ON A METHOD OF HURWITZ AND ITS APPLICATIONS

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Abstract. We give class number formulas for binary cubic and n-ary quadratic forms using a method of Hurwitz. We also show how the same method can be applied to give identities for certain multiple zeta values attached to symmetric cones.

1. Introduction

In two relatively unknown papers, the second of which was published posthumously, Hurwitz developed a method for finding various infinite series representations for class numbers of positive definite integral binary [11] and ternary quadratic forms [12]. In a recent paper [6] we gave corresponding results for indefinite binary quadratic forms using automorphic spectral theory.

That method also yielded a new proof of (the non-Gaussian version of) Hurwitz's formulas in the positive definite binary case. In this paper, we will first recall these formulas and apply them to get class number formulas for certain binary cubic forms. Then we will generalize the Hurwitz formulas to positive definite n-ary forms for all $n \geq 2$ and illustrate how they can be applied to give certain identities relating multiple zeta values for cones to values of the Riemann zeta function. These identities are somewhat similar to the classical sum formula for ordinary multiple zeta values. Another point of the paper is to clarify a basic idea behind the general Hurwitz method, namely the use of an equivariant partition of unity.

Recall that $\Gamma_2 = \mathrm{SL}(2, \mathbb{Z})$ acts on the set of binary quadratic forms of discriminant $d < 0$

$$\mathcal{Q}_d = \{Q(x, y) = (a, b, c) = ax^2 + bxy + cy^2 \mid a, b, c \in \mathbb{Z}, b^2 - 4ac = d\}$$

with finitely many orbits and finite stabilizer subgroup $\Gamma(Q)$. For $d < 0$ the Hurwitz class number is defined as

$$H(d) := \sum_{Q} \tfrac{2}{\#\Gamma_2(Q)} \tag{1}$$

where Q runs over a complete set of Γ_2-representatives of forms in $\mathcal{Q}_d$. The following variant of the first Hurwitz identity for the Hurwitz class number is from [6].

Theorem 1. *For $d < 0$*

$$H(d) = \tfrac{1}{12\pi} |d|^{\frac{3}{2}} \sum_{\substack{a>0 \\ b^2-4ac=d}} \frac{1}{a(a + b + c)c}, \tag{2}$$

Where each term in the infinite sum is positive and the sum converges.

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The convergence of this series is slow but the method yields infinitely many such formulas which, while getting progressively more complicated, have faster convergence. The next one after (2), which is proven in [6], is

$$H(d) = \tfrac{1}{24\pi}|d|^{\frac{5}{2}} \sum_{\substack{a>0 \\ b^2-4ac=d}} \frac{1}{a^2(a+b+c)c^2}. \tag{3}$$

Let $f(x,y)$ be a binary cubic form

$$f(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \tag{4}$$

where a, b, c, d are integers. Two such forms are equivalent if one can be transformed to the other in the usual way by $\gamma \in \Gamma_2$. The discriminant of the form f is

$$\Delta = \Delta(a,b,c,d) = a^2d^2 - 6abcd - 3b^2c^2 + 4ac^3 + 4db^3,$$

which has the same value for equivalent forms. The set of all forms with a fixed non-zero discriminant Δ splits into a finite number of equivalence classes (see e.g. [15, p. 214.]). Define $h_3(\Delta)$ to be the number of $\mathrm{SL}(2,\mathbb{Z})$ equivalence classes of integral binary cubic forms of discriminant Δ where the contribution of each class is divided by $\#\Gamma(f)$ where $\Gamma(f) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma f = f\}$, the order of the isotropy group. Together with some basic invariant theory, Theorem 1 yields formulas for $h_3(\Delta)$ when $\Delta < 0$.

The simplest is the following

Theorem 2. *For $\Delta < 0$ we have that*

$$h_3(\Delta) = \tfrac{|\Delta|^{\frac{3}{2}}}{24\pi} \sum_{\Delta(a,b,c,d)=\Delta} \frac{1}{(b^2-ac)\left((b+c)^2-(a+b)(c+d)\right)(c^2-bd)}. \tag{5}$$

Each term in this sum is positive and the sum is convergent.

The next in the series of such formulas is

$$h_3(\Delta) = \tfrac{|\Delta|^{\frac{5}{2}}}{48\pi} \sum_{\Delta(a,b,c,d)=\Delta} \frac{1}{(b^2-ac)^2\left((b+c)^2-(a+b)(c+d)\right)(c^2-bd)^2}. \tag{6}$$

The method can also be applied to give the class number of certain other binary forms of odd degree with given invariants, for instance binary quintic forms, although the formulas become quite complicated.

Turning to higher dimensional generalizations for integral n with $n > 1$, let $Q(x) = xQx^t$, where Q is an $n \times n$ positive definite integral matrix with determinant D and $x = (x_1, \ldots, x_n)$. The group $\Gamma_n = \mathrm{SL}(n,\mathbb{Z})$ splits the set of all positive definite matrices into classes when $\gamma \in \Gamma_n$ acts on Q by $Q \mapsto \gamma^t Q \gamma$. The (generalized) Hurwitz class number is given by

$$H_n(D) = \sum_Q \frac{\epsilon_n}{\#\Gamma_n(Q)}$$

where $\Gamma_n(Q) = \{\gamma \in \Gamma_n \mid \gamma^t Q \gamma = Q\}$,

$$\epsilon_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{7}$$

and Q runs over a complete set of representatives of forms with $\det Q = D$. The "2" arises since we are using SL instead of PSL. The following is a uniform statement giving the simplest cases of the original Hurwitz formulas.

Theorem 3 (Hurwitz)**.** *For any $D > 0$ and $n = 2, 3$ we have*

$$H_n(D) = c_n D^{\frac{n+1}{2}} \epsilon_n \sum_{\substack{Q>0 \\ det\, Q = D}} \prod_{1\le k\le n} \tfrac{1}{Q(e_k)} \prod_{1\le k<m\le n} \tfrac{1}{Q(e_k - e_m)} \tag{8}$$

with e_k the standard basis vectors for $\mathbb{R}^n$ and

$$c_n^{-1} = \pi^{\frac{n^2-n}{4}} \Gamma(n+2) \prod_{k=0}^{n-1} \Gamma\left(1 + \tfrac{k}{2}\right).$$

We will show in Theorem 8 that (8) continues to hold for $n > 3$ provided we add more terms of the same general shape to the RHS. The terms all arise from a polyhedral decomposition of the cone of positive symmetric matrices $\mathcal{P}_n$ into perfect forms, due to Voronoi [19]. The term in (8) comes from the perfect form A_n, defined in (24), which is the only perfect form when $n = 2, 3$. In general we need all perfect forms and we must use a decomposition of the associated polyhedra into simplices.

The Hurwitz-type class number formulas are well-suited for summation over the determinants D. Doing so leads to identities that are somewhat analogous to certain well-known sum formulas for classical multiple zeta values. The generalized multiple zeta values we encounter have the form

$$\zeta_C(s_1, \ldots, s_m) = \sum_{(n_1,\ldots,n_m)\in C\cap\mathbb{Z}^m} n_1^{-s_1} \cdots n_m^{-s_m} \tag{9}$$

where $C \subset \mathbb{R}^m$ is a certain cone, $s_1, \cdots s_m$ are certain positive integers.

The usual multiple zeta value in m variables $\zeta(s_1, \ldots, s_m)$ is given by (9) when

$$C = \{(x_1, \ldots x_m) \in \mathbb{R}^m : x_1 > x_2 > \cdots > x_m > 0\}$$

and $s_1, \ldots s_m$ are positive integers with $s_1 > 1$. Among the many identities connecting classical multiple zeta values with values of the ordinary Riemann zeta function is the sum formula

$$\sum_{\substack{s_1>1, s_2\ge 1,\ldots,s_m\ge 1 \\ s_1+\cdots+s_m = s}} \zeta(s_1, \ldots, s_m) = \zeta(s),$$

which was discovered in case $m = 2$ by Euler and proven in general in [9].

Identities of this type can be given for the multiple zeta values of (9) that arise from the general version of (8). Here we only consider the case when n is odd. We employ an identity of Ibukiyama and Saito ([13, 14]) for the zeta function whose coefficients are the general class numbers, and this identity is much simpler for odd n. The general identity is given in Theorem 9.

The case $n = 3$ is reasonably simple and completely explicit. For reasons that will become clear in Sections 3 and 4 this requires the use of both covariant and contravariant coordinates with respect to the basis $x^2, y^2, z^2, (x-y)^2, (y-z)^2, (x-z)^2$. Let C be defined by

$$C = \{(x_1, \ldots, x_6) \in (\mathbb{R}^+)^6 \mid \begin{pmatrix} x_1 & \frac{1}{2}(x_1+x_2-x_6) & \frac{1}{2}(x_1+x_3-x_5) \\ * & x_2 & \frac{1}{2}(x_2+x_3-x_4) \\ * & * & x_3 \end{pmatrix} \text{ is positive definite } \}. \tag{10}$$

Then for positive integers $s_1, \ldots, s_6$ with $s_1 + \cdots + s_6 > 6$ the series $\zeta_C(s_1, \ldots, s_6)$ from (9) converges. Consider the determinant

$$D = \det \begin{pmatrix} p_1+p_5+p_6 & -p_6 & -p_5 \\ -p_6 & p_2+p_4+p_6 & -p_4 \\ -p_5 & -p_4 & p_3+p_4+p_5 \end{pmatrix}$$

$$= p_1p_2p_3 + p_1p_2(p_4+p_5) + p_2p_3(p_5+p_6) + p_1p_3(p_6+p_4) + (p_1+p_2+p_3)(p_4p_5+p_5p_6+p_4p_6),$$

which is called a unisignant since the coefficient of each term is positive. For s a non-negative integer define the integers $\left\{ {s_1,\ldots,s_6 \atop 3s} \right\}$ through

$$D^s = \sum_{s_1+\cdots+s_6=3s} \left\{ {s_1,\ldots,s_6 \atop 3s} \right\} \frac{p_1^{s_1}}{s_1!} \cdots \frac{p_6^{s_6}}{s_6!}. \tag{11}$$

We then have the following evaluation.

Theorem 4. *For $s \geq 1$*

$$\sum_{\substack{s_1+\cdots+s_6=3s \\ s_1,\ldots,s_6\geq 0}} \left\{ {s_1,\ldots,s_6 \atop 3s} \right\} \zeta_C(s_1+1,\ldots,s_6+1)$$
$$= \tfrac{3(s+1)!(2s+1)!}{2^{2s}} \big(2\zeta(2)\zeta(s+1)\zeta(2s+3) - \zeta(2)\zeta(s+2)\zeta(2s+2)\big).$$

Note that $\zeta_C(s_1,\ldots,s_6)$ is invariant under any permutation of (s_1,s_2,s_3) provided that the same permutation is performed on (s_4,s_5,s_6).Futhermore we can exchange any two of (s_1,s_2,s_3) with the corresponding elements of (s_4,s_5,s_6). In particular, the case $s=1$ of the formula can be written as

$$\zeta_C(2,2,2,1,1,1) + 3\zeta_C(2,2,1,2,1,1) = \tfrac{9}{4}\big(2\zeta(2)\zeta(2)\zeta(5) - \zeta(2)\zeta(3)\zeta(4)\big) = 7.81059\ldots.$$

The rest of the paper is organized as follows. In the following section we introduce equivariant partitions of unity and give the proof of Theorem 2. Then we develop a general method for constructing such partitions of unity. When specialized to the case of symmetric positive definite matrices these lead to the Hurwitz class number formula of Theorem 3 and its generalizations. This requires the evaluation of a certain integral over the cone of positive definite matrices which is given in the appendix. Finally, the class number formulas are applied to evaluate certain sums of generalized multiple zeta values, in particular those of Theorem 4.

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2. Equivariant partitions of unity

A convenient way to prove Hurwitz-type formulas is to reduce them to the construction of equivariant partitions of unity.

Definition. Let Γ be a group acting on a space X and suppose that we have a function $\rho : X \to \mathbb{C}$ such that

$$\sum_{\gamma\in\Gamma} \rho(\gamma x) = 1, \ \forall x \in X.$$

We will say that ρ induces a Γ-equivariant partition of unity, (or simply an equivariant partition of unity if Γ is understood). When Γ is infinite we make the assumption that the series converges absolutely.

Their application to class number formulas is formulated in the following basic lemma.

Lemma 1. *Assume that Γ acts on X, and $Y \subset X$ is a finite union of h orbits*

$$Y = \bigcup_{j=1}^{h} \Gamma y_j.$$

Assume also that for y_j the stabilizers $\Gamma_j = \{\gamma \in \Gamma \mid \gamma y_j = y_j\}$ are all finite. If ρ induces an equivariant partition of unity then

$$\sum_{j=1}^{h} \tfrac{1}{|\Gamma_j|} = \sum_{y\in Y} \rho(y). \tag{12}$$

Proof.

$$\sum_{y\in Y} \rho(y) = \sum_{j=1}^{h} \sum_{y\in \Gamma y_j} \rho(y) = \sum_{j=1}^{h} \tfrac{1}{|\Gamma_j|} \sum_{\gamma\in\Gamma} \rho(\gamma y_j) = \sum_{j=1}^{h} \tfrac{1}{|\Gamma_j|}.$$

□

For example, Lemma 1 reduces the proof of (1) to showing that

$$\rho(Q) = \tfrac{1}{24\pi} |d|^{\frac{3}{2}} \frac{1}{a(a+b+c)c} \tag{13}$$

induces a Γ_2-equivariant partition of unity, which was done in [6]. Although we did not use this language, and proved this using point-pair invariants on the upper half-plane. This fact also follows easily from a small variation of the proof of Theorem 3 when $n = 2$ that we give below. Generalizing the method of [6] in order to evaluate $H_n(D)$ for $n > 2$ does not seem to be straightforward.

The method of equivariant partition of unity, with some alterations, applies nicely in the proof of Theorem 2. Let Γ, $Y \subset X$, $\rho : X \to \mathbb{C}$ be as above, and assume that we have another Γ-space Z, with an equivariant map $\phi : Z \to Y$ so that

$$\phi(\gamma z) = \gamma\phi(z)$$

for any $\gamma \in \Gamma$, $z \in Z$. Define now

$$G(z) = \rho(\phi(z)),$$

for which we have

$$\sum_{\gamma\in\Gamma} G(\gamma z) = \sum_{\gamma\in\Gamma} \rho(\gamma\phi(z)) = 1.$$

It follows that if Z is the disjoint union of Γz_j for a finite set of z_j, say $z_1, ..., z_m$, then

$$\sum_{z\in Z} G(z) = \sum_{j=1}^{m} \tfrac{1}{|\widetilde{\Gamma}_j|}$$

where $\widetilde{\Gamma}_j = \{\gamma \in \Gamma : \gamma z_j = z_j\}$. The finiteness of these stabilizers follow from the finiteness of stabilizers in Y and the equivariant property of the map ϕ.

Proof of Theorem 2. The proof of Theorem 2 combines the Hurwitz formula (2) with some basic invariant theory. The Hessian covariant of our form f from (4) is given by the positive definite binary quadratic form

$$Q(f) = Q(f; x, y) = -\tfrac{1}{36} \begin{vmatrix} f_{x,x}(x,y) & f_{x,y}(x,y) \\ f_{y,x}(x,y) & f_{y,y}(x,y) \end{vmatrix} = Ax^2 + Bxy + Cy^2,$$

where $A = b^2 - ac, B = bc - ad$, and $C = c^2 - bd$. This form has discriminant

$$D = B^2 - 4AC = \Delta(a, b, c, d).$$

It is a covariant in that

$$\gamma Q(f; x, y) = Q(\gamma f; x, y).$$

Equivalent cubic forms have equivalent Hessians but several inequivalent cubic forms may have the same Hessian. Let $\Gamma(f) = \{\gamma \in \Gamma_2 \mid \gamma f = f\}$. Since $\Gamma(f)$ is a subgroup of $\Gamma(Q(f))$ it is easy to see that

$$\#\Gamma(f) = \begin{cases} 1 & \text{if } Q(f) \text{ is not equivalent to } k(x^2+xy+y^2) \\ 3 & \text{if } Q(f) \text{ is equivalent to } k(x^2+xy+y^2). \end{cases}$$

We apply (2) to count Hessians and the proof of Theorem 2 is now a simple application of the above general set up with $\Gamma = \Gamma_2$, $Y = \mathcal{Q}_d$, $Z =$ the set of binary cubic forms of discriminant D and ϕ the map that takes a binary form f to its Hessian $Q(f)$. Finally, we have that the function

$$G(a,b,c,d) = \rho(Q_f) = \tfrac{1}{24\pi}\tfrac{|D|^{3/2}}{AC(A+B+C)}$$

induces an equivariant partition of unity.

□

The formula (6) follows similarly from (3).

For example, $h_3(3) = 1$. There are 3 classes represented by $f_1 = [0,1,1,0]$, $f_2 = [1,0,-1,1]$, $f_3 = [1,1,0,-1]$. The isotropy subgroup for each $f = f_1, f_2, f_3$ is generated by $\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and has order 3. The sum in (6) restricted to $|a|,|b|,|c|,|d| \le 20, 40$ yields $h_3(3) \sim 0.99841..., 0.998928....$

3. A general construction of partitions of unity

In this section we give a general method of constructing partitions of unity for symmetric cones.

3.1. **Symmetric cones.** We review some basic facts about the geometry of symmetric cones over the reals. A general reference is [7] (see also [8]).

Let V be an m dimensional real vector space. A subset X of V is a cone, if whenever $x \in X$ and $\lambda > 0$ we have that λx is also in X. Assume that V is equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$. The open dual of a cone X is

$$X^* = \{y \in V \mid \langle x, y\rangle > 0, \forall x \in \overline{X} \setminus \{0\}\},$$

where $\overline{X}$ is the closure of X.

A cone X is symmetric if it is self-dual, $X^* = X$, and if the group of linear automorphisms

$$G = \{g \in GL(V) \mid gX = X\} \tag{14}$$

acts transitively on X.

Since V has an inner product there is a well-defined volume form which fixes a normalization μ of the Lebesgue measure. We let G^0 be the connected component of the identity in G, and

$$G_1 = G^0 \cap SL(V).$$

It is also known ([7] pg. 6) that

$$\exists v_0 \in X \text{ such that } G_{v_0} = \{g \in G^0 \mid gv_0 = v_0\} = G^0 \cap O(V). \tag{15}$$

We then have the following easy lemma which we state for later reference.

Lemma 2. *If $v \in X$ we can find $g \in G_1$, $\lambda(v) \in \mathbb{R}^+$ such that $v = \lambda(v)gv_0$. Here $\lambda = \lambda(v)$ is uniquely determined.*

Proof. The existence of such λ, g follows easily from the transitivity of the automorphism group G. For the uniqueness, assume that $\lambda_1 g_1 v_0 = \lambda_2 g_2 v_0$. Then $(\lambda_1/\lambda_2)g_2^{-1}g_1 \in O(V)$ which implies that $\lambda_1 = \lambda_2$. □

3.2. **Polyhedral decompositions.** We will now assume that there is an arithmetic group Γ acting on the symmetric cone X and use a polyhedral decomposition of X to construct step functions which induce a Γ equivariant partition of unity, at least almost everywhere. This is based on [2] but adapted to our needs. (See also the monograph [3].)

To define such an arithmetic group assume that the real vector space V arises from a rational vector space V' via extensions of scalars, in such a way, that the inner product $\langle \cdot, \cdot \rangle$ is rational on V' and that the algebraic group G in (14) is defined by finitely many polynomials with rational coefficients.

Let $L \subset V'$ be a lattice, a discrete subgroup of V' of full rank and

$$\Gamma_L = \{\gamma \in G \mid gL = L\}. \tag{16}$$

Γ is called arithmetic if it is commensurable with Γ_L.

The other input here are polyhedral cones defined as

$$C(v_1, ..., v_t) = \{\sum_{j=1}^{t} \lambda_j v_j \mid \lambda_j > 0 \; \forall j\}.$$

We will assume that the generating set $\{v_1, ..., v_t\}$ is minimal in the sense that the cones $C(v_2, ..., v_t)$,....,$C(v_1, ..., v_{t-1})$ are all properly contained in $C(v_1, ..., v_t)$.

The simplest polyhedral cone is a simplicial one where the generators $v_1, ..., v_m$ form a basis of V. To distinguish them from general cones we will use the notation

$$S = S(v_1, ..., v_m) \tag{17}$$

for simplicial cones. For any cone C we let

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

be the indicator function of C.

The main result of [2] guarantees the existence of a simplicial decomposition of X and leads to the following construction of a non-smooth function which induces a partition of unity except on a set of measure 0.

Theorem 5. *Assume that X is a symmetric cone with a rational structure and with the action of an arithmetic group Γ as described above. There is a finite collection of simplicial cones $S_1 = S(v_{1,1}, ..., v_{m,1})$,..., $S_k = S(v_{1,k}, ..., v_{m,k})$ and integers $r_j = r(S_j)$ such that if we let*

$$w(x) = \sum_{j=1}^{k} \frac{1}{r_j} \chi_{S_j}(x)$$

then

$$\sum_{\gamma \in \Gamma} w(\gamma x) = 1. \tag{18}$$

for almost every $x \in X$. Here the integers $r_j = r(S_j) = \#\{\gamma \in \Gamma : \gamma S_j = S_j\}$

Proof. The theorem is merely a reformulation of the main results of [2]. Since it is generalizing Voronoi's contruction we briefly recall some of the steps so as to refer to it later in the special case of quadratic forms.

It will be convenient to use the notion of a neat group. A matrix is neat if its eigenvalues generate a torsion free subgroup of $\mathbb{C}^*$. A group Γ' is neat if all of its elements are neat. It is shown in [4] that any arithmetic group Γ contains a neat subgroup of finite index.

Let $v \in V'$. The half-line $C = \{\lambda v \mid \lambda > 0\}$ is called cuspidal, if there is $y \in \overline{X}$ such that

$$\{x \mid \langle x, y \rangle \geq 0\} \cap \overline{X} = C \cup \{0\}.$$

If C is a cuspidal half-line we take $v \in L \cap C$ of minimal length and call it a cusp.

Consider now the convex hull Π of all cusps. If F is a face of the boundary of Π then it is the convex hull of finitely many cusps, and so the rays through F form a polyhedral cone. It was shown in [2] that these polyhedral cones can be decomposed into be simplicial cones without introducing new rays. Furthermore if Γ' is neat the cones over the faces of Π are permuted by Γ' with finitely many orbits[1]. That is there is a finite set of simplical cones whose Γ'-translates are disjoint and cover X except on a co-dimension 1 set (coming from the faces). Some of these cones may be equivalent under the bigger group Γ and we select $S_1, ..., S_k$ a set of Γ-inequivalent representatives. We let $r_j = r(S_j) = \#\{\gamma \in \Gamma : \gamma S_j = S_j\}$ and then (18) holds.

□

3.3. **Smoothing the partition of unity via frustum integrals.** Theorem 5 gives a non-smooth partition of unity. In this section we will use integrals over frustums to construct smooth ones. Here by a frustum of a cone C we mean sets of the form

$$F_l(C) = \{x \in C \mid 0 < l(x) < 1\}$$

for some linear functional l. We will assume that l is positive on C, and that the frustum $F_l(C)$ is bounded. Then the integral

$$\int_{F_l(C)} f(x) d\mu \tag{19}$$

is defined for any homogeneous polynomial f, once a normalization μ of the Lebesgue measure is fixed.

Given a basis $v_1, ..., v_m$ of V we will use the system of coordinates $\mathbf{x} = (x_1, ..., x_m)$ such that for any $v \in V$ we have

$$v = \sum_{i=1}^{m} x_i(v) v_i \tag{20}$$

Then there is $\delta(v_1, ..., v_m) \in \mathbb{R}^+$ such that

$$d\mu = \delta(v_1, ..., v_m) d\mathbf{x}. \tag{21}$$

For a multi-index $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we let $\mathbf{x}^{\boldsymbol{\alpha}} = \prod x_i^{\alpha_i}$, $\boldsymbol{\alpha}! = \prod \alpha_i!$, and $|\boldsymbol{\alpha}| = \sum \alpha_i$. Then we have

Proposition 1. *Let $S = S(v_1, ..., v_m)$ be a simplicial cone. Assume that f is a homogeneous polynomial of degree d, which in the coordinate system associated to $v_1, ..., v_m$ is given by $f(v) = \sum_{|\boldsymbol{\alpha}|=d} C_{\boldsymbol{\alpha}}(f) \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}$. Then*

$$\int_{F_l(S)} f d\mu = \frac{\delta(v_1, ..., v_m)}{(m+d)!} \sum_{|\alpha|=d} C_{\boldsymbol{\alpha}}(f) \prod_{i=1}^{m} l(v_j)^{-\alpha_i - 1}.$$

[1]In fact there is more precise geometric description, these cones form a Γ'-admissible decomposition of X which roughly means that all of their lower dimensional faces match up nicely.

Proof. In the coordinate system $\mathbf{x}$

$$\int_{F_l(S)} f d\mu = \delta(v_1, ..., v_m) \int_{F_l(S)} f d\mathbf{x}$$

and the frustum $F_l(S)$ is described by

$$\{(x_1, ..., x_m) \mid x_j \geq 0, 0 < \sum_j l(v_i)x_i \leq 1\}.$$

Consider the map

$$(x_1, ..., x_m) \in \mathbb{R}^m \mapsto (y_1, ..., y_m) = (l(v_1)x_1, ..., l(v_m)x_m).$$

Then the change of variables to $\mathbf{y}$ gives

$$\int_{F_l(C)} f d\mathbf{y} = \sum_{|\boldsymbol{\alpha}|=d} C_{\boldsymbol{\alpha}}(f) \prod_j l(v_j)^{-\alpha_j - 1} \int_{\Delta} \mathbf{y}^{\boldsymbol{\alpha}} d\mathbf{y}$$

where $\Delta = \{(y_1, ..., y_m) \mid y_j \geq 0, \sum_j y_j \leq 1\}$ is the standard simplex. The integral is the well-known multivariate Beta-integral ([1]) with the evaluation $\frac{\boldsymbol{\alpha}!}{(|\boldsymbol{\alpha}|+m)!}$.

□

Given the inner product on V we define g^* for any $g \in GL(V)$ by

$$\langle v, gx \rangle = \langle g^* v, x \rangle,$$

and let

$$\Gamma^* = \{\gamma^* \mid \gamma \in \Gamma\}.$$

Then, since X is self-dual, Γ^* also acts on X and we have the following

Theorem 6. *Let* $\dim V = m$, $X \subset V$ *a symmetric cone, and* Γ *a discrete subgroup of* G *arising from a lattice as described above. Let* r_j *and* $S_j = S(v_{1,j}, ..., v_{m,j})$ *be simplicial cones as in Theorem 5 and* $\delta_j = \delta(v_{1,j}, ..., v_{m,j})$ *as in (21). Let* f *be a* G*-invariant homogeneous polynomial of degree* d *which is positive on* X*, and which in the coordinate system (20) associated to* $v_{1,j}, ..., v_{m,j}$ *is given by* $f(v) = \sum_{|\boldsymbol{\alpha}|=d} C_{\boldsymbol{\alpha},j}(f) \mathbf{x_j}^{\boldsymbol{\alpha}} / \boldsymbol{\alpha}!$.

Define $\rho : X \to \mathbb{C}$ *by*

$$\rho : v \mapsto \frac{1}{A} \frac{\lambda^{m+d}}{(m+d)!} \sum_{j=1}^{k} \frac{\delta_j}{r_j} \sum_{|\boldsymbol{\alpha}|=d} C_{\boldsymbol{\alpha},j}(f) \prod_{i=1}^{m} \langle v, v_{i,j} \rangle^{-\alpha_i - 1}. \tag{22}$$

Here

$$A = \int_{F_0} f(x) d\mu$$

with $F_0 = \{x \in X \mid \langle x, v_0 \rangle \leq 1\}$ *with* v_0 *as in (15) and where* $\lambda = \lambda(v)$ *is defined in Lemma 2.*

Then ρ *induces a smooth* Γ^**-equivariant partition of unity on the space* X.

For the proof we will need the following lemma.

Lemma 3. *Let* f *be* G*-invariant homogeneous polynomial of degree* d, $l(x) = \langle v, x \rangle$ *for some* $v \in X$ *and* $v_0 \in X$ *be as above. Then we have that*

$$\int_{F_l(X)} f(x) d\mu = (\lambda)^{-(m+d)} \int_{F_0} f(x) d\mu$$

where $F_0 = \{x \in X \mid \langle x, v_0 \rangle \leq 1\}$ *and where* $\lambda = \lambda(v)$ *is as in Lemma 2.*

Proof.

$$\int_{F_l(X)} f(x)d\mu = \int_{\substack{x\in X,\\ l(x)\le 1}} f(x)d\mu = \int_{\substack{x\in X,\\ \langle v,x\rangle\le 1}} f(x)d\mu.$$

We have that for some $g \in G_1$

$$\int_{\substack{x\in X,\\ \langle v,x\rangle\le 1}} f(x)d\mu = \int_{\substack{x\in X,\\ \lambda\langle gx,v_0\rangle\le 1}} f(x)d\mu.$$

Since $\det g = 1$ the change of variables $x = \lambda(v)^{-1}g^{-1}y$ then leads to the integral

$$\lambda^{-(m+d)} \int_{\substack{y\in X,\\ \langle y,v_0\rangle\le 1}} f(y)d\mu$$

proving the statement. □

The proof of Theorem 6. Fix $v \in X$ and let $l(x) = \langle v, x\rangle$. For simplicity we will denote the frustum $F_l(X)$ of X by $F_v(X)$. We will evaluate the frustum integral

$$\int_{F_v(X)} f(x)d\mu$$

in two ways. Using Theorem 5 we have that

$$\int_{F_v(X)} f(x)d\mu = \int_{F_v(X)} f(x) \sum_{\gamma\in\Gamma}\sum_{j=1}^{k} \frac{1}{r_j}\chi_{S_j}(\gamma x)d\mu = \sum_{\gamma\in\Gamma}\sum_{j=1}^{k} \frac{1}{r_j}\int_{F_v(\gamma S_j)} f(x)d\mu. \tag{23}$$

Note that $F_v(\gamma S_j) = \{x \in S_j \mid \langle v, \gamma x\rangle \le 1\} = \{x \in S_j \mid \langle \gamma^* v, x\rangle \le 1\} = F_{\gamma^* v}(S_j)$. Since $f(\gamma x) = f(x)$ this gives

$$\int_{F_v(\gamma S_j)} f(x)d\mu = \int_{F_{\gamma^* v}(S_j)} f(x)d\mu$$

Using that $l(v_{i,j}) = \langle v, v_{i,j}\rangle$, Proposition 1 gives

$$\int_{F_{\gamma^* v}(S_j)} f d\mu = \frac{\delta(v_{1,j}, \ldots, v_{m,j})}{(m+d)!} \sum_{|\alpha|=d} C_{\boldsymbol{\alpha}}(f) \prod_{i=1}^{m} \langle \gamma^* v, v_{i,j}\rangle^{-\alpha_i - 1}.$$

This leads to the identity

$$\int_{F_v(X)} f(x)d\mu = \frac{A}{\lambda^{m+d}} \sum_{\gamma^*\in\Gamma^*} \rho(\gamma^* v).$$

For the second evaluation of the frustum integral we use Lemma 3 above, which gives $\int_{F_v(X)} f(x)d\mu = A/\lambda^{m+d}$ and this finishes the proof. □

4. The case of $\mathcal{P}_n$

We now will elaborate on the particular case when the symmetric cone C is $\mathcal{P}_n \subset \mathbb{R}^m$ the space of positive definite symmetric $n \times n$ real matrices, with $m = n(n+1)/2$. Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Then it is known that there are polyhedral decompositions of $\mathcal{P}_n$ for the action of Γ on $\mathcal{P}_n$. For concreteness we will use Voronoi's first reduction [19, 16] which uses perfect forms to give a simplicial decomposition of $\mathcal{P}_n$ as in Theorem 5, but other alternatives are just as suitable. A perfect form or equivalently a perfect lattice in the Euclidean space is a lattice which is completely determined by its shortest vectors. An example of a perfect lattice is given by

$$A_n = \{(x_0, x_1, ..., x_n) \in \mathbb{Z}^{n+1} \mid x_0 + x_1 + ... + x_n = 0\}. \tag{24}$$

Let $P_1, ..., P_r$ represent the Γ-equivalence classes of perfect forms, where we assume that P_1 is of type A_n. Let P be one of these forms with shortest vectors $v_1, ..., v_\ell$. Then the rank 1 symmetric matrices $Q_1 = v_1 v_1^t, ..., Q_\ell = v_\ell v_\ell^t$ span a cone $C_P = C_P(Q_1 \dots Q_\ell)$ in $\mathcal{P}_n$ and the Γ-translates of the union of these cones covers $\mathcal{P}_n$, where each cone has some multiplicity r_i as in Theorem 5. Writing each cone C_P as a finite union of simplicial cones (see [16]) leads to the decomposition

$$\mathcal{P}_n = \bigcup_{\gamma \in \Gamma} \bigcup_i \gamma S_i \tag{25}$$

where

$$S_i = S(Q_{1,i}, ..., Q_{m,i}) \tag{26}$$

are simplicial cones and by the results of [2] we may assume that each $Q_{j,i}$ is of the form $u_{j,i} u_{j,i}^t$ where $u_{j,i}$ is one of the shortest vectors of some perfect form P_i.

In this case

$$\delta_i = \delta_i(Q_{1,i}, \dots Q_{n(n+1)/2,i}) = \frac{1}{\sqrt{\det((u_{j,i}^t u_{k,i})^2)_{j,k=1}^{n(n+1)/2}}}. \tag{27}$$

and

$$\operatorname{tr} Q Q_{j,i} = \operatorname{tr}(Q u_{j,i} u_{j,i}^t) = Q(u_{j,i}). \tag{28}$$

We will make the choice that

$$f(Q) = \det{}^s(Q), \tag{29}$$

for $s = 0, 1, 2, ...$, with degree of homogeneity $d = ns$. If the simplicial cones are S_i as in (26) then for each i we can express

$$\det{}^s Q = \sum_{|\boldsymbol{\alpha}| = ns} C_{\boldsymbol{\alpha},i}(s) \frac{\mathbf{x}_i^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \tag{30}$$

where $\mathbf{x}_i^\alpha = x_{1,i}^{\alpha_1} \cdots x_{m,i}^{\alpha_m}$ the linear functions $x_{j,i} = x_{j,i}(Q)$ are defined by $Q = \sum_{j=1}^m x_{j,i} Q_{j,i}$.

We then have the following reformulation of Theorem 6.

Theorem 7. *Let f be as in (29), S_i, δ_i and $C_{\boldsymbol{\alpha},i}(s)$ be as in (25), (27) and (30) respectively and define $\rho : \mathcal{P}_n \mapsto \mathbb{C}$ by*

$$\rho_s(Q) = B \sum_{i=1}^k \frac{\delta_i}{r_i} \sum_{|\boldsymbol{\alpha}| = ns} C_{\boldsymbol{\alpha},i} \prod_j (\operatorname{tr} Q Q_{j,i})^{-\alpha_j - 1}$$

where r_i is as in Theorem 5,

$$B = \frac{\det^{s+(n+1)/2}(Q)}{C_n(s)(ns + n(n+1)/2)!} \tag{31}$$

with

$$C_n(s) = \frac{1}{2^{n(n-1)/4}} \int_{\substack{Q \in \mathcal{P}_n \\ \operatorname{tr} Q \leq 1}} \det{}^s(Q) \prod_{1 \leq i \leq j \leq n} dq_{ij}, \tag{32}$$

Then ρ induces a partition of unity for $SL_n(\mathbb{Z})$ acting on $\mathcal{P}_n$.

Proof. This is an easy corollary of the general setup for the symmetric cone $\mathcal{P}_n$ where the inner product is normalized so that $\langle Q_1, Q_2\rangle = \operatorname{tr} Q_1Q_2$. The constant λ in Theorem 6 is easily evaluated to be $\lambda = \det Q^{1/n}$. □

We will also need the following evaluation of the constant $C_n(s)$ in Theorem 7 whose proof is postponed to the Appendix.

Proposition 2. *Let $C_n(s)$ be as in (32). Then*

$$C_n(s) = \frac{(\pi/2)^{(n^2-n)/4}}{\Gamma(1+ns+n(n+1)/2)} \prod_{k=0}^{n-1} \Gamma\left(s+1+k/2\right).$$

We finish this section by applying Theorem 7 and Proposition 2 with $s = 0$ to give class number formulas for positive definite n-ary quadratic forms generalizing the theorems of Hurwitz for $n = 2, 3$.

Theorem 8. *With the above setup, for each $D > 0$ we have the convergent identity*

$$H_n(D) = \frac{D^{\frac{n+1}{2}} 2^{n(n-1)/4}\epsilon_n}{\pi^{(n^2-n)/4}\prod_{k=0}^{n-1}\Gamma\left(1+k/2\right)} \sum_{\substack{Q>0\\ \det Q=D}} \sum_{i=1}^{k} \frac{\delta_i}{r_i} \prod_{j=1}^{n(n+1)/2} \frac{1}{Q(u_{j,i})} \tag{33}$$

where ϵ_n is 2 if n is even and 1 if n is odd.

If we use the decomposition (25) in terms of simplicial cones S_i with S_1 corrresponding to the perfect lattice A_n then we can rewrite (33) as

$$H_n(D) = c_n D^{\frac{n+1}{2}}\epsilon_n \sum_{\substack{Q>0\\ \det Q=D}} \prod_{k=1}^{n} \frac{1}{Q(e_k)} \prod_{1\le k<\ell\le n} \frac{1}{Q(e_k-e_\ell)} + \ldots \tag{34}$$

with e_k the standard basis vectors and $c_n = 1/\pi^{(n^2-n)/4}\Gamma(n+2)\prod_{k=0}^{n-1}\Gamma\left(1+k/2\right)$. The distinguished term that is separated arises from the perfect lattice A_n. The cases $n = 2, 3$ are treatable in more explicit form because in those dimensions this is the only perfect lattice leading to Theorem 3.

5. Application to the multiple zeta functions attached to cones

In this section we will show how the results of previous sections can be applied to give special values of multiple zeta functions attached to cones. To this end we start with the following definition.

Definition. Let V be a real vector space of dimension m, $C \subset V$ a cone, and $L \subset V$ a lattice. Assume also that $\mathcal{B}^* = \{l_1, ..., l_m\}$ form a basis of V^* such that the linear forms l_j are positive on C and integral on L for $j = 1, ..., m$. We then define the multiple zeta function associated to C and L as

$$\zeta_{C,L;\mathcal{B}^*}(s_1, ..., s_m) = \sum_{v\in C\cap L} \frac{1}{l_1(v)^{s_1}\cdots l_m(v)^{s_m}}. \tag{35}$$

When the lattice and the basis is well understood we will simply write ζ_C. This definition leads to the multiple zeta values in (9) from the introduction in the case $L = \mathbb{Z}^m$ with $l_j(n_1, ..., n_m) = n_j$.

Remark. Note that if V is equipped with an inner product $\langle \cdot, \cdot \rangle$ that is integral on L we can reformulate this definition as follows. Assume that $v_1, ..., v_m \in L$ form a basis of V. Let $v_1^*, \cdots v_m^*$ be a dual basis in the sense that $\langle v_j^*, v_k \rangle = \delta_{j,k}$ and let $l_j(v) = \langle v_j^*, v \rangle$. Assume that l_j-s are positive on C. Then the multiple zeta function associated to the above data written in these covariant coordinates is

$$\zeta_{C,L}(s_1, ..., s_m) = \sum \frac{1}{n_1^{s_1} \cdots n_m^{s_m}} \tag{36}$$

where the sum is over $\{(n_1, ..., n_m) \in \mathbb{N}^m \mid n_1 v_1^* + ... n_m v_m^* \in C \cap L\}$.

The next theorem is an application of Theorem 7 to the zeta-functions of the type above. More precisely $C = \mathcal{P}_n$ be the cone of positive definite symmetric matrices and L be the lattice of integral symmetric matrices. For $S_i = S_i(v_{1,i}, ..., v_{m,i})$ one of the simplicial cones in Theorem 7. Note that the basis vectors $\{v_{1,i}, ..., v_{m,i}\}$ are in L. We now consider the multiple zeta-function

$$\zeta_i(s_1, ..., s_m) = \zeta_{C,L;\mathcal{B}_i^*}(s_1, ..., s_m)$$

associated to the dual basis $\mathcal{B}_i^*$ formed from $v_{j,i}^*$. Then we have the following application of Theorem 7 which includes a generalization of Theorem 8.

Theorem 9.

$$\sum_{D=1}^{\infty} \frac{H_n(D)}{D^{s+(n+1)/2}} = \frac{\epsilon_n}{\left(\frac{\pi}{2}\right)^{\frac{n^2-n}{4}} \prod_{\ell=0}^{n-1} \Gamma\left(1+\frac{\ell}{2}\right)} \sum_{i=1}^{k} \sum_{|\boldsymbol{\alpha}|=ns} \widetilde{C}_{\boldsymbol{\alpha},i}\, \zeta_i(s_1+1, ..., s_{n(n+1)/2}+1) \tag{37}$$

where $\widetilde{C}_{\boldsymbol{\alpha},i} = \frac{r_j}{\delta_j} C_{\boldsymbol{\alpha},i}$ *with* $C_{\boldsymbol{\alpha},i}$ *coming from (30).*

Proof. We apply Theorem 7 with $f(Q) = |\det Q|^s = D^s$. As in that theorem we let $v_{j,i} = Q_{j,i}$. If now $Q = \sum_{j=1}^m n_i Q_{j,i}^*$ then

$$\prod_{j=1}^{m} (\operatorname{tr} QQ_{j,i})^{-\alpha_i-1} = \prod_{j=1}^{m} n_j^{-\alpha_i-1}.$$

Note that by (28) all of the $n_j = \operatorname{tr} QQ_{j,i} \in \mathbb{Z}^+$. Theorem 7 now gives the following generalization of Theorem 8.

$$\frac{H_n(D)}{D^{s+(n+1)/2}} = \frac{2^{n(n-1)/4}\epsilon_n}{\pi^{(n^2-n)/4} \prod_{\ell=0}^{n-1} \Gamma(1+\ell/2)} \sum_{\substack{Q=\sum n_j Q_{j,i}^* > 0 \\ det(Q)=D}} \sum_{i=1}^{k} \frac{\delta_i}{r_i} \sum_{|\boldsymbol{\alpha}|=ns} C_{\boldsymbol{\alpha},j} \prod_{j=1}^{m} n_j^{-\alpha_i-1}$$

After summing over all D we are lead to $\zeta_{C,i}$ as in (36). This proves the theorem. $\square$

In the case of ternary forms, Theorem 9 can be now used to prove Theorem 4.

The Proof of Theorem 4. To this end we start with the following corollary of Theorem 9. Let V be the 6 dimension vector space of real ternary quadratic forms which we identify with 3×3 symmetric matrices. Inside we have the cone $\mathcal{C} = \mathcal{P}_3$. If we fix the basis

$$v_1 = x^2, v_2 = y^2, v_3 = z^2, v_4 = (x-y)^2, v_5 = (y-z)^2, v_6 = (x-z)^2 \tag{38}$$

then we have the following corollary.

Corollary 1.

$$\frac{12\pi^2(s+1)!(2s+1)!}{2^{2s}}\sum_D \frac{H_3(D)}{D^{s+2}} = \sum_{\substack{s_1+\cdots+s_6=3s\\ s_1,\ldots,s_6\geq 0}} \left\{ {s_1,\ldots,s_6 \atop 3s} \right\} \zeta_{\mathcal{C}}(s_1+1,\ldots,s_6+1) \tag{39}$$

where $\left\{ {s_1,\ldots,s_6 \atop 3s} \right\}$ *defined by (11) and*

$$\zeta_{\mathcal{C}}(s_1,\ldots,s_6) = \sum_{(n_1,\ldots,n_6)} n_1^{-s_1}\cdots n_6^{-s_6}$$

with $n_1,...,n_6$ *satisfying* $\sum n_j v_j^* \in \mathcal{C}$ *with* v_j^* *the dual basis to (38).*

Proof. In the dual basis $v_1^*,...,v_6^*$, the cone $\mathcal{C}$ can be written as

$$\mathcal{C} = \{(x_1,\ldots,x_6)\in\ \mathbb{R}^6 \mid \begin{pmatrix} x_1 & \frac12(x_1+x_2-x_6) & \frac12(x_1+x_3-x_5)\\ * & x_2 & \frac12(x_2+x_3-x_4)\\ * & * & x_3\end{pmatrix} \text{ is positive definite}\}.$$

In these coordinates the summation in $\zeta_{\mathcal{C}}(s_1,...,s_6)=\sum 1/(n_1^{s_1}\cdots n_6^{s_6})$ is over

$$\{(n_1,\ldots,n_6)\in(\mathbb{Z}^+)^6 \mid \begin{pmatrix} n_1 & \frac12(n_1+n_2-n_6) & \frac12(n_1+n_3-n_5)\\ * & n_2 & \frac12(n_2+n_3-n_4)\\ * & * & n_3\end{pmatrix} \text{ is positive definite integral}\}. \tag{40}$$

The results of Selling in [17] (see also [5]) imply that the $SL_3(\mathbb{Z})$ translates of the simplicial cone $S(v_1,...,v_6)$ cover $\mathcal{P}_3$ evenly 24 times. Therefore there is only one term in Theorem 9. To evaluate it let

$$\begin{aligned} Q(x,y,z) &= ax^2+by^2+cz^2+2Cxy+2Bxz+2Ayz\\ &= p_1x^2+p_2y^2+p_3z^2+p_4(y-z)^2+p_5(z-x)^2+p_6(x-y)^2 \end{aligned}$$

so that

$$\begin{vmatrix} a & C & B\\ C & b & A\\ B & A & c\end{vmatrix}^s = \begin{vmatrix} p_1+p_5+p_6, & -p_6, & -p_5\\ -p_6, & p_2+p_4+p_6, & -p_4\\ -p_5, & -p_4, & p_3+p_4+p_5\end{vmatrix}^s. \tag{41}$$

From (41) it is clear that, for $s\in\mathbb{Z}$ there are $s_1,\ldots,s_6\in\mathbb{Z}$ and coefficients $\left\{ {s_1,\ldots,s_6 \atop 3s} \right\}$ such that

$$D^s = \sum_{s_1+\cdots+s_6=3s} \left\{ {s_1,\ldots,s_6 \atop 3s} \right\} \frac{p_1^{s_1}}{s_1!}\cdots\frac{p_6^{s_6}}{s_6!}$$

as in (11). Note that for a multiindex $\alpha=(\alpha_1,...,\alpha_6)$ the coefficients $C(\alpha)$ in (30) used in Theorem 7 satisfy

$$C(s_1,...,s_6) = \left\{ {s_1,\ldots,s_6 \atop 3s} \right\}.$$

Then Theorem 7 implies

$$\frac{(s+1)!(2s+1)!}{2^{2s}}\frac{12\pi^2}{D^{s+2}}\frac{1}{\mu} = \tag{42}$$

$$\sum_{\substack{s_1+\cdots+s_6=3s\\ s_1,\ldots,s_6\geq 0}} \left\{ {s_1,\ldots,s_6 \atop 3s} \right\} \sum_{[Q]} \frac{1}{a^{s_1}b^{s_2}c^{s_3}(b+c-2A)^{s_4}(c+a-2B)^{s_5}(a+b-2C)^{s_6}}$$

here $\mu=|\mathrm{SL}(3,\mathbb{Z})_Q|$ and the sum is over quadratic forms in the class of Q.

Next we sum (42) over all classes and over all D and use the coordinates from (40) to obtain the corollary. □

To prove Theorem 4 we also need a result of Ibukiyama and Saito ([13],[14]) which writes the prehomegeneous vector space zeta functions associated to the $n-$ ary quadratic forms in terms of simple zeta functions. More precisely, let L denote the set of $n \times n$ positive definite symmetric integral matrices with the usual action of $\mathrm{SL}(n, \mathbb{Z})$ and for $x \in L$, let $\mu(x)$ denote the order of the automorphs of x. The pre-homogeneous zeta function associated to L is then

$$\zeta(s, L) := \sum_{L/\sim, x>0} \frac{1}{\mu(x)\det(x)^s} = \sum_{D=1}^{\infty} \frac{1}{D^s} \sum_{x \in L/\sim, \det(x)=D} \frac{1}{\mu(x)} = \sum_{D=1}^{\infty} \frac{H(D)}{D^s} \tag{43}$$

For $n \in \mathbb{N}$ odd, we have the following result of Ibukiyama and Saito.

Theorem 10 (Ibukiyama-Saito). *For n an odd positive integer we have*

$$\begin{aligned} \zeta(s, L) = \frac{\left|\prod_{i=1}^{[n/2]} B_{2i}\right|}{2^{n-1}\left(\frac{n-1}{2}\right)!} \Bigg(& 2^{(n-1)/2} \zeta\left(s - \frac{n-1}{2}\right) \prod_{i=1}^{[n/2]} \zeta(2s - (2i-1)) \\ & + (-1)^{(n^2-1)/8} \zeta(s) \prod_{i=1}^{[n/2]} \zeta(2s - 2i) \Bigg). \end{aligned} \tag{44}$$

Here the Bernoulli numbers B_{2k} are defined by the generating series

$$\frac{x}{e^x - 1} = \sum_{n=1}^{\infty} \frac{B_n x^n}{n!}, \quad \text{and satisfy} \quad B_{2k} = \frac{(-1)^{k+1} 2 (2k)!}{(2\pi)^{2k}} \zeta(2k).$$

Using Theorem 10 with $n = 3$, (43), Corollary 1 together with the identity $B_2 = \zeta(2)/\pi^2$ now gives

$$\begin{aligned} & \sum_{\substack{s_1 + \cdots + s_6 = 3s \\ s_1, \ldots, s_6 \geq 0}} \left\{ {s_1, \ldots, s_6 \atop 3s} \right\} \zeta_C(s_1 + 1, \ldots, s_6 + 1) \\ & \qquad = \tfrac{3(s+1)!(2s+1)!}{2^{2s}} \big(2\zeta(2)\zeta(s+1)\zeta(2s+3) - \zeta(2)\zeta(s+2)\zeta(2s+2) \big) \end{aligned}$$

which in return proves Theorem 4.

Appendix A. The evaluation of $C_n(s)$.

In this Appendix we evaluate the integral that defines the constant $C_n(s)$ in (32).
For this we use the following Lemma.

Lemma 4. *Let $C \in \mathbb{R}^n$ be a cone. Assume that $f(x)$ is a continuous function which is homogeneous of degree d: $f(\lambda x) = \lambda^d f(x)$. Let $\|x\|_1 = \sum |x_i|$. For any $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_0^\infty g(\lambda) \lambda^{d+n-1} d\lambda < \infty$ we have*

$$\int_{\mathbf{x} \in C, \|x\|_1 \leq 1} f(x) d\mu(x) = \frac{\int_C g(\|x\|_1) f(x) d\mu(x)}{(d+n) \int_0^\infty g(\lambda) \lambda^{d+n-1} d\lambda}.$$

Proof. Let $\Sigma = \{x \in \mathbb{R}^n \mid \|x\|_1 = 1\}$ with the surface measure $d\sigma$. We have for any cone C and integrable function f

$$\int_C f(x)d\mu(x) = \int_{\mathbb{R}^+} \int_{\Sigma\cap C} f(\lambda p)\lambda^{n-1} d\sigma(p)d\lambda. \tag{45}$$

We have that $f(\lambda p) = \lambda^d f(p)$ and so

$$\int_{x\in C, \|x\|_1\le 1} f(x)d\mu(x) = \int_0^1 \lambda^{d+n-1}d\lambda \int_{\Sigma\cap C} f(p)d\sigma(p) = \frac{1}{d+n}\int_{\Sigma\cap C} f(p)d\sigma(p).$$

On the other hand consider the function

$$x \mapsto g(\|x\|_1)f(x),$$

which has integral

$$\int_C g(\|x\|_1)f(x)d\mu(x) = \int_0^\infty g(\lambda)\lambda^{d+n-1}d\lambda \int_{\Sigma\cap C} f(p)d\sigma(p)$$

□

As an example let $C = (\mathbb{R}^+)^n$, and $f(x) = \mathbf{x}^\alpha$. This leads to the Beta-integral

$$\int_{\substack{x_1,\dots,x_n>0,\\ x_1+\cdots+x_n<1}} \mathbf{x}^{\boldsymbol{\alpha}} d\mathbf{x}$$

If we now choose $g(\lambda) = e^{-\lambda}$ then

$$\int_{(\mathbb{R}^+)^n} e^{-x_1}x_1^{\alpha_1}\dots e^{-x_n}x_n^{\alpha_n}d\mu(x) = \Gamma(\alpha_1+1)\cdots\Gamma(\alpha_n+1)$$

leading to the evaluation of the multivariate Beta-integral [1]

$$\int_{\substack{x_1,\dots,x_n>0,\\ x_1+\cdots+x_n<1}} \mathbf{x}^{\boldsymbol{\alpha}} d\mathbf{x} = \frac{\prod_{i=1}^n \Gamma(1+\alpha_i)}{\Gamma(1+n+\sum_{i=1}^n \alpha_i)}$$

used in the proof of Proposition 1.

Remark. Lemma 4 is based on an adaptation of polar and spherical coordinates to other norms and in a technical sense is at the core of some of Hurwitz's explicit evaluations of his "projective integrals" in the case $n = 2, 3$ from [11, 12]. More generally assume that a norm $\|x\|$ is defined on an n-dimensional real vector space V and let $B = \{x \in V \mid \|x\| \le 1\}$ be its unit ball. Then B is convex, centrally symmetric and absorbing in the sense that $\cup_{\lambda>0}\lambda B = V$. Let $\Sigma = \{p \in V \mid \|p\| = 1\}$ be the boundary of B, which is the unit "sphere" in the norm $\|\cdot\|$.

If on V we fix a choice of Lebesgue measure $d\mu$ then the $(n-1)$-dimensional Minkowksi content defines a surface measure σ on Σ such that

$$(V \setminus \{0\}, d\mu) \simeq (\mathbb{R}^+, \lambda^{n-1}d\lambda) \times (\Sigma, d\sigma)$$

as measure spaces via the map $\lambda = \|x\|$, $p = x/\|x\|$. With this understanding of $(\Sigma, d\sigma)$ we still have that for any cone C and integrable function f the change of variables (45) and Lemma 4 remains true in this general setup. A number of formulas in [1] Chapter 8 can be evaluated this way with a suitable choice of g.

We now turn back to the

Proof of Proposition 2. Let $C_n(s)$ be as in (32)

$$C_n(s) = \frac{1}{2^{n(n-1)/4}} \int_{\substack{Q\in\mathcal{P}_n \\ \operatorname{tr} Q\le 1}} \det{}^s(Q) \prod_{1\le i\le j\le n} dq_{ij}.$$

We will evaluate this using the Cartan decomposition for $\mathcal{P}_n$. We let $A \simeq (\mathbb{R}^+)^n$ be the subset of diagonal matrices in $\mathcal{P}_n$, and $K = SO(n,\mathbb{R})$. Then we have

$$\mathcal{P}_n = \{k^t a k \mid a \in A, k \in K\}.$$

We choose the Haar measure dk on K so that $\int_K 1dk = 1$. After pulling back the integral to $(\mathbb{R}^+)^n \times K$ we are lead to evaluate (see [18])

$$c_n \int_{\substack{0<a_1,\dots,a_n \\ a_1+\dots+a_n<1}} (a_1...a_n)^s \Delta(a_1,...,a_n) da_1...da_n,$$

where $c_n = \dfrac{\pi^{(n^2+n)/4}}{\prod_{k=1}^n k\Gamma(k/2)}$ and $\Delta(a_1,...,a_n) = \prod_{1\le i<j\le n} |a_j - a_i|$.

The function $(a_1...a_n)^s\Delta(a_1,...,a_n)$ is homogeneous of degree $n(s+\frac{n-1}{2})$. We will now use Lemma 4 with the choice of $g(\lambda) = e^{-\lambda}$ so that

$$\int_{(\mathbb{R}^+)^n} g(\|\mathbf{a}\|_1) f(\mathbf{a}) d\mathbf{a} = \int_{(\mathbb{R}^+)^n} e^{-(a_1+\dots+a_n)} (a_1...a_n)^s \Delta(a_1,...,a_n) da_1...da_n$$

.

On the other hand by Corollary 8.2.2 of [1],

$$\int_{[0,\infty)^n} \Delta(a_1,\dots,a_n) \prod_{k=1}^n a_k^s e^{-a_k} \mathrm{d}a_k = \prod_{k=1}^n \frac{[\Gamma(s+1+(n-k)/2)\Gamma(1+k/2)]}{\Gamma(3/2)}.$$

This together with

$$\int_0^\infty e^{-\lambda} \lambda^{ns+n(n+1)/2-1} d\lambda = \Gamma(ns + n(n+1)/2)$$

and Lemma 4 now gives

$$\int_{\substack{0<a_1,\dots,a_n \\ a_1+\dots+a_n<1}} (a_1...a_n)^s \Delta(a_1,...,a_n) da_1...da_n = \frac{\prod_{k=1}^n \Gamma(s+1+(n-k)/2)\Gamma(1+k/2)}{\Gamma(1+ns+n(n+1)/2)(\Gamma(3/2))^n}.$$

□

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