ON A METHOD OF HURWITZ AND ITS APPLICATIONS

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1. Introduction

In two little-known papers, the second of which was published posthumously, Hurwitz developed a method for finding various infinite series representations for class numbers of positive definite integral binary [12] and ternary quadratic forms [13]. In a recent paper [7] we gave corresponding results for indefinite binary quadratic forms using automorphic spectral theory.

That method also yielded a new proof of (the non-Gaussian version of) Hurwitz’s formulas in the positive definite binary case. In this paper, first we will recall these formulas and apply them to get class number formulas for certain binary cubic forms. Then we will generalize the Hurwitz formulas to positive definite n-ary forms for all \( n \geq 2 \) and illustrate how they can be applied to give certain identities relating multiple zeta values for cones to values of the Riemann zeta function. These identities are somewhat similar to the classical sum formula for ordinary multiple zeta values. Another point of the paper is to clarify a basic idea behind the general Hurwitz method, namely the use of an equivariant partition of unity.

Recall that \( \Gamma_2 = \text{SL}(2, \mathbb{Z}) \) acts on the set of binary quadratic forms of discriminant \( d < 0 \)

\[ Q_d = \{ (Q(x, y) = (a, b, c) = ax^2 + bxy + cy^2 \mid a, b, c \in \mathbb{Z}, b^2 - 4ac = d \} \]

with finitely many orbits and finite stabilizer subgroup \( \Gamma(Q) \). The following variant of the first Hurwitz identity is from [7].

**Theorem 1.** For \( d < 0 \)

\[
\sum_Q \frac{1}{\#\Gamma(Q)} = \frac{1}{24\pi} |d|^{\frac{3}{2}} \sum_{a > 0} \frac{1}{a(a + b + c)c},
\]

where \( Q \) runs over a complete set of \( \Gamma_2 \)-representatives of forms in \( Q_d \). Each term in the infinite sum is positive and the sum converges.

The convergence of this series is slow but the method yields infinitely many such formulas which, while getting progressively more complicated, have faster convergence. The next one after (1) is

\[
h(d) = \frac{1}{24\pi} |d|^{\frac{3}{2}} \sum_{a > 0} \frac{1}{a^2(a + b + c)c^2}.
\]

Let \( f(x, y) \) be a binary cubic form

\[
f(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3
\]
where \( a, b, c, d \) are integers. Two such forms are equivalent if one can be transformed to the other in the usual way by \( \gamma \in \Gamma_2 \). The discriminant of the form \( f \) is
\[
\Delta = \Delta(a, b, c, d) = a^2d^2 - 6abcd - 3b^2c^2 + 4ac^3 + 4db^3,
\]
which has the same value for equivalent forms. The set of all forms with a fixed non-zero discriminant \( \Delta \) splits into a finite number \( h_3(\Delta) \) of equivalence classes (see e.g. [16, p. 214]).

Together with some basic invariant theory, Theorem 1 yields formulas for \( h_3(\Delta) \) when \( \Delta < 0 \). The simplest is the following

Theorem 2. For \( \Delta < 0 \) we have that
\[
h_3(\Delta) = \frac{|\Delta|^\frac{3}{2}}{24\pi} \sum_{\Delta(a,b,c,d)=\Delta} \frac{1}{(b^2-ac)((b+c)^2-(a+b)(c+d))(c^2-bd)} .
\]

Each term in this sum is positive and the sum is convergent.

The next in the series of such formulas is
\[
h_3(\Delta) = \frac{|\Delta|^\frac{3}{2}}{48\pi} \sum_{\Delta(a,b,c,d)=\Delta} \frac{1}{(b^2-ac)^2((b+c)^2-(a+b)(c+d))(c^2-bd)^2} .
\]

The method can also be applied to give class number formulas for certain other binary forms of odd degree with given invariants, for instance binary quintic forms, although they become quite complicated.

Turning to higher dimensional generalizations, for integral \( n \) with \( n > 1 \), let \( Q(x) = x^tQx \), where \( Q \) is an \( n \times n \) positive definite integral matrix with determinant \( D \) and \( x = (x_1, \ldots, x_n) \). The group \( \Gamma_n = \text{SL}(n, \mathbb{Z}) \) splits the set of all positive definite matrices into classes when \( \gamma \in \Gamma_n \) acts on \( Q \) by \( Q \mapsto \gamma^tQ\gamma \). The (generalized) Hurwitz class number is given by
\[
H_n(D) = \sum_{\gamma \in \Gamma_n} \# \Gamma_n \langle Q \rangle
\]
where \( \Gamma_n(Q) = \{ \gamma \in \Gamma_n; \gamma^tQ\gamma = Q \} \),
\[
\epsilon_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}
\]
and \( Q \) runs over a complete set of representatives of forms with \( \det Q = D \). The “2” arises since we are using \( \text{SL} \) instead of \( \text{PSL} \). The following is a uniform statement giving the simplest cases of the original Hurwitz formulas.

Theorem 3 (Hurwitz). For any \( D > 0 \) and \( n = 2, 3 \) we have
\[
H_n(D) = c_nD^{n+1} \sum_{Q>0, \det Q=D} \prod_{1 \leq k \leq n} \frac{1}{Q(e_k)} \prod_{1 \leq k < m \leq n} \frac{1}{Q(e_k-e_m)}
\]
with \( e_k \) the standard basis vectors for \( \mathbb{R}^n \) and
\[
c_n^{-1} = \pi^{\frac{n^2-n}{2}} \Gamma(n+2) \prod_{k=0}^{n-1} \Gamma \left( 1 + \frac{k}{n} \right) .
\]

We will show in Theorem 7 that (7) continues to hold for \( n > 3 \) provided we add more terms of the same general shape to the RHS. The terms all arise from a polyhedral decomposition of the cone of positive symmetric matrices \( P_n \), due to Voronoi [21], into those corresponding to perfect forms. The term in (7) comes from \( A_n \), which gives everything when \( n = 2, 3 \).
In general we need all perfect forms and we must use a decomposition of the associated polyhedra into simplices.

The Hurwitz-type class number formulas are well-suited for summation over the determinants $D$. Doing so leads to identities that are somewhat analogous to certain well-known sum formulas for classical multiple zeta values. The generalized multiple zeta values we encounter have the form

$$\zeta_C(s_1, \ldots, s_m) = \sum_{(n_1, \ldots, n_m) \in C} n_1^{-s_1} \cdots n_m^{-s_m}$$

where $C \subset \mathbb{R}^m$ is a certain cone and $s_1, \cdots, s_m$ are certain positive integers.

The usual multiple zeta value in $m$ variables $\zeta(s_1, \ldots, s_m)$ is given by (8) when

$$C = \{(n_1, \ldots, n_m) : n_1 > n_2 > \cdots > n_m\}$$

and $s_1, \ldots, s_m$ are positive integers with $s_1 > 1$. Among the many identities connecting classical multiple zeta values with values of the ordinary Riemann zeta function is the sum formula

$$\sum_{s_1 > 1, s_2 > 1, \ldots, s_m > 1} \zeta(s_1, \ldots, s_m) = \zeta(s),$$

which was discovered in case $m = 2$ by Euler and proven in general in [10].

Identities of this type can be given for the multiple zeta values of (8) that arise from the general version of (7). Here we only consider the case when $n$ is odd. We employ an identity of Ibukiyama and Sato for the zeta function whose coefficients are the general class numbers, and this identity is much simpler for odd $n$. The general identity is given in Theorem 8.

The case $n = 3$ is reasonably simple and completely explicit. Let $C$ be defined by

$$C = \{(n_1, \ldots, n_6) \in (\mathbb{Z}^+)^6; \left(\begin{array}{ccc}
\frac{n_1}{2} & \frac{n_1 + n_2 - n_6}{2} & \frac{n_1 + n_3 - n_5}{2} \\
\frac{n_2}{2} & \frac{n_2 + n_3 - n_4}{2} & \\
\frac{n_3}{2} & & \\
\end{array}\right) \text{ is positive definite integral}\}.$$ 

Then for positive integers $s_1, \ldots, s_6$ with $s_1 + \cdots + s_6 > 6$ the series $\zeta_C(s_1, \ldots, s_6)$ from (8) converges. Consider the determinant

$$D = \det \left(\begin{array}{cccc}
x_1 + x_5 + x_6 & -x_6 & -x_5 & -x_5 \\
-x_6 & x_2 + x_4 + x_6 & -x_4 & -x_4 \\
-x_5 & -x_4 & x_3 + x_4 + x_5 & \\
\end{array}\right)$$

$$= x_1 x_2 x_3 + x_1 x_2 (x_4 + x_5) + x_2 x_3 (x_5 + x_6) + x_1 x_3 (x_6 + x_4) + \cdots,$$

which is called a unisignant since the coefficient of each term is positive. Define the non-negative integers $\{\{s_1, \ldots, s_6\}_{3s}\}$ through

$$D^s = \sum_{s_1 + \cdots + s_6 = 3s} \zeta_C(s_1, \ldots, s_6) \cdot \frac{s_1^s}{s_1!} \cdots \frac{s_6^s}{s_6!}.$$ 

We have the following evaluation, whose proof relies on a more general version of (7) when $n = 3$.

**Theorem 4.** For $s \geq 1$

$$\sum_{s_1 + \cdots + s_6 = 3s \atop s_1, \ldots, s_6 \geq 0} \zeta_C(s_1 + 1, \ldots, s_6 + 1) = \frac{3(s+1)!2(s+1)!}{2^{2s}} (2\zeta(2)\zeta(s+1)\zeta(2s+3) - \zeta(2)\zeta(s+2)\zeta(2s+2)).$$
Note that $\zeta_C(s_1,\ldots,s_6)$ is invariant under any permutation of $(s_1,s_2,s_3)$ provided that the same permutation is performed on $(s_4,s_5,s_6)$. Furthermore we can exchange any two of $(s_1,s_2,s_3)$ with the corresponding elements of $(s_4,s_5,s_6)$. In particular, the case $s = 1$ of the formula can be written as

$$\zeta_C(2,2,1,1,1,1) + 3\zeta_C(2,2,1,2,1,1) = \frac{9}{2}(2\zeta(2)\zeta(5) - \zeta(2)\zeta(3)\zeta(4)) = 7.81059\ldots.$$ 

In the following section we introduce equivariant partitions of unity and give the proof of Theorem 2. Then we develop a general method for constructing these partitions of unity. When specialized to the case of symmetric positive definite matrices these apply to the Hurwitz class number formula of Theorem 3 and its generalizations. Finally, the class number formulas are applied to evaluate certain sums of generalized multiple zeta values, in particular those of Theorem 4.

2. Equivariant partitions of unity

A convenient way to prove Hurwitz-type formulas is to reduce them to the construction of equivariant partitions of unity.

**Definition.** Let $\Gamma$ be a group acting on a space $X$ and suppose that we have a function $\rho : X \to \mathbb{C}$ such that

$$\sum_{\gamma \in \Gamma} \rho(\gamma x) = 1, \quad \forall x \in X.$$ 

We will say that $\rho$ induces a $\Gamma$-equivariant partition of unity, (or simply an equivariant partition of unity if $\Gamma$ is understood).

Their application to class number formulas is formulated in the following basic lemma.

**Lemma 1.** Assume that $\Gamma$ acts on $X$, and $Y \subset X$ is a finite union of orbits such that the stabilizers $\Gamma_y = \{ \gamma \in \Gamma \mid \gamma y = y \}$ are all finite. If $\rho$ induces an equivariant partition of unity then

$$\sum_{j=1}^{h} \frac{1}{|\Gamma_j|} = \sum_{y \in Y} \rho(y).$$

**Proof.**

$$\sum_{y \in Y} \rho(y) = \sum_{j=1}^{h} \sum_{y \in \Gamma_j y_j} \rho(y) = \sum_{j=1}^{h} \frac{1}{|\Gamma_j|} \sum_{\gamma \in \Gamma} \rho(\gamma y_j) = \sum_{j=1}^{h} \frac{1}{|\Gamma_j|}.$$ 

For example, Lemma 1 reduces the proof of (1) to showing that

$$\rho(Q) = \frac{1}{24\pi^2} \left| d \right|^2 \frac{1}{a(a+b+c)c}$$

induces a $\Gamma_2$-equivariant partition of unity. Although we did not use this language, this was shown in [7] by using point-pair invariants on the upper half-plane. This fact also follows easily from a small variation of the proof of Theorem 3 when $n = 2$ that we give below.

The method of equivariant partition of unity, with some alterations, applies nicely in the proof of Theorem 2. In general, let $\Gamma, Y \subset X$, $\rho : X \to \mathbb{C}$ be as above, and assume that we have another $\Gamma$-space $Z$, with an equivariant map $\phi : Z \to Y$ so that

$$\phi(\gamma z) = \gamma \phi(z).$$
for any $\gamma \in \Gamma$, $z \in Z$. Define now

$$G(z) = \rho(\phi(z)),$$

for which we have

$$\sum_{\gamma \in \Gamma} G(\gamma z) = \sum_{\gamma \in \Gamma} \rho(\gamma \phi(z)) = 1.$$

It follows that if $Z$ is the disjoint union of $\Gamma z_j$ for a finite set of $z_j$, say $z_1, \ldots, z_m$, then

$$\sum_{z \in Z} G(z) = \sum_{j=1}^{m} \frac{1}{|\tilde{\Gamma}_j|},$$

where $\tilde{\Gamma}_j = \{ \gamma \in \Gamma : \gamma z_j = z_j \}$. The finiteness of these stabilizers follow from the finiteness of stabilizers in $Y$ and the equivariant property of the map $\phi$.

**Proof of Theorem 2.** The Hessian covariant of our form $f$ from (3) is given by the positive definite binary quadratic form

$$Q(f) = Q(f; x, y) = -\frac{1}{36} \begin{vmatrix} f_{x,x}(x,y) & f_{x,y}(x,y) \\ f_{y,x}(x,y) & f_{y,y}(x,y) \end{vmatrix} = Ax^2 + Bxy + Cy^2,$$

where $A = b^2 - ac$, $B = bc - ad$, and $C = c^2 - bd$. This form has discriminant

$$D = B^2 - 4AC = \Delta(a,b,c,d).$$

It is a covariant in that

$$\gamma Q(f; x, y) = Q(\gamma f; x, y).$$

Equivalent cubic forms have equivalent Hessians but several inequivalent cubic forms may have the same Hessian. It is well-known that

$$\# \{ \gamma \in \Gamma_2; \gamma f = f \} = 1.$$

We apply (1) to count Hessians and the proof of Theorem 2 is now a simple application of the above general set up with $\Gamma = \Gamma_2$, $Y = \mathcal{Q}_d$, $Z = \text{the set of binary cubic forms of discriminant } D$ and $\phi$ the map that takes a binary form $f$ to its Hessian $Q(f)$. Finally, we have that the function

$$G(a, b, c, d) = \rho(Q_f) = \frac{1}{2\pi} \frac{|D|^{3/2}}{AC(A+B+C)}$$

induces an equivariant partition of unity.

The formula (5) follows similarly from (2).

### 3. A General Construction of Partitions of Unity

In this section we give a general method of constructing partitions of unity for the action of a discrete group using simple properties of cones in Euclidean spaces.
3.1. **Integrals over frustums.** Let $V$ be an $m$ dimensional real vector space. A subset $C$ of $V$ is a cone, if whenever $x \in C$ and $\lambda > 0$ we have that $\lambda x$ is also in $C$.

We will be interested in frustums of the cone $C$, sets of the form

$$F_l(C) = \{ x \in C \mid 0 < l(x) < 1 \}$$

for some linear functional $l$. We will assume that $l$ is positive on $C$, and that the frustum $F_l(C)$ is bounded. Then the integral

$$\int_{F_l(C)} f(x) d\mu$$

is defined for any homogeneous polynomial $f$, once a normalization $\mu$ of the Lebesgue measure is fixed.

The simplest cone is a simplicial one, defined by

$$S = S(v_1, \ldots, v_m) = \{ v = \sum_{j=1}^{m} \lambda_j v_j \mid \lambda_j > 0 \}$$

when the $v_j, j = 1, \ldots, m$ form a basis of $V$. In this case the frustum is bounded for any $l$ for which $l(v_j) > 0$ $j = 1, \ldots, m$.

Given a basis $v_1, \ldots, v_m$ we will use the system of coordinates $x = (x_1, \ldots, x_m)$ such that for any $v \in V$ we have

$$v = \sum_{j=1}^{m} x_j(v) v_j$$

Then there is $\delta(v_1, \ldots, v_m) \in \mathbb{R}^+$ such that

$$d\mu = \delta(v_1, \ldots, v_m) dx.$$

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we let $x^\alpha = \prod x_j^{\alpha_j}$. Also we let $\alpha! = \prod \alpha_j!$, and $|\alpha| = \sum \alpha_j$.

Then we have

**Lemma 2.** Let $S = S(v_1, \ldots, v_m)$ be a simplicial cone. Assume that $f$ is a homogeneous polynomial of degree $d$, which in the coordinate system associated to $v_1, \ldots, v_m$ is given by

$$f(v) = \sum_{|\alpha| = d} C_\alpha(f) x^\alpha.$$ 

Then

$$\int_{F_l(S)} f d\mu = \delta(v_1, \ldots, v_m) \sum_{|\alpha| = d} C_\alpha(f) \frac{\alpha!}{(m+d)!} \prod_j l(v_j)^{-\alpha_j - 1}.$$

**Proof.** In the coordinate system $x$

$$\int_{F_l(S)} f d\mu = \delta(v_1, \ldots, v_m) \int_{F_l(S)} f dx_1 \ldots dx_m$$

and the frustum $F_l(S)$ is described by

$$\{ (x_1, \ldots, x_m) \mid x_j \geq 0, 0 < \sum_j l(v_j)x_j \leq 1 \}.$$

Consider the map

$$(x_1, \ldots, x_m) \in \mathbb{R}^m \mapsto (y_1, \ldots, y_m) = (l(v_1)x_1, \ldots, l(v_m)x_m).$$
Then the change of variables to $y$ gives
\[
\int_{F_i(C)} f \, d\mathbf{y} = \sum_{|\alpha| = d} C_\alpha(f) \prod_j l(v_j)^{-\alpha_j - 1} \int_\Delta y^\alpha \, d\mathbf{y}
\]
where $\Delta = \{(y_1, \ldots, y_m) \mid y_j \geq 0, \sum_j y_j \leq 1\}$ is the standard simplex. The integral is the well-known multivariate Beta-integral ([1]) with the evaluation $\alpha! (\alpha + m)!$.

3.2. Symmetric homogeneous cones. We will now consider the case when the cone is homogeneous, that is the subgroup
\[
\text{Aut}(C) = \{ g \in GL(V) \mid gC = C \}
\]
acts transitively on $C$. The primary example for us is the cone $P_n$ of positive definite matrices in the vector space of $n \times n$ symmetric matrices whose dimension is $m = n(n + 1)/2$. The action of $g \in GL_n(\mathbb{R})$ on $x \in P_n$ via $x \mapsto g^t x g$ is linear and the group $GL_n(\mathbb{R})$ acts transitively this way.

While we are exclusively interested in this particular example some of the calculations are easier to present in the general setup.

Later we will also assume that $C$ is symmetric, in the sense that there is an inner product $\langle \cdot, \cdot \rangle$ on $V$ such that $C^* = C$, where
\[
C^* = \{ x \in V \mid \langle x, y \rangle > 0, \text{ for all } y \in \overline{C} \setminus \{0\} \}.
\]
Again this holds for the cone of positive definite matrices, if we define $\langle Q_1, Q_2 \rangle = \text{tr} Q_1 Q_2$.

If $v \in C$ then its stabilizer
\[
\text{Stab}_v = \{ g \in \text{Aut}(C) \mid gv = v \}
\]
is a maximal compact subgroup of $\text{Aut}(C)$, moreover there exists $v_0 \in C$ such that
\[
\text{Stab}_{v_0} = \text{Aut}(C) \cap O(V).
\]
For a proof of this and many other facts concerning symmetric cones see [8].(For some of the integral evaluations [9] is also useful.) In the case of $P_n$ the statements are obvious, with $v_0 = I$.

The inner product $\langle \cdot, \cdot \rangle$ defines a volume form and hence a normalized Haar measure $\mu$ on $V$ and we let $SL(V)$ be the subgroup of volume (and orientation) preserving linear transformations. We also let
\[
G = \text{Aut}(C) \cap SL(V).
\]
Now if $C' \subset C$ is also a cone, and $g \in G$ then
\[
\int_{F_i(gC')} f(x) \, d\mu = \int_{F_i(C')} f(gx) \, d\mu
\]
where $g^*l$ is the linear functional $g^*l(x) = l(gx)$. (Here we have used that $\det g = 1$.)

In this case if $\Gamma$ is a discrete subgroup of $G$ such that the cones $\gamma C'$ are disjoint, then we may conclude, at least formally that
\[
\int_{F_i(\cup_{\gamma \in \Gamma} \gamma C')} f(x) \, d\mu = \sum_{\gamma \in \Gamma} \int_{F_i(\gamma C')} f(\gamma x) \, d\mu.
\]
For applications we need to evaluate the integrals in (16). To proceed further we need to make some assumptions, both geometric and algebraic on the discrete group $\Gamma$. For evaluating the left hand side of (16) we will need the following geometric assumption.
Assumption 1. $\Gamma \subset G$ is a discrete subgroup and there is a finite collection of polyhedral cones $C_1, ..., C_k$ such that for $\gamma_1, \gamma_2 \in \Gamma$ the translates $\gamma_1 C_i$ and $\gamma_2 C_j$ have disjoint interiors if $\gamma_1 \neq \gamma_2$ and also when $\gamma_1 = \gamma_2$ but $i \neq j$. Moreover this collection satisfies

$$\bigcup_{\gamma \in \Gamma} \bigcup_{i=1}^k \gamma C_i = C.$$ 

Such polyhedral decompositions are shown to exist for a great many situations in [3], see Chapter 2.

For example the assumption is satisfied in the typical situation when there is a polyhedral cone $F \subset C$ that is a fundamental domain for $\Gamma$.

A typical example is the cone in $P_2$ generated by the rank 1 forms

$$(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}), \quad \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \text{ and } \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}.$$ 

After identifying the upper half plane $\mathcal{H}$ with the $2 \times 2$ positive definite matrices of fixed determinant, this becomes the ideal triangle with vertices 0, 1 and $\infty$ in $\mathcal{H}$, which is a fundamental domain for a subgroup of $PSL_2(\mathbb{Z})$ of index 3.

Given such polyhedral cones we define for each each $C_i$

$$r(C_i) = \# \{ \gamma \in \Gamma : \gamma C_i = C_i \}. \tag{17}$$

Furthermore we decompose each $C_i$ into simplicial cones $S_1, ..., S_r$ as in (12) such that $S_1 \cup ... \cup S_r = C_i$ and let $r(S_j)$ have the same value $r(C_i)$ on each of these simplicial cones. Again the existence of such a simplicial decomposition was shown in [2], page 75 item (c). (See also [11].) While these papers assume that the discrete group $\Gamma$ is neat, for us it is sufficient that $\Gamma$ has neat subgroup $\Gamma'$ of finite index. This implies that only need the fact that the decomposition into simplicial cones $\bigcup_{\gamma \in \Gamma'} \bigcup_{i=1}^k \gamma S_i = C$ gives

$$\sum_{\gamma \in \Gamma} \sum_{i=1}^k \frac{1}{|\Gamma : \Gamma'| r_i} \chi_{S_i}(\gamma z) = 1$$

for almost all $z \in C$. In the application we have in mind for $\Gamma = SL_n(\mathbb{Z})$ acting on positive definite symmetric matrices $P_n$ we can take $\Gamma' = \Gamma$.

To evaluate the integrals on the right hand side of (16) we make the following algebraic assumption.

Assumption 2. $f$ is a $G$-invariant polynomial that is positive on $C$.

For example in the case of $P_n$ one may chose $f(x) = (\det x)^s$, for some $s \in \mathbb{Z}^+$.

Then (16) together with Lemma 2 prove the following

Proposition 1. Let $C \subset V$ be a symmetric cone, and $\Gamma$ a discrete subgroup of $G$ which satisfies Assumption 1. Let $S_i = S(v_1(i), ..., v_m(i))$ be simplicial cones in that assumption and let $\delta_i = \delta(v_1(i), ..., v_m(i))$ as in (14) and $r_i$ as above.

If $f$ is a $G$-invariant homogeneous polynomial of degree $d$ which is positive on $C$, and which in the coordinate system (13) associated to $v_1(i), ..., v_m(i)$ is given by $f(v) = \sum_{|a| = d} C_{a,i}(f)x(i)^a$ then

$$\int_{F_i(C)} f(x) d\mu = \sum_{\gamma \in \Gamma} \sum_{i=1}^k \frac{\delta_i}{r_i} \left( \sum_{|a| = d} C_{a,i}(f) \frac{\alpha!}{(m + \alpha)!} \prod_j l(\gamma v_j(i)^{-\alpha_j - 1}) \right). \tag{18}$$
We will now use the extra property of symmetry of $C$. Fix $v_0 \in C$, as in (15) and let $K = \text{Stab}_{v_0}$. By assumption $K \subset O(V)$ and so the bilinear form $\langle v_1, v_2 \rangle$ on $V$ is $K$-invariant. This allows us to identify $V$ with $V^*$ via

$$v \mapsto l_v, \quad l_v(x) = \langle x, v \rangle.$$

$l_v$ is positive on $C$ if and only if $v \in C$. By the assumption of homogeneity of $\text{Aut}(C)$ and the definition of $G = \text{Aut}(C) \cap \text{SL}(V)$, it is clear that we can find $g \in G$, $\lambda(v) \in \mathbb{R}^+$ such that $v = \lambda(v)gv_0$. Note that $\lambda$ is uniquely determined by the $G$-orbit of $v$. Therefore if $v = gw$ then $\lambda(v) = \lambda(w)$. Moreover the symmetry condition implies that if $g \in G$ then so is $g'$, where $\langle g'x, y \rangle = \langle x, gy \rangle$. Hence any $l \in V^*$ that is positive on $C$ is of the form

$$l(x) = \lambda \langle x, g'v_0 \rangle$$

where $g \in G$, in particular $\det(g) = 1$.

We have the following lemma.

**Lemma 3.** Let $f$ be $G$-invariant homogeneous polynomial of degree $d$. Let $v_0 \in C$ be as above and $l(x)$ as in (19). Then we have

$$\int_{F_0(C)} f(x)d\mu = (\lambda)^{-(m+d)} \int_{F_0} f(x)d\mu$$

where $F_0 = \{x \in C \mid \langle x, v_0 \rangle \leq 1\}$.

**Proof.** We have

$$\int_{F_0(C)} f(x)d\mu = \int_{\{l(x) \leq 1\}} f(x)d\mu = \int_{\{\lambda_{(gx,v_0)} \leq 1\}} f(x)d\mu.$$

The change of variables $x = \lambda(v)^{-1}g^{-1}y$ then leads to the integral $\lambda^{-(m+d)} \int_{y \in C, (g,y,v_0) \leq 1} f(y)d\mu$ proving the statement. \hfill \square

Lemma 3 and Proposition 1 now can be used to obtain an equivariant partition of unity for $\Gamma$ acting on $C$.

**Proposition 2.** Let $C \subset V$ be a symmetric cone, and $\Gamma$ a discrete subgroup of $G$ which satisfies Assumption 1. Let $r_i$ and $S_i = S(v_1(i), \ldots, v_m(i))$ be simplicial cones in that assumption, and $\delta_i = \delta(v_1(i), \ldots, v_m(i))$. Let $f$ be a $G$-invariant homogeneous polynomial of degree $d$ which is positive on $C$, and which in the coordinate system (19) associated to $v_1(i), \ldots, v_m(i)$ is given by $f(v) = \sum_{|\alpha| = d} C_{\alpha,i}(f)x(\alpha).$

Define $\rho: C \to \mathbb{C}$ by

$$\rho: v \mapsto A \sum_{i=1}^k \frac{\delta_i}{r_i} \sum_{|\alpha| = d} C_{\alpha,i}(f) \frac{\alpha!}{(m+d)!} \prod_j \langle v, v_j(i) \rangle^{-\alpha_j-1},$$

where

$$A = \frac{\lambda^{m+d}}{\int_{F_0} f(x)d\mu}$$

with $F_0 = \{x \in C \mid \langle x, v_0 \rangle \leq 1\}$ and where $\lambda$ is defined by $v = \lambda g'v_0$, $g \in G$ so that (19) holds.

Then $\rho$ induces a smooth $\Gamma$-equivariant partition of unity on the space $C$. 


Proof. Note that the left hand side of (18) apriori depends on \( l(x) = \langle x, v \rangle \) (and of course on \( f \) and the cone \( C \)). But Lemma 3 shows that in fact it only depends on \( \lambda = \lambda(v) \) which is constant on \( G \)-orbits. Hence once \( C \) and \( f \) are fixed, \( \rho \) induces a partition of unity of \( \Gamma \) acting on \( C \) in view of (16).

4. The case of \( \mathcal{P}_n \)

We now will elaborate on the particular case when the symmetric cone \( C \) is \( \mathcal{P}_n \subset \mathbb{R}^m \), with \( m = n(n+1)/2 \). We will make the choice that \( f(Q) = \det^s(Q) \), for some \( s \in \mathbb{Z}^+ \), with degree of homogeneity \( d = ns \). It is known \([21, 18]\) that there exist a finite index subgroup \( \Gamma \) of \( SL_n(\mathbb{Z}) \) for which a \( \Gamma \)-invariant polyhedral decomposition as required by Assumption 1 exists. The resulting partition of unity for \( \Gamma \) when re-scaled by \( r = \frac{1}{|SL_n(\mathbb{Z});\Gamma|} \) is then a partition of unity for \( SL_n(\mathbb{Z}) \). If the simplicial cones in the assumption are \( S_i = S(Q_1(i), ..., Q_m(i)) \) then for each of the \( S_i \) we can express

\[
\text{det}^s Q = \sum_{|\alpha| = sn} C_{\alpha,i} x(i)^\alpha
\]

where the linear functions \( x_j(i) = x_j(i)(Q) \) are defined by \( Q = \sum_{j=1}^m x_j(i)Q_j(i) \). The Haar moduli \( \delta_i = \delta(Q_1(i), ..., Q_m(i)) \) can be evaluated as

\[
\delta_i = \frac{1}{\sqrt{\text{det}(\text{tr} Q_j(i)Q_k(i)))_{j,k=1}^m}}.
\]

Since \( Q_0 = I_n \) has determinant 1 we have that \( \lambda = \sqrt{\text{det} Q} \). We then have the following reformulation of the Proposition 2.

Theorem 5. Let \( \rho : \mathcal{P}_n \mapsto \mathbb{C} \) be defined by

\[
\rho(Q) = A \sum_{i=1}^k \frac{\delta_i}{r_i} \sum_{|\alpha| = ns} C_{\alpha,i}(f) \frac{\alpha!}{(ns + n(n+1)/2)!} \prod_j (\text{tr} Q_j(i))^{-\alpha_j - 1}
\]

where

\[
A = \frac{\text{det}^{s+(n+1)/2}(Q)}{C_n(s)}
\]

with

\[
C_n(s) = \frac{1}{2^{n(n-1)/4}} \int_{Q \in \mathcal{P}_n} \text{det}^s(Q) \prod_{1 \leq i < j \leq n} dq_{ij},
\]

Then \( \rho \) induces a partition of unity for \( SL_n(\mathbb{Z}) \) acting on \( \mathcal{P}_n \).

The rest of the section is devoted to the evaluation of the constant \( C_n(s) \). For this we use the following Lemma.

Lemma 4. Let \( C \in \mathbb{R}^n \) be a cone. Assume that \( f(x) \) is a continuous function which is homogeneous of degree \( d \): \( f(\lambda x) = \lambda^d f(x) \). Let \( \|x\|_1 = \sum |x_i| \). For any \( g : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) such that \( \int_0^\infty g(\lambda)\lambda^{d+n-1}d\lambda < \infty \) we have

\[
\int_{x \in C, \|x\|_1 \leq 1} f(x)d\mu(x) = \frac{\int_C g(\|x\|_1)f(x)d\mu(x)}{(d+n) \int_0^\infty g(\lambda)\lambda^{d+n-1}d\lambda}.
\]
Proof. Let \( \Sigma = \{ x \in \mathbb{R}^n \mid \| x \|_1 = 1 \} \) with the surface measure \( d\sigma \). We have for any cone \( C \) and integrable function \( f \)

\[
\int_C f(x) d\mu(x) = \int_{\mathbb{R}^+} \int_{\Sigma \cap C} f(\lambda p) \lambda^{n-1} d\sigma(p) d\lambda.
\]

We have that \( f(\lambda p) = \lambda^d f(p) \) and so

\[
\int_{x \in C, \| x \|_1 \leq 1} f(x) d\mu(x) = \int_0^1 \lambda^{d+n-1} d\lambda \int_{\Sigma \cap C} f(p) d\sigma(p).
\]

On the other hand consider the function

\[
x \mapsto g(\| x \|_1) f(x),
\]

which has integral

\[
\int_C g(\| x \|_1) f(x) d\mu(x) = \int_0^\infty g(\lambda) \lambda^{d+n-1} d\lambda \int_{\Sigma \cap C} f(p) d\sigma(p).
\]

As an example let \( C = (\mathbb{R}^+)^n \), and \( f(x) = x^\alpha \). This leads to the Beta-integral

\[
\int_{x_1, \ldots, x_m > 0, \ x_1 + \ldots + x_m < 1} x^\alpha d\mathbf{x}
\]

If we now choose \( g(\lambda) = e^{-\lambda} \) then

\[
\int_{(\mathbb{R}^+)^n} e^{-x_1 x_1^\alpha_1 \ldots e^{-x_n x_n^\alpha_n} d\mu(x) = \Gamma(\alpha_1 + 1) \ldots \Gamma(\alpha_n + 1)}
\]

leading to the evaluation of the multivariate Beta-integral [1] used in the proof of Lemma 2.

Remark. Lemma 4 is based on an adaptation of polar and spherical coordinates to other norms and in a technical sense is at the core of some of Hurwitz’s explicit evaluations of his “projective integrals” in the case \( n = 2, 3 \) from [12, 13]. More generally assume that a norm \( \| x \| \) is defined on an \( n \)-dimensional real vector space \( V \) and let \( X = \{ x \in V \mid \| x \| \leq 1 \} \) be its unit ball. Then \( X \) is convex, centrally symmetric and absorbing in the sense that

\[
\bigcup_{\lambda > 0} \lambda X = V.
\]

Let \( \Sigma \) be the boundary of \( X \), which is the unit ”sphere” \( \{ p \in V \mid \| p \|_X = 1 \} \) in the norm \( \| \cdot \|_X \).

If on \( V \) we fix a choice of Lebesgue measure \( d\mu \) then the \((n-1)\)-dimensional Minkowski content defines a surface measure \( \sigma \) on \( \Sigma \) such that

\[
(V \setminus \{ 0 \}, d\mu) \simeq (\mathbb{R}^+, \lambda^{n-1} d\lambda) \times (\Sigma, d\sigma)
\]

as measure spaces via the map \( \lambda = \| x \|, p = x/\| x \| \). With this understanding of \( (\Sigma, d\sigma) \) we still have that for any cone \( C \) and integrable function \( f \) the change of variables (23) and Lemma 4 remains true in this general setup. A number formulas in [1] Chapter 8 can be evaluated this way with a suitable choice of \( g \).

We now turn back to the evaluation of \( C_n(s) \) in Theorem 5.

**Proposition 3.** Let \( C_n(s) \) be as in (22). Then

\[
C_n(s) = \frac{1}{2^n(n-1)/4 \Gamma(1 + ns + n(n+1)/2)} \prod_{k=0}^{n-1} \Gamma(s + 1 + k/2)
\]
Proof. We will evaluate
\[ C_n(s) = \frac{1}{2^{n(n-1)/4}} \int_{Q \in P_n} \det^s(Q) \prod_{1 \leq i \leq j \leq n} dq_{ij} \]
using the polar coordinates for \( P_n \). We let \( \widetilde{A} \simeq (\mathbb{R}^+)^n \) be the subset of diagonal matrices in \( P_n \), and \( K = SO(n, \mathbb{R}) \). Then we have
\[ P_n = \{ a[k] \mid a \in \widetilde{A}, k \in K \}. \]
We choose the Haar measure \( dk \) on \( K \) so that \( \int_K 1 \, dk = 1 \). After pulling back the integral to \((\mathbb{R}^+)^n \times K \) we are lead to evaluate
\[ c_n \int_{0 < a_1, \ldots, a_n} (a_1 \ldots a_n)^s \Delta(a_1, \ldots, a_n) da_1 \ldots da_n, \]
where \( c_n = \frac{\pi^{(n^2+n)/4}}{\prod_{k=1}^n k!^{1/2}} \) and \( \Delta(a_1, \ldots, a_n) = \prod_{1 \leq i < j \leq n} |a_j - a_i| \).

The function \( (a_1 \ldots a_n)^s \Delta(a_1, \ldots, a_n) \) is homogeneous of degree \( n(s + \frac{n-1}{2}) \). We will now use Lemma 4 with the choice of \( g(\lambda) = e^{-\lambda} \) so that
\[ \int_{(\mathbb{R}^+)^n} g(\|a\|_1) f(a) da = \int_{(\mathbb{R}^+)^n} e^{-(a_1 + \ldots + a_n)} (a_1 \ldots a_n)^s \Delta(a_1, \ldots, a_n) da_1 \ldots da_n. \]

On the other hand by Corollary 8.2.2 of [1],
\[ \int_{[0, \infty)^n} \Delta(a_1, \ldots, a_n) \prod_{k=1}^n a_k^k e^{-a_k} da_k = \prod_{k=1}^n \frac{\Gamma(s + 1 + (n-k)/2) \Gamma(1+k/2)}{\Gamma(3/2)}. \]
This together with
\[ \int_0^\infty e^{-\lambda} \lambda^{ns+n(n+1)/2-1} d\lambda = \Gamma(ns + n(n + 1)/2) \]
and Lemma 4 now gives
\[ \int_{0 < a_1, \ldots, a_n} (a_1 \ldots a_n)^s \Delta(a_1, \ldots, a_n) da_1 \ldots da_n = \prod_{k=1}^n \frac{\Gamma(s + 1 + (n-k)/2) \Gamma(1+k/2)}{\Gamma(1+ns + n(n + 1)/2)(\Gamma(3/2))^n}. \]
\[ \square \]

5. Application to class number Formulas

5.1. Binary quadratic forms. Let
\[ V = \{ Q = (\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}) \mid a, b, c \in \mathbb{R} \} \]
be the vector space of symmetric matrices. An element \( Q \) of \( V \) is identified with the binary quadratic form \([a, b, c] \in P_d\) via
\[ ax^2 + 2bxy + cy^2 = Q(x, y) = (x \ y) \left( \begin{smallmatrix} a & b \\ b & c \end{smallmatrix} \right) (x \ y). \]
We let \( C \) be the cone of positive definite matrices;
\[ C = \{ [a, b, c] \mid a > 0, ac - b^2 > 0 \}. \]
Moreover if \( \Gamma = SL_2(\mathbb{Z}) \) then the simplicial cone
\[ (24) \quad S = S \left( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \]
covers \( C \) evenly 6 times.

This shows that all the conditions in Assumptions 1 and 2 about the group action are satisfied, and if we choose \( f(Q) = (\det Q)^l \), it is invariant, and homogeneous of degree \( d = 2l \). For Theorem 1 we only need the case \( l = 0 \) when \( f(Q) = 1 \). We apply Theorem 5 and Proposition 3 from the previous section with \( Q_1 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \), \( Q_2 = \left( \begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix} \right) \), \( Q_3 = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and this shows that

\[
\rho(Q) = \frac{|D|^{3/2}}{3\pi} \prod_j \langle Q, Q_j \rangle^{-1} = \frac{|D|^{3/2}}{3\pi} \frac{1}{a(a-2b+c)c}
\]

induces a partition of unity. Note that the \( Q_1, Q_2, Q_3 \) correspond respectively to the minimal vectors \((1,0), (1,-1)\) and \((0,1)\) of the lattice \( A_2 \).

### 5.2. Ternary quadratic forms

Let \( V \) be the space of ternary quadratic forms

\[
Q(x, y, z) = ax^2 + by^2 + cz^2 + 2Cxy + 2Bxz + 2Ayz
\]

\[
= p_1 x^2 + p_2 y^2 + p_3 z^2 + p_4 (y-z)^2 + p_5 (z-x)^2 + p_6 (x-y)^2
\]

We identify \( V \) with the space of symmetric \( 3 \times 3 \) matrices

\[
V = \left\{ \begin{pmatrix} a & C & B \\ C & b & A \\ B & A & c \end{pmatrix} \mid a, b, c, A, B, C \in \mathbb{R} \right\}
\]

\[
= \left\{ \begin{pmatrix} p_1 + p_5 + p_6 & -p_6 & -p_5 \\ -p_6 & p_2 + p_4 + p_6 & -p_4 \\ -p_5 & -p_4 & p_3 + p_4 + p_5 \end{pmatrix} \mid p_1, p_2, \ldots, p_6 \in \mathbb{R} \right\}
\]

We let \( C \) be the cone of positive definite matrices;

\[
C = \{ Q = [a, b, c, A, B, C] \mid Q > 0 \}.
\]

Similar to the case of binary forms, \( SL_3(\mathbb{R}) \) acts on \( V \) via

\[
Q \mapsto Q[g] = g^t Q g
\]

and we clearly have that

\[
C = \{ I[g] \mid g \in GL_3^+(\mathbb{R}) \}.
\]

and the stabilizer of \( I \) is \( K = SO(3, \mathbb{R}) \), a maximal compact subgroup.

As a \( K \)-invariant inner product we choose

\[
\langle Q_1, Q_2 \rangle = \text{tr} Q_1 Q_2 = a_1 a_2 + b_1 b_2 + c_1 c_2 + 2A_1 A_2 + 2B_1 B_2 + 2C_1 C_2.
\]

Next we choose as a basis for \( V \) the following forms

\[
Q_1 = \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), \quad Q_2 = \left( \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right), \quad Q_3 = \left( \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right), \quad Q_4 = \left( \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix} \right), \quad Q_5 = \left( \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right), \quad Q_6 = \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)
\]

Then the elements of \( V \) can also be written in the dual basis associated to the inner product

(25) and the basis (26) as

\[
V = \left\{ \begin{pmatrix} n_1 & \frac{1}{2} (n_1 + n_2 - n_6) & \frac{1}{2} (n_1 + n_3 - n_5) \\ n_2 & n_2 & \frac{1}{2} (n_2 + n_3 - n_4) \\ n_3 & n_3 & n_3 \end{pmatrix} \mid n_1, n_2, \ldots, n_6 \in \mathbb{R} \right\}
\]

If \( \Gamma = PSL_3(\mathbb{Z}) \) then as the simplicial cone we take

(28) \( S = S(Q_1, Q_2, \ldots, Q_6) \)
which covers \( C \) evenly 24 times. This is the Selling fundamental domain for \( \text{PSL}_3(\mathbb{Z}) \) as the elements of \( S \) are the ternary forms \([p_1 + p_2 + p_3, p_2 + p_4 + p_6, p_3 + p_4 + p_5, -p_4, -p_5, -p_6]\) with \( p_1, p_2, \ldots, p_6 > 0 \) (see [19]). Note that the rank 1 forms \( Q_1, \ldots, Q_6 \) correspond to the minimal vectors \( e_1, e_2, e_3, e_1 - e_3, e_2 - e_3, e_1 - e_2 \) of the lattice \( A_3 \).

We now apply Theorem 5 with \( f(Q) = 1 \), so that \( m = 0 \), and \( n = \dim V = 6 \). If we use the simplicial cone \( S \) from (28) and Proposition 3 we get

\[
\rho(Q) = \frac{D^2}{12\pi^2} \frac{1}{abc(a + b - 2C)(b + c - 2A)(c + a - 2B)},
\]

induces a partition of unity for \( \text{SL}_3(\mathbb{Z}) \) which proves Theorem 3.

5.3. Positive definite \( n \)-ary quadratic forms and the proof of Theorem 7. We now turn to Theorem 7. Let \( \Gamma = \text{SL}_n(\mathbb{Z}) \). Then it is known that there is a polyhedral fundamental domain for \( \Gamma \). For concreteness we will use Voronoi’s reduction [21, 18], but other alternatives are just as suitable. Let \( P_1, \ldots, P_r \) represent the \( \Gamma \)-equivalence classes of perfect forms, where we assume that \( P_1 \) is of type \( A_n \). Let the shortest vectors of \( P_i \) be \( v_{1,i}, \ldots, v_{t,i} \). Then the rank 1 symmetric matrices \( Q_j(i) = v_{i,j}v_{i,j}^T, 1 \leq j \leq t \) span a cone \( C_i = C_i(Q_1(i) \ldots Q_t(i)) \) in \( P_n \) and the \( \Gamma \)-translates of the union of these cones covers \( P_n \), where each cone has some multiplicity \( r_i \) as in (17). If one now takes a decomposition of these cones into simplicial ones, so that \( P_n = \bigcup_{\gamma \in \Gamma} \bigcap_{i=1}^{k} \gamma S_i \), then Theorem 5 is applicable leading to the formula

\[
H_n(D) = \frac{D^{n+1}}{\pi^{(n^2-n)/4}} \frac{2^{n(n-1)/4} \epsilon_n}{\prod_{k=0}^{n-1} \Gamma(1 + k/2)} \sum_{Q > 0} \sum_{\det Q = D} \prod_{i=1}^{k} \prod_{j=1}^{\delta_i} \frac{1}{Q(v_{i,j})}.\]

This proves Theorem 7.

6. Proof of Theorem 4

As before let

\[
Q(x, y, z) = ax^2 + by^2 + cz^2 + 2Cxy + 2Bxz + 2Ay z = p_1x^2 + p_2y^2 + p_3z^2 + p_4(y - z)^2 + p_5(z - x)^2 + p_6(x - y)^2
\]

so that

\[
\begin{vmatrix}
  a & C & B \\
  C & b & A \\
  B & A & c
\end{vmatrix}^s = \begin{vmatrix}
  p_1 + p_5 + p_6, & -p_6, & -p_5 \\
  -p_6, & p_2 + p_4 + p_6, & -p_4 \\
  -p_5, & -p_4, & p_3 + p_4 + p_5
\end{vmatrix}.
\]

From (31) it is clear that, for \( s \in \mathbb{Z} \) there are \( s_1, \ldots, s_6 \in \mathbb{Z} \) and coefficients \( \{s_1, \ldots, s_6\} \) such that

\[
D^s = \sum_{s_1 + \ldots + s_6 = 3s} \left\{ s_1, \ldots, s_6 \right\} \frac{p_1^{s_1}}{s_1!} \ldots \frac{p_6^{s_6}}{s_6!}.
\]

To prove Theorem 4 we first apply Theorem 5 with \( f(Q) = |\det Q|^s = D^s \). Note that the coefficients \( C(\alpha) \) in there for a multi index \( \alpha = (\alpha_1, \ldots, \alpha_6) \) satisfy

\[
C(s_1, \ldots, s_6)s_1! \ldots s_6! = \left\{ s_1, \ldots, s_6 \right\}
\]

and we obtain
The evaluation of the above integral, using Proposition 3 gives the following theorem from Hurwitz[13] (section 6, equation (23)).

**Theorem 6.**

\[ \frac{(s + 1)!(2s + 1)!}{2^{2s}} \frac{12\pi^2}{D^{s+2}} \mu = \sum_{s_1 + \cdots + s_6 = 3s \atop s_1, \ldots, s_6 \geq 0} \{ s_1, \ldots, s_6 \} Z_Q(s_1 + 1, \ldots, s_6 + 1) \]

here \( \mu = |\text{SL}(3, \mathbb{Z})_Q| \), and

\[ Z_Q(s_1, \ldots, s_6) = \sum_{[Q]} \frac{1}{a^{s_1}b^{s_2}c^{s_3}(b + c - 2A)^{s_4}(c + a - 2B)^{s_5}(a + b - 2C)^{s_6}} \]

where the sum is over all forms in the class of \( Q \).

Next we sum (32) over all classes and over all \( D \) and use the coordinates from (27) to obtain the following

**Lemma 5.**

\[ \frac{12\pi^2(s + 1)!(2s + 1)!}{2^{2s}} \sum_{D} \frac{H(D)}{D^{s+2}} = \sum_{s_1 + \cdots + s_6 = 3s \atop s_1, \ldots, s_6 \geq 0} \{ s_1, \ldots, s_6 \} \zeta_C(s_1 + 1, \ldots, s_6 + 1) \]

where \( C = (\mathbb{R}^+)^6 \) and

\[ \zeta_C(s_1, \ldots, s_m; L) = \sum_{(n_1, \ldots, n_m) \in C \cap L} n_1^{-s_1} \cdots n_m^{-s_m} \]

with \( L \) the lattice of all positive definite ternary quadratic forms.

To prove Theorem 4 we also need a result of Ibukiyama and Saito which writes the prehomogeneous vector space zeta function associated to the ternary quadratic forms in terms of simple zeta functions. More precisely, let \( L \) denote the space of positive definite integral ternary quadratic forms with the usual action of \( \text{SL}(3, \mathbb{Z}) \) and for \( x \in L \), let \( \mu(x) \) denote the order of the automorphs of \( x \). The prehomogenous vector space zeta function associated to \( L \) is then (up to some constant) the left hand side of (34). Indeed

\[ \zeta(s, L) := \sum_{L/\sim, x \geq 0} \frac{1}{\mu(x) \text{det}(x)^s} = \sum_{D=1}^{\infty} \frac{1}{D^s} \sum_{x \in L/\sim, \text{det}(x) = D} \frac{1}{\mu(x)} = \sum_{D=1}^{\infty} \frac{H(D)}{D^s} \]

On the other hand Theorem ?? of Ibukiyama and Sato for \( n = 3 \) gives

\[ \zeta(s, L) = \frac{3(s + 1)!(2s + 1)!}{2^{2s}}(2\zeta(2)\zeta(s + 1)\zeta(2s + 3) - \zeta(2)\zeta(s + 2)\zeta(2s + 2)) \]

Now (34) and (35) together with (36) finishes the proof of Theorem 4.

7. **Formulas and Theorems that need to be incorporated, rewritten and checked**

**Theorem 7.** With the above setup, for each \( D > 0 \) we have the convergent identity
\[ H_n(D) = \frac{D^{n+1} \epsilon_n}{(\frac{n}{2})^{\frac{n^2-n}{2}}} \prod_{k=0}^{n-1} \Gamma \left( 1 + \frac{k}{2} \right) \sum_{Q > 0} \sum_{\det Q = D} \prod_{i=1}^{k} \frac{\delta_i}{r_i} \prod_{j=1}^{\ell} \frac{1}{Q(u_{i,j})} \]

where \( \epsilon_n \) is 2 if \( n \) is even and 1 if \( n \) is odd.

Do we need to do the binary, ternary cases of (5.1),(5.2) seperately? We have a general proof

**Theorem 8.**

\[ \sum_{D=1}^{\infty} \frac{H_n(D)}{D^{s+n(n+1)/2}} = \sum_{i=1}^{k} \sum_{|\alpha|=nS} \tilde{C}_{\alpha,i} \zeta_s(s_1 + 1, \ldots, s_{n(n+1)/2} + 1) \]

where \( S_i^* \) are dual cones to the ones appearing in the polyhedral decomposition above and \( \tilde{C}_{\alpha,i} = \frac{\delta_i}{\delta_i} C_{\alpha,i} \) with \( C_{\alpha,i} \) coming from (31).

On the other hand, when \( n \) is odd, we have the following theorem of Ibukiyama and Sato

For \( n \in \mathbb{N} \) odd, we have

\[ \sum_{D=1}^{\infty} \frac{H_n(D)}{D^{s+n(n+1)/2}} = \frac{\prod_{i=1}^{[n/2]} B_{2i}}{2^{n-1} (n-1)!} \left( 2^{(n-1)/2} \zeta \left( s - \frac{n-1}{2} \right) \prod_{i=1}^{[n/2]} \zeta(2s - (2i - 1)) + (-1)^{(n^2-1)/8} \zeta(s) \prod_{i=1}^{[n/2]} \zeta(2s - 2i) \right) \]

Is this Ibu-Saito formula correct? It does not seem to give the ternary case. This Theorem should be formulated as a single identity.

It might be good to formulate the first term in the sum of multiple zetas that corresponds to \( A_n \). In any case the proof of Theorem 4 should be attached to the proof of the general multiple zeta result and not given as though it is a different result.

**References**

ON A METHOD OF HURWITZ AND THEIR APPLICATION


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