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Hyperbolic distribution problems and half-integral weight Maass forms

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1. Introduction

Recently Iwaniec [15] has given an estimate for the Fourier coefficients of a holomorphic cusp form of weight half an odd integer which improves upon that corresponding to the trivial bound for Salié sums. More precisely, let

$$F(z)=\sum_{n=1}^{\infty} a(n)e(nz)$$

be a holomorphic cusp form of weight $k=\frac{1}{2}+\ell$ for $\Gamma=\Gamma_0(N)$ with

$$\int_{\Gamma\backslash H} |F(z)|^2 y^{k-2}\,dx\,dy=1\,,$$

where $\ell\in\mathbb{Z}^+$ and $N\equiv 0 \pmod 4$. For any $n\geqq 1$ and $\varepsilon>0$ the trivial bound is

$$a(n) \underset{k,\varepsilon}{\ll} n^{k/2-1/4+\varepsilon}. \tag{1.1}$$

The exponent $k/2-1/4$ cannot be reduced if $F(z)$ comes from the subspace spanned by theta functions when $k=3/2$. But for $k\geqq 5/2$ or square-free n we expect that it may be replaced by $(k-1)/2$, corresponding to the Ramanujan conjecture for integral k. Iwaniec proves that for $k\geqq 5/2$ and n square-free

$$a(n) \underset{k,\varepsilon}{\ll} n^{k/2-2/7+\varepsilon}. \tag{1.2}$$

(Actually n^ε is replaced by $d(n)\log^2 2n$ where $d(n)$ is the divisor function.)

A striking application of (1.2) is to give the uniform distribution of certain lattice points in $\mathbb{Z}^3$ on a sphere centered at the origin with increasing radius, without imposing Linnik's condition and in a quantitative sense. Motivated by the corresponding problem in negative curvature, which is the distribution of Heegner points and closed geodesics on $PSL_2(\mathbb{Z})\backslash H$, one is led to establish the analogue of

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(1.2) for certain non-holomorphic forms of weight k for $\Gamma_0(N)$. Before describing this result and its applications in more detail we will first briefly indicate how (1.2) controls the distribution of lattice points on S^2.

Consider the set $V_n = \{m/|m| \in S^2;\ m \in \mathbb{Z}^3, |m|^2 = n\}$ where $|m|^2 = m_1^2 + m_2^2 + m_3^2$. For n square-free it follows from a celebrated result of Gauss (see e.g., [3]) that

$$r_3(n) = \# V_n = \frac{24h(d)}{w(d)}\left[1 - \left(\frac{d}{2}\right)\right] \tag{1.3}$$

where $d, h(d)$, and $w(d)$ are the discriminant, class number, and number of units of $\mathbb{Q}(\sqrt{-n})$. Now the Fourier coefficients of the theta function given by $\theta(z;u) = \sum_{m \in \mathbb{Z}^3} u(m) e(z|m|^2)$, with $u(x)$ a spherical harmonic of degree ℓ are $a(n) = n^{\ell/2} r_3(n) W_u(n)$, where

$$W_u(n) = \frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi)$$

are "Weyl sums". Since $\theta(z;u)$ is a holomorphic cusp form for $\Gamma_0(4)$ of weight $3/2 + \ell$ for $\ell \geqq 1$ (see [41]) we see that by Siegel's (ineffective) estimate $r_3(n) \gg_\varepsilon n^{1/2-\varepsilon}$ and (1.2) we get

$$W_u(n) \ll_{u,\varepsilon} n^{-1/28+\varepsilon}$$

provided n is square-free and $n \not\equiv 7 \pmod 8$. By the analogue of Weyl's citerion we deduce the uniform distribution (with a non-trivial error term if (1.2) is made uniform in k) of V_n on S^2 as $n \to \infty$ through such n. We remark that by combining (1.2) with the Shimura lift and Deligne's estimate the conditions on n may be relaxed to $n \equiv 1, 2, 3, 5, 6 \pmod 8$. This should be compared with Linnik's result ([20] p. 38) giving the uniform distribution of these V_n only if n is subjected to the further condition $\left(\frac{-n}{p}\right) = 1$ for some fixed odd prime p. Previously this unnatural condition could only be removed subject to certain unproved hypotheses concerning the zeros of Dirichlet L-functions (see [20] and [25]).

The main object of this paper is to extend this method to study the distributions of Heegner points and closed geodesics on $PSL_2(\mathbb{Z})\backslash H$. For this we need a non-holomorphic generalization of (1.2). This is given as Theorem 5 in Sect. 5. That such an extension is possible is indicated in [15]. The "beef" of the proof is an estimate of Iwaniec for a certain sum of Kloosterman sums over varying levels. When this is applied to Proskurin's generalization of the Kuznetsov sum formula given in Sect. 3, Theorem 5 follows is much the same way as does (1.2).

The other ingredient is the expression of the appropriate "Weyl sums" in terms of Fourier coefficients of certain non-holomorphic forms of weight 1/2. This is achieved through a theta-correspondence given essentially by Maass as an extension of Siegel's celebrated work on the analytic theory of indefinite quadratic forms. Although we only need it in a particular case here we will develop this correspondence in general, with future applications in mind.

To describe our main application, let $Q = Q(x,y) = ax^2 + bxy + cy^2$ be a primitive irreducible integral binary quadratic form with discriminant $d = b^2 - 4ac$.

Denote by $h(d)$ the number of (proper) classes of such forms. If $d>0$, Pell's equation is $x^2-dy^2=4$. Let (x_d, y_d) with $x_d, y_d>0$ be the fundamental solution and set $\varepsilon_d=(x_d+\sqrt{d}\,y_d)/2$. Set $\hat{\Gamma}=PSL_2(\mathbb{Z})$ and denote by F the standard fundamental region for $\hat{\Gamma}$ in the upper half-plane H. If $d<0$ the $h(d)$ *Heegner points* are given by

$$\Lambda_d=\left\{z_Q=\frac{-b+\sqrt{d}}{2a};\quad b^2-4ac=d,\quad z_Q\in F\right\}.$$

If $d<0$ is a fundamental discriminant then these points correspond to the ideal classes in $\mathbb{Q}(\sqrt{d})$. If $d>0$ the points $(-b\pm\sqrt{d})/2a$ determine the endpoints of a geodesic in H, with respect to the metric $ds^2=y^{-2}(dx^2+dy^2)$, which induces a unique primitive, positively oriented closed geodesic in $\hat{\Gamma}\backslash H$ of length $\log\varepsilon_d$ or $2\log\varepsilon_d$, according as Q is or is not equivalent to $-Q$. For a given $d>0$ denote by Λ_d the set of all such distinct geodesics. The total length of geodesics in Λ_d is $h(d)\log\varepsilon_d$ and every primitive, positively oriented closed geodesic in $\hat{\Gamma}\backslash H$ occurs in exactly one Λ_d. This classification follows from [36] (see also [43]).

Let $d\mu(z)=\frac{3}{\pi}\,dx\,dy/y^2$ so $\mu(F)=1$. Suppose $\Omega\subset F$ is convex (in the non-Euclidean sense) with a piece-wise smooth boundary. We shall prove the following uniform distribution statements in Sect. 6.

Theorem 1. *Suppose d is a fundamental discriminant. Then for some $\delta>0$ depending only on Ω*

i) $$\frac{\#\Lambda_d\cap\Omega}{\#\Lambda_d}=\mu(\Omega)+0(|d|^{-\delta})\quad \textit{as}\quad d\to-\infty\quad \textit{and}$$

ii) $$\frac{\sum_{C\in\Lambda_d}|C\cap\Omega|}{\sum_{C\in\Lambda_d}|C|}=\mu(\Omega)+0(d^{-\delta})\quad \textit{as}\quad d\to+\infty$$

where $|C|$ is the (non-Euclidean) length of C and the 0-constants depend only on δ and Ω, though ineffectively.

Remarks: (1) Part (i) with the error term $0(\log^{-A}|d|)$ for some $A>0$ but subject to the additional condition $(d/p)=1$ for some fixed odd prime p with suitable Ω was proved by Linnik [20] using his ergodic method.

(2) If Gauss' conjecture that $h(d)=1$ for infinitely many fundamental $d>0$ holds, then by (ii) we deduce that for any $\varepsilon>0$ there is a closed geodesic C in $\hat{\Gamma}\backslash H$ such that

$$\left|\frac{|C\cap\Omega|}{|C|}-\mu(\Omega)\right|<\varepsilon.$$

(3) Using results of [33] it is possible to give similar results for certain co-compact arithmetic subgroups of $PSL_2(\mathbb{R})$.

A different type of application of (1.2) reported on in [15] is to give upper bounds for critical values of twisted L-functions attached to certain holomorphic cusp forms via the Waldspurger theorem. We observe that Theorem 5 includes the interesting case of holomorphic cusp forms of weight 3/2 which is not covered directly in [15], although standard convergence producing modifications there will yield it as well. To describe such an application of this case let E_q be the elliptic curve with complex multiplication given by $y^2=x^3-q^2x$ and consider its (essential) Hasse-Weil L-function (see e.g., [16])

$$L(s,E_q)=\prod_{p\nmid 2q}(1-a_p(q)p^{-s}+p^{1-2s})^{-1}$$

where $\#E_q(\mathbb{Z}/p\mathbb{Z})=p+1-a_p(q)$ and $q\in\mathbb{Z}^+$ is square-free. Then by a theorem of Tunnell [48] and Theorem 5 we deduce the estimate

$$L(1,E_q)\underset{\varepsilon}{\ll} q^{3/7+\varepsilon} \tag{1.4}$$

for the critical values as $q\to\infty$. Here the trivial bound is $q^{1/2+\varepsilon}$ from (1.1) or from the functional equation for $L(s,E_q)$ and a convexity argument, while that corresponding to Ramanujan and Lindelöf is q^ε.

Similarly, when Theorem 5 is applied to half-integral weight Eisenstein series in case $N=4$ and $D=-4$ it follows from [7] that

$$L(\tfrac{1}{2}+it,\chi_d)\underset{t,\varepsilon}{\ll}|d|^{3/14+\varepsilon} \tag{1.5}$$

for Dirichlet L-functions $L(s,\chi_d)$. Here $d\in\mathbb{Z}$ is a fundamental discriminant and $\chi_d=\left(\frac{d}{\cdot}\right)$ is Kronecker's symbol. Burgess' famous result [4] replaces 3/14 by 3/16. Nevertheless, (1.5) gives an indication of the depth of Theorem 5 since Ramanujan here would give *the* Lindelöf hypothesis in d-aspect.

Finally, we mention that the estimate in Theorem 5 for weight 3/2 has *a* certain application in Galois theory (see [2]) and, when combined with results of *R*. Schulze-Pillot [38], contributes to the classical problem of determining the asymptotic behavior of the representation numbers of an arbitrary positive definite integral ternary quadratic form.

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2. Maass forms of half-integral weight

The following conventions will be observed throughout this paper. For $z\in\mathbb{C}$, $z\neq 0$ and $v\in\mathbb{R}$ we define z^v by

$$z^v=|z|^v\exp(iv\arg z)\quad\text{where always}\quad \arg z\in(-\pi,\pi].$$

The extended Kronecker symbol $\left(\frac{c}{d}\right)$ is defined as in [10] for $c,d\in\mathbb{Z}$ with $c\equiv 0,1 \pmod 4$ and $d>0$, but we set $\left(\frac{c}{d}\right)=\operatorname{sgn}(c)\left(\frac{c}{|d|}\right)$ for $d<0$ and $\left(\frac{1}{0}\right)=1$. $\left(\frac{c}{d}\right)$ is then extended as in [41] to $c \equiv 2,3 \pmod 4$ and *odd* d by letting $\left(\frac{c}{d}\right)$ be the Jacobi symbol for $d>0$ and $(c,d)=1$ and setting $\left(\frac{c}{d}\right)=0$ if $(c,d)>1$, $\left(\frac{c}{d}\right)=\operatorname{sgn}(c)\left(\frac{c}{|d|}\right)$ for $d<0$, and $\left(\frac{0}{\pm 1}\right)=1$. The convention concerning $\left(\frac{c}{d}\right)$ for $d<0$ differs from that in [10] if $c<0$ but insures that

$$\left(\frac{-1}{d}\right)=(-1)^{(d-1)/2}$$

holds for all odd d.

The theory of Maass forms with general weights and multiplier systems was developed by Selberg (see [39] and [40]). Detailed treatments are given in [12] and [34]. Another useful reference is [35] and we will employ its notation as much as possible.

Let $k \in \frac{1}{2}\mathbb{Z}$ be given and suppose $D \in \mathbb{Z}$ is a fundamental discriminant (or 1) which is even if $2k$ is odd and otherwise satisfies $(-1)^k D>0$. Let $N \in \mathbb{Z}^+$ be such that $D|N$ (so $4|N$ if $2k$ is odd) and define the *generalized theta multiplier* for $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ by

$$\chi_{k,D}(\gamma)=\left(\frac{c}{d}\right)^{2k}\left(\frac{-1}{d}\right)^{-k}\left(\frac{|D|}{d}\right) \tag{2.1}$$

where, if $2k$ is even this is read as $\left(\frac{D}{d}\right)$. In case $k=1/2$ and $D=-4$ this is the multiplier for $y^{1/4}\theta(z)=y^{1/4}\sum_{n\in\mathbb{Z}} e(n^2 z)$ on $\Gamma_0(4)$ (see [16]). For $2k$ odd it follows that $\chi_{k,D}(\gamma)$ defines a consistent multiplier system for $\Gamma_0(N)$ since by [10] $\left(\frac{|D|}{\cdot}\right)$ is an even character mod N. For $2k$ even $\left(\frac{D}{\cdot}\right)$ gives a character mod N which satisfies the consistency condition $\left(\frac{D}{-1}\right)=(-1)^k$. Clearly $\chi_{k,D_1}=\chi_{k,D_2}$ iff $|D_1|=|D_2|$.

Throughout this section we will denote $\Gamma_0(N)$ by Γ. A *Maass form of weight* k *and discriminant* D *for* Γ is an eigenfunction $f(z)$ on $H=\{z=x+iy; y>0\}$ of

$$\Delta_k=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-iky\frac{\partial}{\partial x} \tag{2.2}$$

which satisfies for all $\gamma\in\Gamma$ acting on H by $\gamma z=(az+b)/(cz+d)$ the transformation rule

$$f(\gamma z)=\chi_{k,D}(\gamma)\, e^{ik\arg(cz+d)} f(z) \tag{2.3}$$

and has a polynomial growth condition in the cusps of $\hat{\Gamma}\backslash H$ where $\hat{\Gamma}=\Gamma/\{\pm I\}$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\in\Gamma$, $f(z)$ has a Fourier expansion at the cusp at ∞ given by

$$f(z)=c(0,y)+\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \varrho(n)\, W_{k/2\operatorname{sgn}(n),s-1/2}(4\pi|n|y)e(nx) \tag{2.4}$$

where $c(0,y)=\varrho(0)y^s+\varrho'(0)y^{1-s}$, $\lambda=s(1-s)$ with $\mathrm{Re}(s)\geqq 1/2$ is the eigenvalue defined by $\Delta_k f+\lambda f=0$, and $W_{\alpha,\beta}(z)$ is the standard Whittaker function (see [24]). It should be noted that in case $k=0$, $\varrho(n)$ is usually defined as the coefficient of the K-Bessel function, having the effect of multiplying $\varrho(n)$ by $2|n|^{1/2}$. If $f(z)=y^{k/2}F(z)$ where $F(z)$ is a holomorphic form of weight k so f has eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$ then (2.4) reads

$$F(z)=\sum_{n=0}^{\infty} a(n)e(nz) \tag{2.5}$$

where $a(n)=(4\pi n)^{k/2}\varrho(n)$ for $n>0$. This follows since $\varrho(n)=0$ for $n<0$ in (2.4) if $F(z)=y^{-k/2}f(z)$ is holomorphic and by [15] p. 305.

Define the *Maass operator* Λ_k by $\Lambda_k=(z-\bar{z})\dfrac{\partial}{\partial\bar{z}}+\dfrac{k}{2}=iy\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}+\dfrac{k}{2}$. If $f(z)$ is a Maass form of weight k for Γ then $(\Lambda_k f)(z)$ is one of weight $k-2$ with the same eigenvalue and discriminant. This follows by applying Λ_k to (2.3) and (2.4). Using [24] p. 302, we see that if $\{\varrho(n)\}$ are the Fourier coefficients of f and $\{\hat{\varrho}(n)\}$ are those of $\Lambda_k f$ then

$$\hat{\varrho}(n)=\begin{cases}\varrho(n), & n<0\\ (s-k/2)(k/2-1+s)\varrho(n), & n>0,\end{cases}$$

$$\hat{\varrho}(0)=(k/2-s)\varrho(0), \tag{2.6}$$

and

$$\hat{\varrho}'(0)=(k/2-1+s)\varrho'(0).$$

Clearly $\Lambda_k f=0$ iff $y^{-k/2}f(z)$ is holomorphic of weight k. Also, $\bar{f}(z)$ is a Maass form of weight $-k$ and discriminant D with Fourier coefficients $\bar{\varrho}(-n)$ for $n\neq 0$. Thus, in order to study $\varrho(n)$ for any non-holomorphic $f(z)$ and any n it is enough to assume that $k\in\{0,1/2,1,3/2\}$. Finally, by (1.9) in [35] if $\|f\|^2=\int\limits_{\Gamma\backslash H}|f|^2\dfrac{dxdy}{y^2}=1$ then

$$\|\Lambda_k f\|^2=\lambda-\frac{k}{2}\left(1-\frac{k}{2}\right)=\left(\frac{k}{2}-s\right)\left(\frac{k}{2}-1+s\right). \tag{2.7}$$

In preparation for Kuznetsov's sum formula we next introduce the Eisenstein series. Let $\{\infty=\kappa_1,\kappa_2,\ldots,\kappa_h\}$ be a complete set of Γ-inequivalent cusps of $\hat{\Gamma}\backslash H$. For each such κ_ℓ choose $\sigma_\ell\in SL_2(\mathbb{Z})$ s.t. $\sigma_\ell(\kappa_1)=\kappa_\ell$ and $\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell=\Gamma_1$, where $\Gamma_\ell=\{\gamma\in\Gamma:\gamma\kappa_\ell=\kappa_\ell\}$ is the stabilizer of κ_ℓ. For each $\ell=1,\ldots,h$ and $\mathrm{Re}(s)>1$ define the *Eisenstein series* by

$$E_\ell(z,s;N,k,D)=\sum_{\gamma\in\Gamma_\ell\backslash\Gamma}\bar{\chi}_{k,D}(\sigma_\ell^{-1}\gamma)e^{-ik\arg j(\sigma_\ell^{-1}\gamma,z)}y(\sigma_\ell^{-1}\gamma z)^s$$

where $j(\gamma,z)=cz+d$, $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. For each such s $E_\ell(z,s;N,k,D)$ is a Maass form of weight k and discriminant D for Γ with eigenvalue $s(1-s)$ and Fourier coefficients given by

$$\varrho_{\ell s}(n)=\frac{\pi^s e(-k/4)|n|^{s-1}}{\Gamma(s+k/2\,\mathrm{sgn}(n))}\phi_{\ell n}(s)\quad\text{for}\quad n\neq 0,$$

$$\varrho_\ell(0)=\delta_{\ell 1},\quad\text{and}\quad \varrho'_{\ell s}(0)=\frac{\pi 4^{1-s}e(-k/4)\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)}\phi_{\ell 0}(s), \tag{2.8}$$

where

$$\phi_{\ell n}(s)=\sum_{c>0} c^{-2s} \sum_{\substack{0\leqq d<c\\ \gamma\in\sigma_\ell^{-1}\Gamma}} \bar{\chi}_{k,D}(\gamma)e(nd/c)$$

is a "singular series".

By Selberg's general theory $E_\ell(z,s;N,k,D)$ has a meromorphic continuation in s with no poles for $\mathrm{Re}\, s\geqq 1/2$ except for possibly finitely many simple ones in $(1/2,1]$. In fact $\phi_{\ell n}(s)$ may be evaluated in terms of Dirichlet L-functions with the consequence that no such poles may exist except possibly at $s=3/4$ when $2k$ is odd or at $s=1$ when $k=0$. In cases $k=\pm(3/2+2\ell)$ for $\ell=0,1,2,\ldots$, it follows from the presence of $\Gamma(s\mp k/2)$ in (2.8) that no pole exists. Now $E_\ell(z,1/2+it;N,k,D)$ for $\ell=1,\ldots,h$ furnish the continuous spectrum $[1/4,\infty)$ with multiplicity h of Δ_k on $\mathbf{H}_{k,D}$, the Hilbert space of functions $f(z)$ on H satisfying (2.3) and $\|f\|<\infty$. A non-zero residue of $E_\ell(z,s;N,k,D)$ at $s=3/4$ ($s=1$) is an eigenfunction in $\mathbf{H}_{k,D}$ with eigenvalue $\lambda=3/16$ ($\lambda=0$) belonging to the discrete spectrum. The rest of the discrete spectrum is provided by the Maass cusp forms, which are Maass forms whose zeroth Fourier coefficients in all cusps vanish. In general, we have that the discrete spectrum of Δ_k on $\mathbf{H}_{k,D}$ is contained in $[k/2(1-k/2),\infty)$ and any form with eigenvalue $\lambda=k/2(1-k/2)$ comes from a holomorphic one of weight k. This follows from (2.7). Selberg showed that if $k\in 2\mathbb{Z}$ the first positive eigenvalue is $\geqq 3/16$ while Goldfeld and Sarnak [35] observed in case $2k$ is odd that the first eigenvalue $>3/16$ is $\geqq 15/64$.

Definition. A Maass form $f(z)$ is *spectral* if either $\|f\|=1$ or $f(z)=E_\ell(z,\frac{1}{2}+it;N,k,D)$ for some $t\in\mathbb{R}$ and $\ell\in\{1,\ldots,h\}$.

3. Kuznetsov sum formula

We assume in this section that $k\in\{0,1/2,1,3/2\}$. As noted above this is sufficient for our purposes. We state Kuznetsov's sum formula in these cases as given (actually more generally) by Proskurin in [31]. Let $\{u_j(z)\}_{j=0}^{\infty}$ be an orthonormal basis of Maass forms corresponding to the discrete spectrum $0\leqq\lambda_0\leqq\lambda_1<\ldots$ of Δ_k on $\mathbf{H}_{k,D}$ with Fourier coefficients given by $\{\varrho_j(n)\}$ for $n\neq 0$, $\varrho_j(0)=0$, and $\varrho_j'(0)=0$ except when $u_j(n)$ comes from a residue. Note that $\lambda=3/16$ corresponds to the holomorphic forms for both $k=1/2$ and $k=3/2$. Let $\{f_{ij}\}_{i=1}^{d_j}$ be an orthonormal basis for the space of holomorphic cusp forms for $\Gamma_0(N)$ of weight $k+2j$ with multiplier $\chi_{k,D}$ where $j=1,2,\ldots$ with Fourier coefficients $a_{ij}(n)$.

Next let $\phi(x)$ be a smooth function on $[0,\infty)$ such that $\phi(0)=\phi'(0)=0$ and for some $\varepsilon>0$ $\phi(x)\ll_\varepsilon x^{-1-\varepsilon}$, $\phi^{(\ell)}(x)\ll_\varepsilon x^{-2-\varepsilon}$ for $\ell=1,2,3$ as $x\to\infty$. Define the *Kuznetsov transforms* for $t\in\mathbb{R}$ or $it\in[-1/4,1/4]$ by

$$\tilde{\phi}(t)=\int_0^\infty J_{t-1}(x)\phi(x)\frac{dx}{x} \tag{3.1}$$

and

$$\hat{\phi}(t)=\beta(t)\,(\phi^*(t)-\phi^*(-t)) \tag{3.2}$$

where

$$\beta(t)=\frac{\pi^2 e(1/4+k/4)}{(sh\pi t)(ch2\pi t+\cos\pi k)\Gamma(1/2-k/2+it)\Gamma(1/2-k/2-it)}$$

and

$$\phi^*(t)=\cos\pi(k/2+it)\int_0^\infty J_{2it}(x)\phi(x)\frac{dx}{x}.$$

Observe that $\hat{\phi}(-t)=\hat{\phi}(t)$ so that $\hat{\phi}(t_j)$ is well-defined for $t_j=\sqrt{\lambda_j-1/4}$. Set for $n,m\geqq 1$

$$V_1(m,n)=4(mn)^{1/2}\sum_{\lambda_j>0}\bar{\varrho}_j(m)\varrho_j(n)\hat{\phi}(t_j)/ch\pi t_j, \tag{3.3}$$

$$V_2(m,n)=\sum_{j=1}^{h}\int_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{it}\frac{\bar{\phi}_{jm}(1/2+it)\phi_{jn}(1/2+it)}{ch\pi t|\Gamma(1/2+k/2+it)|^2}\hat{\phi}(t)dt, \tag{3.4}$$

and

$$V_3(m,n)=4\sum_{j\geqq 1}\frac{\Gamma(k+2j)e(k/4+j/2)\tilde{\phi}(k+2j)}{(4\pi)^{k+2j}(mn)^{k/2+j-1/2}}\sum_{i=1}^{d_j}\bar{a}_{ij}(m)a_{ij}(n). \tag{3.5}$$

Finally, the generalized Kloosterman sum in question is given by

$$K_{k,D}(m,n;c)=K(m,n;c)=\sum_{d\bmod c}\bar{\chi}_{k,D}(\gamma)e\left(\frac{m\bar{d}+nd}{c}\right). \tag{3.6}$$

We may now state Kuznetsov's sum formula.

Theorem 2 (Proskurin [31]). *For $k\in\{0,1/2,1,3/2\}$, $n,m\geqq 1$, and $\phi(x)$ as above* (3.1) *we have*

$$\sum_{\substack{c>0\\ c\equiv 0(N)}}c^{-1}K(m,n;c)\phi(4\pi\sqrt{mn}/c)=\sum_{\ell=1}^{3}V_\ell(m,n).$$

4. A correspondence of Maass

This section is devoted to the construction of Maass forms as integrals of Siegel theta functions against certain automorphic eigenfunctions. This is done by extending some results of Siegel and Maass from the 1950's (see [22, 44] and also [13]). For holomorphic forms similar methods have been used extensively to simplify and systematize various lifts, for example those of Shimura and Doi-Naganuma (see [18, 27], and [42]).

A recent application of the non-holomorphic version of Shimura's correspondence is Goldfeld and Sarnak's 15/64 eigenvalue bound mentioned at the end of Sect. 2. The inverse form of this correspondence is actually a special case of that considered below. Here certain Fourier coefficients of the resulting Maass form of weight 1/2 are essentially Weyl-type sums for the distribution of certain lattice points and closed geodesics on the appropriate hyperbolic surface. A non-trivial bound for such Fourier coefficients then gives uniformity results for these distributions. These matters are pursued in Sect. 5 and 6.

We first need to introduce Siegel's theta function for an indefinite quadratic form. Let S be an $m\times m$ matrix with elements $s_{ij}\in\frac{1}{2}\mathbb{Z}$ and diagonal elements $s_{ii}\in\mathbb{Z}$. Then S has a uniquely determined signature $(n, m-n)$ for $m\geqq n\geqq 0$. Now for $z=x+iy$ with $y>0$ and $P\in H_S$, the majorant space of S (see [46]) let $R=xS+iyP$. Here $P={}^tP$, $P>0$, and $PS^{-1}P=S$. *Siegel's theta function is* defined for $\alpha\in\mathbb{Q}^n$ with $2S\alpha\in\mathbb{Z}^m$ by

$$\theta_\alpha(z,P)=\sum_{h\in\mathbb{Z}^m} e(R[h+\alpha]) \tag{4.1}$$

where $A[B]={}^tBAB$. This sum is absolutely convergent for fixed z since $y>0$ and $P>0$.

Our first task is to extend the considerations in [44] concerning the transformation formula for $\theta(z,P)=\theta_0(z,P)$ in z under $\Gamma_0(N)$ for a suitable N. Specifically, let $\delta=|\det 2S|$ and

$$\delta'=\begin{cases}\delta, & m \text{ even}\\ 2\delta, & m \text{ odd}\end{cases}.$$

Write $\delta'=ef^2$ where e is square-free and set

$$\omega(e)=\begin{cases}0, & e\equiv 1 \pmod 4\\ 1, & \text{otherwise}\end{cases}$$

and

$$D=\begin{cases}-(-1)^{\omega(e)}4e, & m \text{ odd}\\ \sigma 4^{\omega(\sigma e)}e, & m \text{ even}\end{cases} \tag{4.2}$$

where $\sigma=(-1)^k$ with $k=\dfrac{m}{2}-n$. Let N be the *level* of $2S$, i.e. $N\in\mathbb{Z}^+$ is minimal such that $N(2S)^{-1}$ is integral with even diagonal entries.

Theorem 3. *$D|N$ and D is a fundamental discriminant or* 1 *which is even if $2k$ is odd and satisfies $(-1)^kD>0$ otherwise. For $\begin{pmatrix}a&b\\c&d\end{pmatrix}=\gamma\in\Gamma_0(N)$ we have*

$$\theta(\gamma z,P)=\chi_{-k,D}(\gamma)\,(cz+d)^{n/2}(c\bar z+d)^{(m-n)/2}\,\theta(z,P) \tag{4.3}$$

where $\chi_{k,D}$ is given in (2.1), *provided $d=1$ if $c=0$. If $\delta=1$ then $4|k$.*

Proof. By Siegel's [44] Hilfssatz 1 we have for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ with $c>0$ and $r,q,h\in\mathbb{Z}^m$

$$\begin{aligned}(cz+d)^{-n/2}&(c\bar z+d)^{(n-m)/2}\,\theta_{h/N}(\gamma z,P)\\ &=\varepsilon\delta^{-1/2}c^{-m/2}\sum_{\substack{q \bmod N\\ 2Sq\equiv 0\,(N)}}\phi_\gamma(h,q)\,\theta_{q/N}(z,P)\end{aligned} \tag{4.4}$$

where $\phi_\gamma(h,q)=\sum\limits_{\substack{r\equiv h(N)\\ r \bmod cN}} e((aS[r]+2{}^trSq+dS[q])/cN^2)$ and $\varepsilon=e((m-2n)/8)$. Thus we have

$$(cz+d)^{-n/2}(c\bar{z}+d)^{(n-m)/2}\theta_{h/N}(\gamma z,P)$$
$$=\varepsilon\delta^{-1/2}c^{-m/2}\sum_{\substack{q \bmod N\\ 2Sq\equiv 0\,(N)}} e(-b(dS[q]+2{}^thSq)/N^2)\phi_\gamma(h+dq,0)\theta_{q/N}(z,P) \tag{4.5}$$

by using the identity

$$\phi_\gamma(h,q)=e(-b(dS[q]+2{}^thSq)/N^2)\phi_\gamma(h+dq,0).$$

By a standard method (see [29, 37]) we get that for $c\equiv 0 \,(\bmod N)$ and $d>0$,

$$(cz+d)^{-n/2}(c\bar{z}+d)^{(n-m)/2}\theta(\gamma z,P)=g(c,d)\theta(z,P) \tag{4.6}$$

where $g(c,d)=d^{-m/2}\sum_{r \bmod d} e(-cS[r]/d)$ is a Gauss sum.

To complete the proof we need to evaluate $g(c,d)$. To this end note that if in (4.6) we take $\gamma=\begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix}\gamma'$ for $\ell\in\mathbb{Z}$ then since $\theta(z+\ell,P)=\theta(z,P)\neq 0$ we get for $c\equiv 0\,(\bmod N)$ and $d,d+c\ell>0$ that $g(c,d+\ell c)=g(c,d)$. Since

$$\begin{pmatrix}a & bc'\\ N & d\end{pmatrix}\in\Gamma_0(N) \quad\text{if}\quad \begin{pmatrix}a & b\\ Nc' & d\end{pmatrix}\in\Gamma_0(N)$$

we see that for $d,d+\ell N>0$, $g(c,d)=g(c,d+\ell N)$.

Now choose an odd prime $p\equiv d\,(\bmod c)$ so $g(c,d)=g(c,p)$, when $d>0$. We may transform $S[x]$ to diagonal form mod p since $p\nmid\delta$. Thus, for some $m\times m$ integral matrix T with $|T|\equiv t\not\equiv 0\,(\bmod p)$ we have $S[x]=\sum_{j=1}^m a_ju_j^2$ where $a_j\in\mathbb{Z}$ with $p\nmid a_j$ and $x=Tu$. Clearly we may take $a_j>0$ for $j\leqq n$ and $a_j<0$ for $j>n$. Then $|S|\equiv t^2\prod_{j=1}^m a_j\,(\bmod p)$, and also $\delta\equiv 2^mt^2\prod_{j=1}^m |a_j|\,(\bmod p)$.

Thus

$$g(c,p)=p^{-m/2}\sum_{\substack{u \bmod p\\ u\in\mathbb{Z}^m}} e\left(-c\sum_{j=1}^m a_ju_j^2/p\right)$$
$$=p^{-m/2}\prod_{j=1}^m\sum_{u_j \bmod p} e(-ca_ju_j^2/p) \tag{4.7}$$
$$=\left(\frac{c}{p}\right)^{-2k}\left(\frac{-1}{p}\right)^k\left(\frac{2^m\delta}{p}\right)$$

using the familiar evaluation for $p\nmid y$

$$\sum_{u \bmod p} e(yu^2/p)=\left(\frac{y}{p}\right)\left(\frac{-1}{p}\right)^{1/2}p^{1/2}.$$

Suppose that m is odd. Clearly $D=-(-1)^{\omega(e)}4e$ is an even fundamental discriminant and by (4.7)

$$g(c,d)=g(c,p)=\left(\frac{c}{p}\right)^{-2k}\left(\frac{-1}{p}\right)^{k}\left(\frac{|D|}{p}\right).$$

As in Lemma 1 of [47] it follows that $4|N$ and that $\left(\frac{|D|}{\cdot}\right)$ defines a character mod N. Also,

$$\left(\frac{c}{d+c\ell}\right)\left(\frac{-1}{d+c\ell}\right)^{-1/2}=\left(\frac{c}{d}\right)\left(\frac{-1}{d}\right)^{-1/2}$$

for $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(4)$ and $\ell\in\mathbb{Z}$. Thus $g(c,d)=\chi_{-k,D}(\gamma)$ for $\gamma\in\Gamma_0(N)$ and $d>0$.

Now suppose m is even. Then D is a fundamental discriminant such that $(-1)^k D>0$. Also,

$$g(c,d)=g(c,p)=\left(\frac{\sigma\delta}{p}\right)=\left(\frac{\sigma 4^{\omega(\sigma e)}e}{p}\right)=\left(\frac{D}{p}\right)$$

which as above is a character mod N, so $g(c,d)=\left(\frac{D}{d}\right)$.
If $d=0$ then $\delta=1$ and from (4.4)

$$z^{-n/2}\bar{z}^{(n-m)/2}\theta(-1/z,P)=\varepsilon\theta(z,P)$$

so by applying the transformation $z\to z+1$ we get

$$(z+1)^{-n/2}(\bar{z}+1)^{(n-m)/2}\theta\left(\frac{-1}{z+1},P\right)=\varepsilon\theta(z,P).$$

Thus from (4.6) $\varepsilon=g(1,1)=1$ so (4.3) holds in this case as well. Also, since $\varepsilon=e(k/4)$ we have that $\delta=1$ implies that $4|k$.

If $d<0$ then replacing γ by $-\gamma$ in (4.3) gives the same result, provided $c\neq 0$. This completes the proof.

We are now ready to define Maass' correspondence. As above denote by H_S the space of majorants of S. Thus H_S consists of positive symmetric $m\times m$ matrices P such that $PS^{-1}P=S$. H_S is the symmetric space of dimension $n(m-n)$ attached to the group $G=SO(S)=\{g\in \mathrm{SL}_m(\mathbb{R});\ S[g]=S\}$ acting by $P\to P[g]$, since every majorant has the form $P[g]$ for $P\in H_S$ fixed and some $g\in G$ and since $G\cap SO(P)$ is a maximal compact subgroup of G (see [46]). A similar result holds for G_0, the connected component of G containing the identity I. An invariant metric on H_S is given by $ds^2=\frac{1}{8}\operatorname{tr}(P^{-1}dPP^{-1}dP)$.

The group of units of S is $\Gamma_S'=\mathrm{SL}_m(\mathbb{Z})\cap G$. We will denote by Γ' any subgroup of finite index in Γ_S' and let

$$\hat{\Gamma}'=\begin{cases}\Gamma'/\{\pm I\}, & -I\in\Gamma',\\ \Gamma', & \text{otherwise}.\end{cases}$$

$\hat{\Gamma}'$ acts discontinuously on H_S. $S[x]$ is said to be a *zero form* if $S[h]=0$ for some $h\in\mathbb{Z}^m$, $h\neq 0$. If $S[x]$ is not a binary zero form then Siegel [46] has shown that $v(\hat{\Gamma}'\backslash H)<\infty$, where dv is the invariant measure induced by ds^2. We shall assume that this is the case. Let Δ_S be the Laplace-Beltrami operator on H_S induced by ds^2 and suppose $u(P)$ is an eigenfunction of Δ_S on $\hat{\Gamma}'\backslash H_S$ with eigenvalue λ' defined by $\Delta_S u+\lambda' u=0$. For ϕ_1,ϕ_2 functions on $\hat{\Gamma}'\backslash H_S$ define the inner product $\langle\phi_1,\phi_2\rangle=\int_{\hat{\Gamma}'\backslash H_S}\phi_1\bar{\phi}_2\,dv$. By (4.1) it follows that $\theta(z,\cdot)$ is defined on $\hat{\Gamma}'\backslash H_S$. Suppose $\langle u(\cdot),\theta(z,\cdot)\rangle$ is absolutely convergent for each $z\in H$, that $\langle u,u\rangle<\infty$, and let D and N be defined in and below (4.2).

Theorem 4. *Under the above assumptions* $f(z)=y^{m/4}\langle u(\cdot),\theta(z,\cdot)\rangle$ *is a Maass form of weight* $k=(m/2)-n$ *and discriminant* D *for* $\Gamma_0(N)$ *with eigenvalue* $\lambda=\frac{1}{4}\left(\lambda'+m-\frac{m^2}{4}\right)$ $f(z)$ *is a cusp form (possibly zero) unless* $u(P)=c_0$ *for* c_0 *a nonzero constant.*

Proof. By Theorem 3 $f(z)$ is seen to satisfy (2.3). It follows from [22] (110) that $f(z)$ has a Fourier expansion at ∞ given by (2.4) where $s(1-s)=\frac{1}{4}\left(\lambda'+m-\frac{m^2}{4}\right)$. Thus either $f(z)=0$ or it is an eigenfunction of Δ_k with eigenvalue λ. By (4.1) it follows that $f(z)=y^{m/4}<u,1>+o(1)$ as $y\to\infty$ and we have $\lambda'<u,1>=0$. Thus $c(0,y)=0$ in (2.4) unless $\lambda'=0$. If $\lambda'=0$ then $u(P)=c_0\neq 0$ and $c(0,y)=c_0y^{m/4}$. Using (4.4) we see that the zeroth Fourier coefficient of $f(z)$ in every cusp vanishes unless $u(P)=c_0$, in which case $f(z)$ is easily seen to satisfy a polynomial growth condition in each cusp. This proves Theorem 4.

For small values of m there are "accidental" isomorphisms for G which allow us to apply the above correspondence to classical situations. For example, if $m=2$ and $n=1$ then for a certain choice of S and Γ' H_S may be identified with a semi-circle in H and $u(P)$ with a Grössencharakter for $\mathbb{Q}(\sqrt{D})$ for $D>0$. Then $f(z)$ is essentially the weight zero Maass form for $\Gamma_0(D)$ with character $\left(\frac{D}{\cdot}\right)$ constructed in Maass' celebrated paper [21]. The special case alluded to in the beginning of this section is for $m=3$ and $n=1$. Then $G_0\simeq \mathrm{PSL}_2(\mathbb{R})$ and H_S may be identified with H. $\Gamma'\subset G_0$ includes many important arithmetic subgroups, in particular all congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. The Hilbert modular groups for quadratic fields and their congruence subgroups are included in case $m=4$, with $n=1$ giving the imaginary quadratic ones and $n=2$ giving the real quadratic ones. Finally, the case $m=5$ and $n=2$ includes the Siegel modular group of genus 2 (see [18]).

From the point of view of distribution problems, the case $m=3$ is the most interesting. In this case also the Fourier coefficients of $f(z)$ may be explicitly evaluated and are found to control the distributions of interest. Thus for the remainder of the paper we will be concerned with the Fourier coefficients in case k is half an odd integer.

5. An estimate for Fourier coefficients

In this section we will extend Iwaniec's result (1.2) to include all spectral Maass forms of weight half an odd integer and discriminant D. For the applications given it is important to keep the estimate uniform in the eigenvalue λ. We remark that the generalized Ramanujan/Lindelöf conjecture would replace 2/7 by 1/2 and 5/4 by ε in the following result.

Theorem 5. *Let $\{\varrho(n)\}$ be the Fourier coefficients of a spectral Maass form $f(z)$ of weight $k=1/2+\ell$ and (even) discriminant D for $\Gamma_0(N)$, where $\ell\in\mathbb{Z}$ and $N\equiv 0 \pmod D$, with eigenvalue $\lambda=1/4+t^2$. We have the estimate*

$$\varrho(n) \underset{k,D,\varepsilon}{\ll} |\lambda|^A ch(\pi t/2)|n|^{-2/7+\varepsilon}$$

as $|n|\to\infty$, provided n is square-free or a fundamental discriminant. Here we may take $A=5/4-k/4\ \mathrm{sgn}(n)$.

Proof. We shall assume in the proof that $D=-4$ which is the case required for Theorem 1. The needed estimates in [15] easily extend to cover the general case. Now, observe that we may assume that $k=1/2$ *or* $3/2$ *and* $n\geqq 1$. This follows by (1.2), (2.5), (2.6), and the remarks following (2.6). Here we are using (2.6)–(2.8) to verify that $\Lambda_k f(z)/(k/2-s)$ is spectral if $\lambda\geqq 1/4$.

Next we must choose a test function $\phi(x)$ for which $\hat{\phi}(t)>0$ for $t\in\mathbb{R}$ or $it\in[-1/4,1/4]$. For $c_0=-4e(-k/4)/\pi^2\Gamma(5/2)$ let

$$\phi(x)=c_0x^{-3/2}J_{9/2}(x)$$

where $J_{9/2}(x)$ is the usual Bessel function. This ϕ satisfies the condition of Theorem 2 and a calculation using the Weber-Schafheitlin integral ([9] p. 692) shows that

$$\hat{\phi}(t)=\frac{1/4+t^2}{ch(2\pi t)\Gamma(4+it)\Gamma(4-it)\Gamma(1/2-k/2+it)\Gamma(1/2-k/2-it)}.$$

Clearly $\hat{\phi}(t)>0$ for $t\in\mathbb{R}$ or $it\in[-1/4,1/4]$, the value for $it=1/4$ defined by taking the limit. By Stirling's formula we see that as $\lambda\to\infty$

$$\hat{\phi}(t)\sim\frac{1}{2\pi^2}|t|^{k-5}.$$

By Theorem 2 we have for $f(z)$ spectral and $n\geqq 1$ that

$$n|\varrho(n)|^2\ll\lambda^{(5-k)/2}ch\pi t(|S_N|+|V_3(n,n)|) \tag{5.1}$$

where

$$S_N=\sum_{\substack{c\equiv 0(N)\\ c>0}}(c/n)^{3/2}c^{-1}K(n,n;c)J_{9/2}(4\pi n/c).$$

We now employ Iwaniec's device of averaging over the level. Setting for $P>(4\log 2n)^2$

$$\bar{Q}=\{pN;P<p\leqq 2P, p\nmid 2n\}$$

we have that $[\Gamma_0(Q):\Gamma_0(N)]^{-1/2}f(z)$ is a spectral Maass form for $\Gamma_0(Q)$ when $Q\in\bar{Q}$.

Since $[\Gamma_0(Q):\Gamma_0(N)]\leqq p+1$ we get from (5.1) upon summing over $\bar{Q}$:

$$n|\varrho(n)|^2 \ll \lambda^{(5-k)/2}(ch\pi t)\log P \sum_{Q\in\bar{Q}} (|S_Q|+|V_3(n,n)|). \tag{5.2}$$

By the proof of (1.2) in [15] we have

$$\sum_{Q\in\bar{Q}} |V_3(n,n)| \underset{\varepsilon,k}{\ll} |n|^{3/7+\varepsilon}. \tag{5.3}$$

Next,

$$\sum_Q |S_Q| \leqq \sum_Q \left| \sum_{\substack{c\equiv 0\,(Q)\\ c\leqq n}} c^{-1}K(n,n;c)J_{9/2}(4\pi n/c)\right|$$

$$+\sum_Q \left| \sum_{\substack{c\equiv 0\,(Q)\\ c>n}} (c/n)^{3/2}c^{-1}K(n,n;c)J_{9/2}(4\pi n/c)\right|.$$

These sums may be treated almost exactly as in [15] since 9/2 is half-integral and since for $x>n$

$$n^{-3/2}(xJ_{9/2}(4\pi n/x))' \ll n x^{-5/2}.$$

Thus it follows by Theorem 3 in [15], (5.2), and (5.3) that for $N\equiv 0 \pmod 8$ and $n\geqq 1$ square-free

$$n|\varrho(n)|^2 \underset{\varepsilon,k}{\ll} \lambda^{(5-k)/2} ch\pi t\,[(n/P)^{1/2}+(nP)^{3/8}+n^{3/7}]n^{\varepsilon}.$$

Choosing $P=n^{1/7}$ we get

$$|\varrho(n)| \underset{\varepsilon,k}{\ll} \lambda^{(5-k)/4} ch(\pi t/2) n^{-2/7+\varepsilon}.$$

If $N\equiv 4 \pmod 8$ the same holds by passing to $\Gamma_0(2N)$. If n is a fundamental discriminant then a slight alteration of Theorem 3 in [15] gives the same result. This completes the proof.

6. Ternary forms and hyperbolic surfaces

We end this final section with the proof of Theorem 1. First we give Maass' [22] evaluation of the Fourier coefficients of $f(z)$ from Theorem 4 in case $m=3$ and $n=1$.

Let $S[x]$ be an integral ternary quadratic form of signature (1,2). Recall the discussion above Theorem 4 and suppose that $\Gamma'\subset\Gamma''=G_0\cap \mathrm{SL}_3(\mathbb{Z})$. Now for some nonsingular matrix C,

$$S[C]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

After Klein and Fricke H_S may be identified with H and $\Gamma'=\hat{\Gamma}'$ with $\hat{\Gamma}$, where Γ is a co-finite subgroup of $\mathrm{SL}_2(\mathbb{R})$ containing $-I$. More precisely, $\mathrm{PSL}_2(\mathbb{R})\cong SO_0(1,2)$

through the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to g_0 = \begin{pmatrix} (a^2+b^2+c^2+d^2)/2 & (-a^2+b^2-c^2+d^2)/2 & -ab-cd \\ (-a^2-b^2+c^2+d^2)/2 & (a^2-b^2-c^2+d^2)/2 & ab-cd \\ -ac-bd & ac-bd & ad+bc \end{pmatrix} \quad (6.1)$$

and $\Gamma = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : C g_0 C^{-1} \in \Gamma' \right\}$.

To $h \in \mathbb{R}^3$, $h \neq 0$, we associate the points given by

$$z_h^{\pm} = \frac{-q_3 \pm (-S[h])^{1/2}}{q_1 + q_2}$$

where $q = C^{-1}h$, ${}^t q = (q_1, q_2, q_3)$ provided $q_1 + q_2 \neq 0$. If $q_1 + q_2 = 0$ let $z_h^{-\operatorname{sgn}(q_3)} = i\infty$ and $z_h^{\operatorname{sgn}(q_3)} = (q_2 - q_1)/2q_3$. Let $\Gamma_h' = \{g \in \Gamma'; gh = h\}$ and Γ_h be the corresponding subgroup of Γ containing $-I$. Then Γ_h is the stabilizer of $z_h^{\pm}$ in Γ since a calculation using (6.1) shows that $h_1 = gh$ iff $z_{h_1}^{\pm} = \gamma z_h^{\pm} = (az_h^{\pm} + b)/(cz_h^{\pm} + d)$ and $S[h_1] = S[h]$, where $g \in G_0$ corresponds to $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. Clearly $\Gamma_h = \Gamma_{nh}$ for $n \in \mathbb{Z}$, $n \neq 0$.

Define the *S-discriminants*

$$\mathbf{D}_S = \{d \in \mathbb{Z}; d \neq 0 \text{ and } d = -S[h] \text{ for some } h \in \mathbb{Z}^3 \text{ where, if } d > 0,\ \#\Gamma_h'' \neq 1 \text{ for all such } h\}. \quad (6.2)$$

For $d \in \mathbf{D}_S$ with $d < 0$ consider the image of the set $\{z_h^+ : -S[h] = d, h \in \mathbb{Z}^3\}$ under the map $H \to \hat{\Gamma} \backslash H$. Denote this image by Λ_d^+ and note that $\# \Lambda_d^+ < \infty$. If $d > 0$ then for $d = -S[h]$, $\hat{\Gamma}_h$ is an infinite cyclic group of hyperbolic transformations. Since Γ_h is the stabilizer of $z_h^{\pm}$ in Γ, $\hat{\Gamma}_h$ is generated by a primitive hyperbolic transformation δ. Denote by C_h the unique primitive (clockwise), oriented closed geodesic in $\hat{\Gamma}/H$ contained in those formed by reducing the geodesic joining z_h^+ and z_h^- mod $\hat{\Gamma}_h$. C_h has length $|C_h| = \frac{1}{2}|\log N(\delta)|$ or $|\log N(\delta)|$, according as $-h = gh$ for some $g \in \Gamma'$ or not. Let $\Lambda_d^+ = \{C_h;\ -S[h] = d, h \in \mathbb{Z}^3\}$. Then $\# \Lambda_d^+ < \infty$ since $C_{h_1} = C_{h_2}$ iff $\pm h_1 = g h_2$ for some $g \in \Gamma'$.

Now let u be a Maass cusp form (of weight 0) on $\hat{\Gamma} \backslash H$ with $\Delta_0 u + \lambda' u = 0$ and $\lambda' = \frac{1}{4} + t^2$. Then $\langle u(\cdot), \theta(z, \cdot) \rangle$ converges absolutely so by Theorem 4 $f(z) = y^{3/4} \langle u(\cdot), \theta(z, \cdot) \rangle$ is a Maass cusp form of weight $1/2$ and discriminant D for $\Gamma_0(N)$ with eigenvalue $\lambda = \frac{1}{4}\left(\lambda' + \frac{3}{4}\right) = \frac{1}{4} + \left(\frac{t}{2}\right)^2$ (it is possibly zero). In this case Maass has determined the Fourier coefficients $\{\varrho(n)\}$. Actually, Maass only considers co-compact Γ, but since we assume that u is a cusp form his proof extends easily. See also [13]. Using the notation above we have

Theorem 6 (Maass [22]) *Let S have signature* (1,2). *For* $f(z) = y^{3/4} \langle u(\cdot), \theta(z, \cdot) \rangle$ *and* $d \in \mathbf{D}_S$, *the* d^{th} *Fourier coefficient of* $f(z)$ *in* (2.4) *is given by*

$$\varrho(d) = \frac{\pi^{-\operatorname{sgn}(d)/4}}{\sqrt{2}} |d|^{-3/4} M_u(d)$$

where

$$M_u(d)=\begin{cases}\sum'_{z\in\Lambda_d^+} u(z), & d<0\\ \sum_{C\in\Lambda_d^+}\int_C u(z)ds, & d>0\end{cases}$$

and the prime indicates that $u(z)$ *is divided by the order of the stabilizer of* z *in* $\hat{\Gamma}$.

We now proceed to prove Theorem 1 in the Introduction. Let

$$S=\begin{bmatrix}0&0&2\\0&-1&0\\2&0&0\end{bmatrix} \quad\text{and}\quad \Gamma=\mathrm{SL}_2(\mathbb{Z})\ .$$

Then $D=-4$, $N=4$, and $\mathbf{D}_S$ consists of the usual ring discriminants. By Siegel's (ineffective) estimate

$$\#\Lambda_d \underset{\varepsilon}{\gg} |d|^{1/2-\varepsilon} \tag{6.3}$$

as $d\to-\infty$ and

$$\sum_{C\in\Lambda_d}|C| \underset{\varepsilon}{\gg} |d|^{1/2-\varepsilon} \tag{6.4}$$

as $d\to+\infty$.

By the spectral decomposition of $L^2(\hat{\Gamma}\backslash H)$ (see [12]) the "Weyl sums" in this setting are of two types:

$$W_{\mathrm{Eis}}(d,t)=\begin{cases}\dfrac{1}{\#\Lambda_d}\sum_{z\in\Lambda_d} E(z,\tfrac12+it)\ , & d<0\\ \dfrac{1}{\Sigma|C|}\sum_{C\in\Lambda_d}\int_C E(z,\tfrac12+it)ds\ , & d>0\end{cases}$$

and

$$W_{\mathrm{cusp}}(d,t)=\begin{cases}\dfrac{1}{\#\Lambda_d}\sum_{z\in\Lambda_d} u(z)\ , & d<0\\ \dfrac{1}{\Sigma|C|}\sum_{C\in\Lambda_d}\int_C u(z)ds\ , & d>0\,.\end{cases}$$

Here $E(z,s)=E_\infty(z,s;1,0,1)$. By (6.3), (6.4), and classical formulas of Dirichlet and Hecke (see [45]) we have for d a fundamental discriminant

$$W_{\mathrm{Eis}}(d,t) \underset{\varepsilon}{\ll} |t|^A|d|^{-1/4+\varepsilon}L(\tfrac12+it,\chi_d)$$

as $|d|\to\infty$ where $A>0$ is constant (which may vary in different expressions). By Theorem 5 and [7] we get as in (1.5)

$$W_{\mathrm{Eis}}(d,t) \underset{\varepsilon}{\ll} |t|^A|d|^{-1/28+\varepsilon}. \tag{6.5}$$

Combining Theorems 4, 5, and 6 with (6.3) and (6.4) we have for fundamental d

$$W_{\text{cusp}}(d, t) \underset{\varepsilon}{\ll} |t|^A |d|^{-1/28+\varepsilon}. \tag{6.6}$$

Theorem 1 now follows in a standard way from (6.5), (6.6) and the spectral decomposition theorem.

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