0$, Kohnen (see [15]) showed how to obtain in this way cusp forms in S_k. It was observed in [5] that a similar procedure works for $m < 0$. Petersson [21] had explicitly computed the Fourier expansions of $P_{k,m}$ in terms of Bessel functions and Kloosterman sums, and the projections $g_{k,m}$ of $P_{k,m}$ to $M_k^!$ have Fourier expansions that are simple modifications of these. To give them, for $m, n \in \mathbb{Z}$ and $c \in \mathbb{Z}^+$ with $c \equiv 0 \pmod 4$ let

$$K_k(m, n; c) = \sum_{a (\mathrm{mod}\ c)} \left(\tfrac{c}{a}\right) \varepsilon_a^{2k} e\left(\tfrac{ma + n\bar{a}}{c}\right) \tag{13}$$

be the Kloosterman sum, where $\left(\frac{c}{a}\right)$ is the extended Legendre symbol and

$$\varepsilon_a = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4 \\ i & \text{if } a \equiv 3 \pmod 4). \end{cases}$$

Also, let $\delta_{\text{odd}}(n) = 1$ if n is odd and $\delta_{\text{odd}}(n) = 0$ otherwise.

Proposition 2. *Suppose that $k = s + 1/2$ where $s \geq 2$. Then, for any nonzero integer m with $(-1)^s m \equiv 0, 1 \pmod 4$, there exists a form $g_{k,m} \in M_k^!$ with Fourier expansion*

$$g_{k,m}(\tau) = q^m + \sum_{\substack{n \geq 1 \\ (-1)^s n \equiv 0,1 (4)}} b_k(m, n) q^n$$

where for $(-1)^s \equiv 0, 1 \pmod 4$ the coefficient $b_k(m, n)$ is given explicitly by the absolutely convergent sum

$$b_k(m, n) = 2\pi i^{-k} \left|\tfrac{n}{m}\right|^{\frac{k-1}{2}} \sum_{\substack{c \equiv 0 (4) \\ c > 0}} \left(1 + \delta_{\text{odd}}(\tfrac{c}{4})\right) c^{-1} K_k(m, n; c) \cdot \begin{cases} I_{k-1}(\frac{4\pi\sqrt{|mn|}}{c}), & \text{if } m < 0; \\ J_{k-1}(\frac{4\pi\sqrt{|mn|}}{c}), & \text{if } m > 0. \end{cases}$$

A similar formula holds when $m = 0$ that can be further evaluated to give Cohen's formulas. Also, a modified version holds when $s = 1$ (see [5]).

Of course, $g_{k,-m}$ can be expressed in terms of the basis elements $f_{k,m}$. If there are no nonzero cusp forms in M_{2s}, then $g_{k,-m} = f_{k,m}$ for all m. In general, however,

$$g_{k,-m} - f_{k,m} \in S_k \tag{14}$$

is a nonzero cusp form. It seems likely that the Fourier coefficients $b_k(m, n)$ of $g_{k,m}$ are irrational, even transcendental, in general.

3. Weak Maass forms

Next we will show that for $f \in M_{2-2s}^!$ with $s \in \mathbb{Z}^+$, the function $\partial^{s-1} f$ is a weak Maass form and compute its Fourier expansion. Recall that ∂^{s-1} was defined in (5). Then we express $\partial^{s-1} f_{2-2s,m}$ in terms of certain Poincaré series. We need the following result which, in essence, is due to Maass (see also [17, p. 250]).

Proposition 3. *Suppose that $f(\tau) = \sum_n a(n)q^n \in M^!_{2-2s}$ for integral $s \geq 1$. Then $\partial^{s-1} f$ is a weak Maass form for Γ with eigenvalue $s(1-s)$. Explicitly, we have*

$$\partial^{s-1} f(\tau) = 2\pi y^{\frac{1}{2}} \sum_{n>0} a(-n) n^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi n y) e(-nx) + (-1)^{s-1}\Big(\pi^{\frac{1}{2}-s}\Gamma(s-\tfrac{1}{2}) y^{1-s} a(0) + 2y^{\frac{1}{2}} \sum_{n\neq 0} a(n)|n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)\Big), \tag{15}$$

where I and K are the usual Bessel functions.

Proof. By induction it is readily shown that for $n > 0$

$$\partial^{s-1} e(-n\tau) = n^{s-1} \sum_{m=0}^{s-1} \frac{(s-1+m)!}{m!(s-1-m)!} (-4\pi n y)^m e(-n\tau).$$

Standard formulas for Bessel functions with half-integral parameter (see e.g. [12]) yield

$$\begin{aligned} \partial^{s-1} e(-n\tau) &= 2\, n^{s-\frac{1}{2}} y^{\frac{1}{2}} \Big(\pi I_{s-\frac{1}{2}}(2\pi n y) + (-1)^{s-1} K_{s-\frac{1}{2}}(2\pi n y)\Big) e(nx) \\ \partial^{s-1} e(n\tau) &= 2(-1)^{s-1}\, n^{s-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi n y) e(nx), \\ \partial^{s-1}(1) &= (-1)^{s-1} \pi^{\frac{1}{2}-s} \Gamma(s-\tfrac{1}{2}) y^{1-s}. \end{aligned}$$

These formulas easily give (15), thus finishing the proof of Proposition 3. □

We next express the weak Maass form $\partial^{s-1} f_{2-2s,m}$ associated to the basis element $f_{2-2s,m}$ in terms of certain Poincaré series, when $s \geq 2$ and $2s = 12\ell + k'$ as before. For $m \in \mathbb{Z}$ with $m \neq 0$ consider the Poincaré series (see [20])

$$F_m(\tau, s) = 2\pi |m|^{s-\frac{1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m \operatorname{Re} \gamma\tau)(\operatorname{Im} \gamma\tau)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |m| \operatorname{Im} \gamma\tau), \tag{16}$$

which converges absolutely for $\operatorname{Re} s > 1$. Here Γ_∞ is the subgroup of translations in Γ. Clearly $F_m(\gamma\tau, s) = F_m(\tau, s)$ for $\gamma \in \Gamma$ and $\Delta F_m(\tau, s) = s(1-s) F_m(\tau, s)$.

Proposition 4. *For integral $s \geq 2$ we have for $m \geq \ell + 1$*

$$\partial^{s-1} f_{2-2s,m}(\tau) = F_{-m}(\tau, s) + \sum_{0<n<\ell+1} a_{2-2s}(m, -n) F_{-n}(\tau, s). \tag{17}$$

Proof. To prove Proposition 4, we need the Fourier expansion of F_m. This can be found, for instance, in [10]. Let

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(s/2) \zeta(s).$$

Then we have

$$F_m(\tau, s) = 2\pi |m|^{s-\frac{1}{2}} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |m| y) e(mx) + \frac{4\pi \sigma_{2s-1}(|m|)}{(2s-1)\xi(2s)} y^{1-s} + 4\pi |m|^{s-\frac{1}{2}} \sum_{n\neq 0} c(m, n; s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx), \tag{18}$$

where

$$c(m, n; s) = \sum_{c>0} c^{-1} K_0(m, n; c) \cdot \begin{cases} I_{2s-1}(4\pi \sqrt{|mn|}\, c^{-1}) & \text{if } mn < 0 \\ J_{2s-1}(4\pi \sqrt{|mn|}\, c^{-1}) & \text{if } mn > 0 \end{cases} \tag{19}$$

and

$$K_0(m,n;c)=\sum_{a(\mathrm{mod}\ c)^*} e\big(\tfrac{ma+n\bar{a}}{c}\big)$$

is the usual Kloosterman sum, the * restricting the sum to $(a,c)=1$. Consider the Maass form

$$\phi(\tau)=\partial^{s-1}f_{2-2s,m}(\tau)-\Big(F_{-m}(\tau,s)+\sum_{0<n<\ell+1}a_{2-2s}(m,-n)F_{-n}(\tau,s)\Big).$$

By Proposition 3 and (18) we have that

$$\phi(\tau)=c(0)y^{1-s}+\sum_{n\neq 0}c(n)y^{\frac12}K_{s-\frac12}(2\pi|\,n|\,y)e(nx).$$

where the $c(n)$ are explicitly computable in terms of the $c_s(m,n)$ and the $a_{2-2s}(m,n)$. Since $\phi\in L^2(\Gamma\backslash\mathcal{H})$ with eigenvalue $s(1-s)$, it must be equal to 0. □

In the case $s=1$ the Poincaré series $F_m(\tau,1)$ is defined through analytic continuation (see e.g. [20]) and Proposition 4 continues to hold in the modified form

$$f_{0,m}(\tau)=j_m(\tau)=F_{-m}(\tau,1)-24\sigma(m)\qquad\text{for}\qquad m\geq 1.$$

4. Preliminary formulas for the trace

For the proof of Theorem 1, we will need to compute the trace of $\partial^{s-1}f_{2-2s,m}$ in terms of the coefficients of the basis elements $f_{s+1/2,m}$. In view of Proposition 4, we are reduced to computing $\mathrm{Tr}_{d,D}(F_m(\cdot,s))$, where $F_m(\tau,s)$ is the Poincaré series defined in (16). When $D=s=1$ it was shown in [8] that this trace may be expressed in a simple way in terms of a certain exponential sum. In general we need the exponential sum introduced in [15]:

$$S_m(d,D;c)=\sum_{\substack{b\,(\mathrm{mod}\ c)\\ b^2\equiv Dd\,(\mathrm{mod}\ c)}}\chi\big(\tfrac{c}{4},b,\tfrac{b^2-Dd}{c}\big)e\left(\tfrac{2mb}{c}\right),$$

where χ is defined in (3) and $c\equiv 0\pmod 4$. Clearly

$$S_{-m}(d,D;c)=\overline{S_m}(d,D;c)=S_m(d,D;c).$$

We have the following identity.

Proposition 5. *Let $s\geq 2$ and $m\neq 0$. Suppose that D is fundamental and that $dD<0$. Then*

$$\mathrm{Tr}_{d,D}(F_m(\cdot,s))=\sqrt{2}\pi|m|^{s-\frac12}\,|d|^{\frac14}\,|D|^{\frac14}\sum_{c\equiv 0\,(\mathrm{mod}\ 4)}c^{-\frac12}S_m(d,D;c)I_{s-\frac12}\Big(\tfrac{4\pi\sqrt{m^2|dD|}}{c}\Big).$$

Proof. We have the absolutely convergent expression

$$\mathrm{Tr}_{d,D}(F_m(\cdot,s))=2\pi|m|^{s-\frac12}\sum_Q\frac{\chi(Q)}{\omega_Q}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}e(m\mathrm{Re}\,\gamma\tau_Q)(\mathrm{Im}\,\gamma\tau_Q)^{\frac12}I_{s-\frac12}(2\pi|m|\ \mathrm{Im}\,\gamma\tau_Q)$$

$$=\sqrt{2}\pi|m|^{s-\frac12}\,|d|^{\frac14}\,|D|^{\frac14}\sum_{a=1}^{\infty}a^{-\frac12}I_{s-\frac12}\Big(\tfrac{\pi\sqrt{m^2|dD|}}{a}\Big)\Big[\sum_Q\frac{\chi(Q)}{\omega_Q}\sum_\gamma e\big(m\,\mathrm{Re}(\,\gamma\tau_Q)\big)\Big],\tag{20}$$

where the sum over γ is over all $\gamma\in\Gamma_\infty\backslash\Gamma$ with

$$\mathrm{Im}\,\gamma\tau_Q=\frac{\sqrt{|Dd|}}{2a}.$$

Consider the sum in brackets in (20). For fixed $a > 0$, the values of $2a\mathrm{Re}(\gamma\tau_Q)$ run over the $\pmod{2a}$-incongruent solutions to the quadratic congruence

$$b^2 \equiv dD \pmod{4a}$$

with multiplicity w_Q as γ and Q run over their respective representatives. Thus we have

$$\sum_Q \tfrac{\chi(Q)}{\omega_Q} \sum_\gamma e\big(m\,\mathrm{Re}(\gamma\tau_Q)\big) = \tfrac{1}{2} S_m(d,D;4a).$$

Substitute this in (20) and write c for $4a$ to finish the proof. □

We need to express the traces in terms of the Fourier coefficients of modular forms. This is done by applying an identity originally due to Salié in a special case to transform the sum of exponential sums in Proposition 5 into a sum of Kloosterman sums. This sum may then be interpreted in terms of the Fourier coefficients of half-integral weight Poincaré series. This technique goes back to Zagier [25], who applied it in the context of base-change. Kohnen [15] applied it to the Shimura lift of cusp forms. More recently, this method has proven to be fruitful in the context of weakly holomorphic forms. In [8] it was applied to give a new proof of Zagier's original identity for traces of singular moduli. The technique has since been extended in various ways in [13] and [3]. In particular, the following formula for the trace of $F_m(\tau,s)$ in terms of the coefficients $b_k(m,n)$ of half integral weight Poincaré series was given in [3] when $m=-1$ and $(-1)^s D < 0$.

Proposition 6. *Suppose that $m \neq 0$, $s \geq 2$ and $dD < 0$ with D fundamental. Then*

$$\mathrm{Tr}_{d,D}(F_m(\cdot,s)) = \begin{cases} \varepsilon\,|d|^{\frac{s}{2}}\,|D|^{\frac{1-s}{2}} \sum_{n|m} \chi_D(n) n^{s-1} b_{s+\frac{1}{2}}\big(-|d|, \frac{m^2|D|}{n^2}\big), & \text{if } (-1)^s D > 0, \\ \varepsilon\,|d|^{\frac{1-s}{2}}\,|D|^{\frac{s}{2}}\,|m|^{2s-1} \sum_{n|m} \chi_D(n) n^{-s} b_{s+\frac{1}{2}}\big(\frac{-m^2|D|}{n^2}, |d|\big), & \text{if } (-1)^s D < 0, \end{cases}$$

where the sum $n|m$ is over the positive divisors of m, $\varepsilon = (-1)^{\lfloor \frac{s+1}{2} \rfloor}$, and $b_{s+1/2}$ was defined in Proposition 2.

Proof. Recall the Kloosterman sum associated to modular forms of half integral weight defined in (13). It is clear that replacing k with $k+2$ does not change this sum; each $K_{s+\frac{1}{2}}(m,n,c)$ is equal to $K_{\frac{1}{2}}(m,n;c)$ or $K_{\frac{3}{2}}(m,n;c)$, depending on whether s is even or odd, respectively. In fact, we have the relations

$$K_{\frac{1}{2}}(m,n;c) = i \cdot K_{\frac{3}{2}}(-m,-n;c) = K_{\frac{1}{2}}(n,m;c). \tag{21}$$

We have the following identity for the Kloosterman sums, which can be proved by a slight modification of the proof of Kohnen in [15, Prop. 5, p. 258] (see also [8], [13] and [24]).

Lemma 1. *For integers $m \neq 0$ and $c > 0$ with $4|c$, an integer d with $d \equiv 0,1 \pmod 4$ and D a fundamental discriminant, we have the identity*

$$S_m(d,D;c) = (1-i) \sum_{n|\left(m,\frac{c}{4}\right)} \big(1+\delta_{\mathrm{odd}}(\tfrac{c}{4n})\big)\chi_D(n)\sqrt{\tfrac{n}{c}}\; K_{\frac{1}{2}}\left(d, \tfrac{m^2 D}{n^2}; \tfrac{c}{n}\right).$$

By Proposition 5 and Lemma 1 we quickly derive that

$$\mathrm{Tr}_{d,D}(F_m(\cdot,s)) = \sqrt{2}\pi(1-i)|m|^{s-\frac12}\,|d|^{\frac14}\,|D|^{\frac14}\sum_{n|m}\chi_D(n)n^{-\frac12}$$
$$\cdot\sum_{c\equiv 0(4)} c^{-1}(1+\delta_{\mathrm{odd}}(\tfrac{c}{4}))K_{\frac12}\big(d,\tfrac{m^2D}{n^2};c\big)I_{s-\frac12}\left(\tfrac{4\pi}{c}\sqrt{m^2\,|Dd|\,/n^2}\right).$$

Comparison with Proposition 2 and use of (21) finishes the proof of Proposition 6. □

5. THE ZAGIER LIFT

In this section we give the proof of Theorem 1. The following proposition gives an explicit formula for the Zagier lift of $f \in M^!_{2-2s}$ when $(-1)^sD > 0$. In its proof we make repeated use of the classical Shimura lift, integral and half-integral weight duality from (9) and (12), and the fact that the constant term of a form in $M^!_2$ must vanish. Write $2s = 12\ell + k'$ with $k' \in \{0,4,6,8,10,14\}$ as above.

Proposition 7. *Suppose that $s \geq 2$ is an integer and that $f(\tau) = \sum_n a(n)q^n \in M^!_{2-2s}$. Suppose that D is a fundamental discriminant with $(-1)^sD > 0$. Then the D^{th} Zagier lift of f is given by*

$$\mathfrak{Z}_D f = \sum_{m>0} a(-m)\sum_{n|m}\chi_D(n)n^{s-1}f_{\frac32-s,\frac{m^2|D|}{n^2}}. \tag{22}$$

Proof. Recall that when $(-1)^sD > 0$ the Zagier lift was defined by

$$\mathfrak{Z}_D f(\tau) = \sum_{m>0} a(-m)\sum_{n|m}\chi_D(n)n^{s-1}q^{-\frac{m^2|D|}{n^2}} + \tfrac12 L(1-s,\chi_D)a(0) + \sum_{d:\, dD<0}\mathrm{Tr}^*_{d,D}(f)q^{|d|},$$

where

$$\mathrm{Tr}^*_{d,D}(f) = (-1)^{\lfloor\frac{s-1}{2}\rfloor}\,|d|^{-\frac{s}{2}}\,|D|^{\frac{s-1}{2}}\;\mathrm{Tr}_{d,D}(\partial^{s-1}f).$$

We prove Proposition 7 by comparing the Fourier coefficients of $\mathfrak{Z}_D f$ with those of the function on the right hand side of (22), which we will denote simply by F. We do this for the positive coefficients, the principal parts and the constant terms separately.

Consider first the positive coefficients. By Propositions 4 and 6, we have for $m > \ell$ that

$$-\mathrm{Tr}^*_{d,D}(f_{2-2s,m}) = \sum_{n|m}\chi_D(n)n^{s-1}b_{s+\frac12}\big(-|d|\,;\tfrac{m^2|D|}{n^2}\big) + \sum_{j=1}^{\ell} a_{2-2s}(m,-j)\sum_{h|j}\chi_D(h)h^{s-1}b_{s+\frac12}\big(-|d|\,;\tfrac{j^2|D|}{h^2}\big). \tag{23}$$

From (14) we have the cusp form $C(\tau) = g_{s+\frac12,-|d|}(\tau) - f_{s+\frac12,|d|}(\tau) = \sum_{n\geq1} c(n)q^n$. Thus

$$b_{s+\frac12}\big(-|d|,\tfrac{j^2|D|}{h^2}\big) = a_{s+\frac12}\big(|d|\,,\tfrac{j^2|D|}{h^2}\big) + c\big(\tfrac{j^2|D|}{h^2}\big).$$

However, $\mathscr{S}_D C$, the D^{th} Shimura lift of C, is a cusp form of weight $2s$ with jth coefficient

$$\sum_{h|j}\chi_D(h)h^{s-1}c\big(\tfrac{j^2|D|}{h^2}\big).$$

The contribution to $-\mathrm{Tr}^*_{d,D}(f_{2-2s,m})$ in (23) from coefficients of C, which is

$$\sum_{n|m}\chi_D(n)n^{s-1}c\left(\tfrac{m^2|D|}{n^2}\right) + \sum_{j=1}^{\ell} a_{2-2s}(m,-j)\sum_{h|j}\chi_D(h)h^{s-1}c\left(\tfrac{j^2|D|}{h^2}\right),$$

can be interpreted as the constant term of $(\mathscr{S}_D C)f_{2-2s,m} \in M_2^!$, which must be zero. Thus we have

$$-\mathrm{Tr}^*_{d,D}(f_{2-2s,m}) = \sum_{n|m} \chi_D(n) n^{s-1} a_{s+\frac{1}{2}}\big(\,|d|\,,\tfrac{m^2|D|}{n^2}\big) \tag{24}$$

$$+\sum_{j=1}^{\ell} a_{2-2s}(m,-j)\sum_{h|j}\chi_D(h)h^{s-1}a_{s+\frac{1}{2}}\big(\,|d|\,,\tfrac{j^2|D|}{h^2}\big). \tag{25}$$

By duality, $\mathrm{Tr}^*_{d,D}(f_{2-2s,m})$ is the coefficient of $q^{|d|}$ in the Fourier expansion of

$$\sum_{n|m}\chi_D(n)n^{s-1}f_{\frac{3}{2}-s,\frac{m^2|D|}{n^2}} - \sum_{j=1}^{\ell} a_{2s}(-j,m)\sum_{h|j}\chi_D(h)h^{s-1}f_{\frac{3}{2}-s,\frac{j^2|D|}{h^2}}.$$

For an arbitrary form $f=\sum a(m)q^m \in M^!_{2-2s}$ we have

$$f=\sum_{m>\ell} a(-m) f_{2-2s,m}, \qquad \text{and so} \qquad \mathrm{Tr}^*_{d,D}(f)=\sum_{m>\ell} a(-m)\mathrm{Tr}^*_{d,D}(f_{2-2s,m})$$

is the coefficient of $q^{|d|}$ in

$$\sum_{m>\ell} a(-m)\Big(\sum_{n|m}\chi_D(n)n^{s-1}f_{\frac{3}{2}-s,\frac{m^2|D|}{n^2}} - \sum_{j=1}^{\ell} a_{2s}(-j,m)\sum_{h|j}\chi_D(h)h^{s-1}f_{\frac{3}{2}-s,\frac{j^2|D|}{h^2}}\Big). \tag{26}$$

For $1\le j\le \ell$ we have, once again using that the constant of a form in $M_2^!$ vanishes, that

$$a(-j) = -\sum_{m>\ell} a(-m)a_{2s}(-j,m).$$

Thus the form in (26) simplifies to F.

Next consider the principal parts. The properties of the basis elements given in Section 2 show that

$$f_{\frac{3}{2}-s,\frac{m^2|D|}{n^2}} = 0 \qquad \text{if} \qquad \tfrac{m^2|D|}{n^2} < C$$

for some C that depends only on the weight $\frac{3}{2}-s$. We use this and the Fourier expansion

$$f_{\frac{3}{2}-s,\frac{m^2|D|}{n^2}}(\tau) = q^{\frac{-m^2|D|}{n^2}} + \sum_h a_{\frac{3}{2}-s}\big(\tfrac{m^2|D|}{n^2},h\big)q^h$$

to write the negative powers of q appearing in the Fourier expansion of F as

$$\sum_{m>0} a(-m)\sum_{n|m}\chi_D(n)n^{s-1}q^{\frac{-m^2|D|}{n^2}} \tag{27}$$

$$-\sum_{m>0} a(-m)\sum_{n|m,\ \frac{m^2|D|}{n^2}<C}\chi_D(h)n^{s-1}q^{\frac{-m^2|D|}{n^2}} + \sum_{m,h>0} a(-m)\sum_{n|m}\chi_D(n)n^{s-1}a_{\frac{3}{2}-s}\big(\tfrac{m^2|D|}{n^2},-h\big)q^{-h}.$$

The first sum is the principal part of $\mathfrak{Z}_D f$, so we must prove that the expression in the second line of (27), call it S, vanishes. By duality,

$$S = -\sum_{m>0} a(-m)\Big(\sum_{n|m,\ \frac{m^2|D|}{n^2}<C}\chi_D(n)n^{s-1}q^{\frac{-m^2|D|}{n^2}} + \sum_{h>0}\sum_{n|m}\chi_D(n)n^{s-1}a_{s+\frac{1}{2}}\big(-h,\tfrac{m^2|D|}{n^2}\big)q^{-h}\Big).$$

Now for any $h > 0$, the coefficient of q^m in the Shimura lift $\mathscr{S}_D f_{s+\frac12,-h}$ of the cusp form $f_{s+\frac12,-h}$ is given by

$$\sum_{n|m} \chi_D(n) n^{s-1} \cdot \left(a_{s+\frac12}\big(-h, \tfrac{m^2|D|}{n^2}\big) + \begin{cases} 1, & \text{if } \frac{m^2|D|}{n^2} = h, \\ 0, & \text{otherwise.} \end{cases} \right)$$

(The last term here arises from the initial q^h in the Fourier expansion of $f_{s+\frac12,-h}$, since $a_{s+\frac12}(-h,h)$ is zero by definition.) From this, it is clear that the coefficient of q^{-h} in S for each $h > 0$ can be interpreted as the constant term of $(\mathscr{S}_D f_{s+\frac12,-h}) f \in M_2^!$, so $S = 0$.

Finally we evaluate the constant term of F, again using duality, as

$$\sum_{m>0} a(-m) \sum_{n|m} \chi_D(n) n^{s-1} a_{\frac32-s}\big(\tfrac{m^2|D|}{n^2}, 0\big) = -\sum_{m>0} a(-m) \sum_{n|m} \chi_D(n) n^{s-1} a_{s+\frac12}\big(0, \tfrac{m^2|D|}{n^2}\big).$$

Since $s \geq 2$ we have by [16]

$$\mathscr{S}_D f_{s+\frac12,0}(\tau) = \tfrac12 L(1-s,\chi_D) + \sum_{m>0} \Big(\sum_{n|m} \chi_D(n) h^{s-1} a_{s+\frac12}\big(0, \tfrac{n^2|D|}{h^2}\big)\Big) q^m$$

and the constant term of $(\mathscr{S}_D f_{s+\frac12,0}) f \in M_2^!$ is

$$\tfrac12 L(1-s,\chi_D) a(0) + \sum_{m>0} a(-m) \Big(\sum_{n|m} \chi_D(n) n^{s-1} a_{s+\frac12}\big(0, \tfrac{m^2|D|}{n^2}\big)\Big) = 0,$$

as desired. This concludes the proof of Proposition 7. □

We also need the corresponding statement if $(-1)^s D < 0$.

Proposition 8. *Suppose that $s \geq 2$ is an integer and that $f \in M^!_{2-2s}$ has Fourier coefficients $a(n)$. Suppose that D is a fundamental discriminant with $(-1)^s D < 0$. Then the D^{th} Zagier lift of f is given by*

$$\mathfrak{Z}_D f = \sum_{m>0} a(-m) m^{2s-1} \sum_{n|m} \chi_D(n) n^{-s} f_{s+\frac12, \frac{m^2|D|}{n^2}} + g, \tag{28}$$

where $g \in S_{s+1/2}$ is the unique cusp form whose Fourier coefficients $b(n)$ match those of $\mathfrak{Z}_D f$ for the first ℓ positive values of n with $(-1)^s n \equiv 0, 1 \pmod 4$.

Proof. Using Propositions 4 and 6 as before, we find that

$$\begin{aligned} \mathrm{Tr}^*_{d,D}(f_{2-2s,m}) =& m^{2s-1} \sum_{n|m} \chi_D(n) n^{-s} b_{s+\frac12}\big(-\tfrac{m^2|D|}{n^2}; |d|\big) \\ &+ \sum_{j=1}^{\ell} a_{2-2s}(m,-j) j^{2s-1} \sum_{h|j} \chi_D(h) h^{-s} b_{s+\frac12}\big(-\tfrac{j^2|D|}{h^2}; |d|\big). \end{aligned} \tag{29}$$

Thus, the trace $\mathrm{Tr}^*_{d,D}(f)$ of an arbitrary form

$$f = \sum a(m) q^m = \sum_{m>\ell} a(-m) f_{2-2s,m}$$

is given by

$$\sum_{m>\ell} a(-m) \Big(m^{2s-1} \sum_{n|m} \chi_D(n) n^{-s} b_{s+\frac12}\big(\tfrac{-m^2|D|}{n^2}; |d|\big)$$

$$+\sum_{j=1}^{\ell} a_{2-2s}(m,-j)j^{2s-1}\sum_{h|j}\chi_D(h)h^{-s}b_{s+\frac{1}{2}}\left(\tfrac{-j^2|D|}{h^2};|d|\right)\Big)$$
$$=\sum_{m>0} a(-m)m^{2s-1}\sum_{n|m}\chi_D(n)n^{-s}b_{s+\frac{1}{2}}\left(\tfrac{-m^2|D|}{n^2};|d|\right),$$

where we have simplified as before. This is just the coefficient of $q^{|d|}$ in the modular form $F\in M^!_{s+\frac{1}{2}}$ given by

$$F=\sum_{m>0} a(-m)m^{2s-1}\sum_{n|m}\chi_D(n)n^{-s}g_{s+\frac{1}{2},\frac{-m^2|D|}{n^2}}.$$

Now since $g_{s+\frac{1}{2},\frac{-m^2|D|}{n^2}}-f_{s+\frac{1}{2},\frac{m^2|D|}{n^2}}\in S_{s+\frac{1}{2}}$ from (14), we find that

$$F=\sum_{m>0} a(-m)m^{2s-1}\sum_{n|m}\chi_D(n)n^{-s}f_{s+\frac{1}{2},\frac{m^2|D|}{n^2}}+g$$

for a certain cusp form g, and, arguing as in Proposition 7, the principal part of F matches the principal part of $\mathfrak{Z}_D f$. Since the constant term and positive coefficients of F match those of $\mathfrak{Z}_D f$, Proposition 8 now follows. □

The first statement of Theorem 1 follows from Propositions 7 and 8. The statement on integrality follows from Proposition 1 in the case $(-1)^s D>0$. Otherwise we can reduce to this case using the following identity, which holds if $(-1)^s D<0$ and D' is fundamental with $(-1)^s D'>0$:

$$\mathrm{Tr}^*_{m^2D',D}(f)=-m^{2s-1}\sum_{a|m}\mu(a)\chi_{D'}(a)\sum_{b|ma^{-1}}\chi_D(b)(ab)^{-s}\mathrm{Tr}^*_{(\frac{m}{ab})^2D,D'}(f).$$

This identity is a consequence of the following lemma.

Lemma 2. *For D and D' fundamental discriminants with $DD'<0$ and $m\in\mathbb{Z}^+$ we have*

$$\mathrm{Tr}_{m^2D',D}=\sum_{a|m}\mu(a)\chi_{D'}(a)\sum_{b|ma^{-1}}\chi_D(b)\mathrm{Tr}_{(\frac{m}{ab})^2D,D'}.$$

Lemma 2 is obtained by writing the trace as a sum of sums over primitive quadratic forms, noting that $\chi_D=\chi_{D'}$ for such primitive forms, and applying Möbius inversion.

We now briefly indicate how one shows that the Zagier lift is compatible with the Hecke operators. If $k\in 2\mathbb{Z}>0$ and p is a prime, the weight k Hecke operator $|_kT(p)$ acts on a modular form $f(\tau)=\sum_n a(n)q^n\in M^!_k$ by

$$f|_kT(p)=\sum_n\Big(a(pn)+p^{k-1}a\big(\tfrac{n}{p}\big)\Big)q^n.$$

If $k\in 2\mathbb{Z}\le 0$, we multiply this by p^{1-k} so that $|_kT(p)$ preserves the integrality of Fourier coefficients.

When $0<s\in\mathbb{Z}$, the half integral weight Hecke operator $|_{s+1/2}T(p^2)$ acts on a form $g(\tau)=\sum_n b(n)q^n\in M^!_{s+1/2}$ by

$$g|_{s+1/2}T(p^2)=\sum_n\Big(b(p^2n)+\Big(\tfrac{(-1)^s n}{p}\Big)p^{s-1}b(n)+p^{2s-1}b\big(\tfrac{n}{p^2}\big)\Big)q^n.$$

Again, for $s\le 0$, we normalize this by multiplying by p^{1-2s}.

It is straightforward to see that for any prime p,

$$(\mathfrak{Z}_D f)|_{3/2-\hat{s}}T(p^2) = \mathfrak{Z}_D(f|_{2-2s}T(p)).$$

In the case that $(-1)^s D > 0$, we need only use the explicit Fourier expansion of the Zagier lift to compare principal parts. If $(-1)^s D < 0$, though, we must also show that

(30)
$$\mathrm{Tr}^*_{(-1)^s n,D}(f|_{2-2s}T(p)) = \mathrm{Tr}^*_{(-1)^s np^2,D}(f) + \left(\tfrac{(-1)^s n}{p}\right)p^{s-1}\mathrm{Tr}^*_{(-1)^s n,D}(f) + p^{2s-1}\mathrm{Tr}^*_{(-1)^s n/p^2,D}(f)$$

for the first ℓ positive values of n with $(-1)^s n \equiv 0, 1 \pmod 4$. To see that this holds, we argue as in the proof of [26, Theorem 5(ii)] to show that $\mathrm{Tr}_{(-1)^s n,D}((\partial^{s-1}f)|_0 T(p))$ equals

$$\mathrm{Tr}_{(-1)^s np^2,D}(\partial^{s-1}f) + \left(\tfrac{(-1)^s n}{p}\right)\mathrm{Tr}_{(-1)^s n,D}(\partial^{s-1}f) + p\mathrm{Tr}_{(-1)^s n/p^2,D}(\partial^{s-1}f),$$

and use the fact that if $k < 0$, then $\partial_k(f|_k T(p)) = p \cdot (\partial_k f)|_{k+2}T(p)$ to obtain equation (30).

6. The Shimura lift

In this final section we prove Theorem 2. For this we need the following two propositions.

Proposition 9. *Suppose that $s \in \mathbb{Z}^+$ and $\tau \in \mathcal{H}$. As a function of $z \in \mathcal{H}$,*

$$\partial^{s-1}\left(\frac{f_{2s}(z)f_{2-2s}(\tau)}{j(\tau)-j(z)}\right)$$

is a meromorphic modular form of weight $2s$ with poles of order at most s that only occur at points equivalent to τ under Γ.

Proof. Observe first that if f has weight k and g has weight 0 then by (4)

$$\partial_k(fg) = g\,\partial_k(f) + f\,\mathcal{D}(g).$$

Apply this repeatedly with $g(\tau) = (j(\tau)-j(z))^{-n}$ for $1 \le n < s$. We derive that

$$\partial^{s-1}\left(\frac{f_{2-2s}(\tau)}{j(\tau)-j(z)}\right) = \sum_{n=1}^{s}\frac{g_n(\tau)}{(j(z)-j(\tau))^n}$$

for $g_n \in M_0^!$, from which the result follows easily. □

Theorem 2 is a consequence of Proposition 9 together with the following explicit formula for the D^{th} Shimura lift of $f_{s+1/2,|d|}$. Write $2s = 12\ell + k'$ as above.

Proposition 10. *Suppose that $s \ge 2$, $(-1)^s D > 0$ and that $dD < 0$. Then*

$$\mathscr{S}_D f_{s+1/2,|d|}(z) = \mathrm{Tr}^*_{d,D}\left(\frac{f_{2s}(z)f_{2-2s}(\tau)}{j(\tau)-j(z)}\right) + f(z),$$

where $f \in M_{2s}$ is the unique holomorphic form whose Fourier coefficients $a(n)$ match those of $\mathscr{S}_D f_{s+1/2,|d|}$ for $n = 0, \ldots, \ell$.

Proof. By (7) we have, writing $r = e(z)$,

(31)
$$\mathscr{S}_D\, f_{s+1/2,|d|}(z) = \tfrac{1}{2}L(1-s,\chi_D)a_{s+1/2}(|d|,0) + \sum_{m>0}\Big(\sum_{n|m}\chi_D(n)n^{s-1}a_{s+1/2}\big(|d|,\tfrac{m^2|D|}{n^2}\big)\Big)r^m.$$

By (24) and (31) we have

$$-\sum_{m>\ell} \mathrm{Tr}^*_{d,D}(f_{2-2s,m})r^m = \mathscr{S}_D f_{s+1/2,|d|}(z) - \tfrac{1}{2}L(1-s,\chi_D)a_{s+1/2}(|d|,0) \tag{32}$$

$$- \sum_{0<m\leq \ell}\Big(\sum_{n|m}\chi_D(n)n^{s-1}a_{s+1/2}\big(|d|,\tfrac{m^2|D|}{n^2}\big)\Big)r^m$$

$$+\sum_{j=1}^{\ell}\sum_{m>\ell} a_{2-2s}(m,-j)r^m \sum_{h|j}\chi_D(h)h^{s-1}a_{s+\frac{1}{2}}\big(\,|d|\,,\tfrac{j^2|D|}{h^2}\big). \tag{33}$$

By integral weight duality (9) the term in (33) is

$$-\sum_{j=1}^{\ell}\sum_{m>\ell} a_{2s}(-j,m)r^m \sum_{h|j}\chi_D(h)h^{s-1}a_{s+\frac{1}{2}}\big(\,|d|\,,\tfrac{j^2|D|}{h^2}\big)$$

$$= -\sum_{j=1}^{\ell}(f_{2s,-j}(z)-r^j)\sum_{h|j}\chi_D(h)h^{s-1}a_{s+\frac{1}{2}}\big(\,|d|\,,\tfrac{j^2|D|}{h^2}\big),$$

so by (32) we get after some cancellation that

$$-\sum_{m>\ell} \mathrm{Tr}^*_{d,D}(f_{2-2s,m})r^m = \mathscr{S}_D f_{s+1/2,|d|}(z) - f(z). \tag{34}$$

The identity of Proposition 10 follows from (8) and (34), at least when $\operatorname{Im} z > \max_Q \operatorname{Im}\tau_Q$. Proposition 10 now follows by analytic continuation. $\square$

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7. Appendix

Table 1 gives explicit formulas for the first two basis elements $f_{b+\frac{1}{2}}, f^*_{b+\frac{1}{2}}$ of weight $b+1/2$ for all $b \in \{6,8,9,10,11,12,13,14,15,16,17,19\}$ as polynomials in the weight $1/2$ theta function $\theta = \sum_{n\in\mathbb{Z}} q^{n^2}$ and the weight 2 Eisenstein series on $\Gamma_0(4)$ given by

$$F(z) = \sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1}.$$

Both θ and F have integral Fourier coefficients.

The space of holomorphic modular forms on $\Gamma_0(4)$ of weight $s+1/2$ is generated by the forms $F^n\theta^{2s+1-4n}$, where $0 \leq n \leq \lfloor\frac{2s+1}{4}\rfloor$ (see [7]). Thus, to construct these basis elements we examine the Fourier expansion of the form

$$f = \sum_{n=0}^{\lfloor\frac{2s+1}{4}\rfloor} A(n)F^n\theta^{2s+1-4n}$$

and choose the coefficients $A(n)$ so that f is in the plus space $M^!_{s+\frac{1}{2}}$ and has the appropriate leading terms in its Fourier expansion. Table 1 shows that all of the $A(n)$ are integral for

TABLE 1

$$
\begin{array}{lcll}
f_{\frac{13}{2}} & = & F\theta^9 - 18F^2\theta^5 + 32F^3\theta & = \; q + O(q^4) \\
f^*_{\frac{13}{2}} & = & \theta^{13} - 26F\theta^9 + 156F^2\theta^5 & = \; 1 + O(q^4) \\
f_{\frac{17}{2}} & = & F\theta^{13} - 26F^2\theta^9 + 152F^3\theta^5 + 128F^4\theta & = \; q + O(q^4) \\
f^*_{\frac{17}{2}} & = & \theta^{17} - 34F\theta^{13} + 340F^2\theta^9 - 816F^3\theta^5 & = \; 1 + O(q^4) \\
f_{\frac{19}{2}} & = & F^3\theta^7 - 16F^4\theta^3 & = \; q^3 + O(q^4) \\
f^*_{\frac{19}{2}} & = & \theta^{19} - 38F\theta^{15} + 456F^2\theta^{11} - 1672F^3\theta^7 & = \; 1 + O(q^4) \\
f_{\frac{21}{2}} & = & F\theta^{17} - 34F^2\theta^{13} + 336F^3\theta^9 - 800F^4\theta^5 + 512F^5\theta & = \; q + O(q^4) \\
f^*_{\frac{21}{2}} & = & \theta^{21} - 42F\theta^{17} + 588F^2\theta^{13} - 2912F^3\theta^9 + 2496F^4\theta^5 & = \; 1 + O(q^4) \\
f_{\frac{23}{2}} & = & F^3\theta^{11} - 12F^4\theta^7 - 64F^5\theta^3 & = \; q^3 + O(q^4) \\
f^*_{\frac{23}{2}} & = & \theta^{23} - 46F\theta^{19} + 736F^2\theta^{15} - 4600F^3\theta^{11} + 8096F^4\theta^7 & = \; 1 + O(q^4) \\
f_{\frac{25}{2}} & = & F^4\theta^9 - 16F^5\theta^5 & = \; q^4 + O(q^5) \\
f^*_{\frac{25}{2}} & = & F\theta^{21} - 42F^2\theta^{17} + 584F^3\theta^{13} - 2808F^4\theta^9 + 1792F^5\theta^5 + 2048F^6\theta & = \; q + O(q^5) \\
f_{\frac{27}{2}} & = & F^3\theta^{15} - 32F^4\theta^{11} + 272F^5\theta^7 - 256F^6\theta^3 & = \; q^3 + O(q^4) \\
f^*_{\frac{27}{2}} & = & \theta^{27} - 54F\theta^{23} + 1080F^2\theta^{19} - 9576F^3\theta^{15} & \\
 & & +34048F^4\theta^{11} - 26752F^5\theta^7 & = \; 1 + O(q^4) \\
f_{\frac{29}{2}} & = & F^4\theta^{13} - 36F^5\theta^9 + 320F^6\theta^5 & = \; q^4 + O(q^5) \\
f^*_{\frac{29}{2}} & = & F\theta^{25} - 50F^2\theta^{21} + 896F^3\theta^{17} - 6664F^4\theta^{13} & \\
 & & +16672F^5\theta^9 - 3072F^6\theta^5 + 8192F^7\theta & = \; q + O(q^5) \\
f_{\frac{31}{2}} & = & F^4\theta^{15} - 30F^5\theta^{11} + 224F^6\theta^7 & = \; q^4 + O(q^7) \\
f^*_{\frac{31}{2}} & = & F^3\theta^{19} - 38F^4\theta^{15} + 444F^5\theta^{11} - 1408F^6\theta^7 - 1024F^7\theta^3 & = \; q^3 + O(q^7) \\
f_{\frac{33}{2}} & = & F^4\theta^{17} - 32F^5\theta^{13} + 272F^6\theta^9 - 256F^7\theta^5 & = \; q^4 + O(q^5) \\
f^*_{\frac{33}{2}} & = & F\theta^{29} - 58F^2\theta^{25} + 1272F^3\theta^{21} - 12824F^4\theta^{17} & \\
 & & +56064F^5\theta^{13} - 71552F^6\theta^9 - 4096F^7\theta^5 + 32768F^8\theta & = \; q + O(q^5) \\
f_{\frac{35}{2}} & = & F^4\theta^{19} - 38F^5\theta^{15} + 440F^6\theta^{11} - 1408F^7\theta^7 & = \; q^4 + O(q^7) \\
f^*_{\frac{35}{2}} & = & F^3\theta^{23} - 46F^4\theta^{19} + 724F^5\theta^{15} - 4240F^6\theta^{11} & \\
 & & +5632F^7\theta^7 - 4096F^8\theta^3 & = \; q^3 + O(q^7) \\
f_{\frac{39}{2}} & = & F^4\theta^{23} - 46F^5\theta^{19} + 720F^6\theta^{15} - 4064F^7\theta^{11} + 3584F^8\theta^7 & = \; q^4 + O(q^7) \\
f^*_{\frac{39}{2}} & = & F^3\theta^{27} - 54F^4\theta^{23} + 1068F^5\theta^{19} - 9120F^6\theta^{15} & \\
 & & +28608F^7\theta^{11} - 6144F^8\theta^7 - 16384F^9\theta^3 & = \; q^3 + O(q^7)
\end{array}
$$

the first two basis elements of each half integral weight, so it follows that all of the $f_{b+\frac{1}{2}}$ and $f^*_{b+\frac{1}{2}}$ have integral Fourier coefficients.

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