Fourth Moments of L-Functions Attached to Newforms

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1. Introduction and Statement of the Main Result

In 1923 Hardy and Littlewood gave the estimate

$$\int_{-T}^{T} \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 dt \ll T \log^4 T \tag{1.1}$$

as $T \to \infty$ for the fourth moment of the Riemann ζ-function on its critical line (see [10], p. 194). Three years later Ingham [15] showed this to be sharp by providing the asymptotic result

$$\int_{-T}^{T} \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 dt = \pi^{-2} T \log^4 T + O(T \log^3 T).$$

For the Dedekind ζ-function of a number field k of degree $n \geqq 2$ no comparable result is known, to give one being essentially equivalent in difficulty to sharply bounding $\int_{-T}^{T}|\zeta(\frac{1}{2} + it)|^{4n}\, dt$. However, for $k = \mathbb{Q}(\sqrt{-1})$, Sarnak [26] has recently given the following sharp average version of Hardy and Littlewood's estimate:

$$\sum_{|m| \leqq T} \int_{-T}^{T} \left|\zeta\left(\tfrac{1}{2} + it, \lambda^m\right)\right|^4 dt \ll T^2 \log^4 T \tag{1.2}$$

as $T \to \infty$, where, for $m \in \mathbb{Z}$, λ^m is the "Grössencharakter" mod 1 defined for an ideal $\mathfrak{a} = (\alpha)$ by $\lambda^m(\mathfrak{a}) = (\alpha/|\alpha|)^{4m}$. His method is a development of one first employed by Titchmarsh in 1928 for the Riemann ζ-function (see [28], p. 143) and then by Potter [23] in 1940 for the mean square of Dirichlet series of signature à la Hecke. It is based on the transformation properties of an associated automorphic form, specifically the derivative of an Eisenstein series on hyperbolic 3-space. This method avoids the use of approximate functional equations and the large sieve for Grössencharaktere, which yield for general k the estimate (see [6])

$$\sum_{|m| \leqq T} \int_{-T}^{T} \left|\zeta\left(\tfrac{1}{2} + it, \chi\lambda^m\right)\right|^4 dt \ll_{k,\mathfrak{q}} T^n \log^A T \tag{1.3}$$

as $T \to \infty$, where $\chi\lambda^m$ for $m \in \mathbb{Z}^{n-1}$ is a certain Grössencharakter mod $\mathfrak{q}$. Here and throughout A denotes a positive but unspecified constant. As remarked in

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[26] it should be possible to extend Sarnak's method to general k, giving in (1.3) the sharp bound $T^n \log^4 T$.

My object here is to extend this method in a different direction which includes functions attached to certain newforms which are *not of complex multiplication type*. In order to simplify algebraic aspects we shall restrict our attention in this application to forms on $\Gamma_0(2)$, but expect our result to extend to $\Gamma_0(N)$ for $N \in \mathbb{Z}^+$ at least for N not *a square* by using results from [21]. To state this, let, for $k \in 2\mathbb{Z}^+$, $\mathcal{N}_k$ denote the set of normalized newforms of weight k on $\Gamma_0(2)$ (see [2] for definitions and basic results). If $f \in \mathcal{N}_k$ has the Fourier expansion $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$, where $e(z) = \exp\{2\pi i z\}$, then the L-function attached to f is given for $\mathcal{Re}(s) > \frac{1}{2}(k+1)$ by the absolutely convergent Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

$L_f(s)$ has the Euler product expansion (see [2], p. 151)

$$L_f(s) = (1 \pm 2^{k/2-1-s})^{-1} \prod_{p>2} (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}, \tag{1.4}$$

also for $\mathcal{Re}(s) > \frac{1}{2}(k+1)$, and continues to an entire function of order 1 which satisfies the functional equation

$$\xi_f(s) = (\sqrt{2}\pi)^{-s} \Gamma(s) L_f(s) = \pm(-1)^{k/2} \xi_f(k-s), \tag{1.5}$$

where the $\pm$'s in (1.4) and (1.5) correspond (see [2], p. 149). Hence the "critical line" for $L_f(s)$ is $\mathcal{Re}(s) = \frac{1}{2}k$. In Section 3 we shall prove

THEOREM 1. *As* $T \to \infty$,

$$\sum_{\substack{8 \le k \le T \\ k \text{ even}}} \sum_{f \in \mathcal{N}_k} \int_{-T}^{T} \left| L_f(\tfrac{1}{2}k + it) \right|^4 dt \ll T^3 \log^4 T$$

with the implied constant absolute.

It is remarked that for a single L-function the methods of Chandrasekharan and Narasimhan [4] would give $\int_{-T}^{T} |L_f(\frac{1}{2}k + it)|^4 \, dt \ll T^2 \log^A T$ as $T \to \infty$ for some $A > 0$. Thus Theorem 1 gives an improvement to $T \log^4 T$ "on average" for $k \le T$, since the number of nonzero integrals being summed is $\gg T^2$. In fact it is sharp since a similar lower bound can be obtained. As above, the uniformity difficulties associated with the use of approximate functional equations and a large sieve inequality, in this case for the Fourier coefficients (see [5]), are avoided by the Fourier analysis of a certain Maass form restricted to a "cone" in hyperbolic 5-space which leads to Theorem 1. This analysis is carried out more generally in the next section.

Before proceeding to this we shall briefly reveal the identity of the first newform on $\Gamma_0(2)$. This occurs for weight $k = 8$. We refer to Rankin's book [25], § 10.3, for the details. We have $\mathscr{N}_8 = \{F_8^*\}$, where

$$F_8^*(z) = e(z)\prod_{n=1}^{\infty}(1 - e(nz) - e(2nz) + e(3nz))^8 \tag{1.6}$$
$$= \sum_{n=1}^{\infty}\Theta(n)e(nz)$$

in honor of Glaisher, who in 1907 first studied $\Theta(n)$ in connection with the number of representations $r_{16}(n)$ of $n \in \mathbb{Z}^+$ as the sum of 16 integral squares. In fact, for $n \in \mathbb{Z}^+$,

$$r_{16}(n) = \tfrac{32}{17}\left[\beta(n) + 16(-1)^{n-1}\Theta(n)\right],$$

where

$$\beta(n) = \begin{cases}\sigma_7(n) & \text{if } n \text{ is odd},\\ \sum_{d|n}(-1)^d d^7 & \text{if } n \text{ is even}.\end{cases}$$

Also, $\Theta(2) = -8$ from (1.6) so that the $+$ sign holds in (1.4) and (1.5). In passing we note that an analysis of the trace of a certain representation occurring in Section 3 below leads to the remarkable formula

$$\Theta(n) = \left(16 + 8(-1)^n\right)^{-1}$$
$$\cdot \sum_{m\in\mathbb{Z}} r_3(n - m^2)\left[(2m)^6 - 5(2m)^4 n + 6(2m)^2 n^2 - n^3\right],$$

where $r_3(n)$ is the number of representations of $n \geqq 0$ as the sum of 3 integral squares.

2. The Mean Square of Certain Dirichlet Series

In this section a general mean square estimate will be given which will be applied in the next section to prove Theorem 1. The Dirichlet series to be considered are analogous to those introduced by Hecke in 1936 ([11], p. 591, see also [12]). Here one says a function $\phi(s)$ *has signature* $\{\lambda, k, \gamma\}$ for $\lambda, k > 0$ and $\gamma = \pm 1$ if

(2.1)

(i) $\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ for some $s \in \mathbb{C}$,

(ii) $(s - k)\phi(s)$ is entire of finite genus,

(iii) $R(s) = (2\pi/\lambda)^{-s}\Gamma(s)\phi(s) = \gamma R(k - s)$.

Soon after their introduction, Potter [23] showed that for $\phi(s)$ with signature $\{\lambda, k, \gamma\}$ such that

$$\sum_{n \leqq x} |a(n)|^2 \sim bx^k \log^c x$$

for some $b, c \geqq 0$ we have that, as $T \to \infty$,

$$\int_0^T \left|\phi\left(\frac{1}{2}k + it\right)\right|^2 dt \sim \frac{2kb}{c+1} T \log^{c+1} T. \tag{2.2}$$

We shall establish a version of this result for a class of functions which have two gamma factors in their functional equations, at the expense of averaging over certain "twists." Actually we will give only an upper bound, not an asymptotic result. Nevertheless, the analysis shows that under appropriate assumptions this bound is sharp, and with further work an asymptotic result should be obtainable by the same method.

Let $\Lambda \subset \mathbb{R}^n$ for $n > 1$ be a full lattice and $r \in \mathbb{R}$. Say $\phi(s)$ *has signature* $\langle \Lambda, n, r \rangle$ if

(2.3)

(i) $\phi(s) = \sum_{\beta \in \Lambda}' a(\beta)|\beta|^{-2s}$ for some $a(\beta)$, $s \in \mathbb{C}$,

(ii) $(s - \frac{1}{2}(n + ir))(s - \frac{1}{2}(n - ir))\phi(s)$ is entire of finite genus and, for any degree m spherical harmonic P_m on S^{n-1}, the twist of $\phi(s)$ by P_m,

$$\phi(s, P_m) = \sum_{\beta \in \Lambda}' a(\beta) P_m(\beta/|\beta|)|\beta|^{-2s},$$

is entire of finite genus for $m \geqq 1$,

(iii) for any P_m with $m \geqq 0$,

$$R(s, P_m) = \pi^{-2s}\Gamma(s + \tfrac{1}{2}(m + ir))\Gamma(s + \tfrac{1}{2}(m - ir))\phi(s, P_m)$$

$$= (-1)^m R(\tfrac{1}{2}n - s, P_m'),$$

where

$$P_m'((\omega_1, \cdots, \omega_n)) = P_m((\omega_1, -\omega_2, \cdots, -\omega_n)).$$

Thus $\mathscr{Re}(s) = \frac{1}{4}n$ is the "critical line" for each $\phi(s, P_m)$. Here as usual a prime in a sum indicates that the term with zero argument should be omitted.

In order to state our result we first need to discuss the spherical harmonics P_m. These are eigenfunctions of the Laplace-Beltrami operator on S^{n-1} with eigenvalues $\lambda_m = \lambda(P_m)$ given by $-\lambda_m = \tau^2 - \frac{1}{4}(n-2)^2$, where $\tau = m + \frac{1}{2}n - 1$. The dimension of the eigenspace $\mathscr{P}_m$ of λ_m is given by (see [30], p. 445)

$$(2.4)\quad d_m = \frac{(2m+n-2)\Gamma(m+n-2)}{\Gamma(m+1)\Gamma(n-1)} \sim \frac{2}{(n-2)!} m^{n-2} \quad \text{as} \quad m \to \infty.$$

Denote by $\sum^*_{\mathscr{P}_m}$ the sum over any orthonormal basis $\{P_m\}$ of $\mathscr{P}_m$ with respect to the inner product of $L^2(d\omega)$, where $d\omega$ is the usual invariant measure on S^{n-1} normalized so that $\omega(S^{n-1}) = 1$. We now give the analogue of Potter's result.

THEOREM 2. *Suppose $\phi(s)$ has signature $\langle \Lambda, n, r \rangle$ and that, for some $c \geq 0$,*

$$(2.5)\quad \sum_{|\beta| \leq x}{}' |a(\beta)|^2 \ll x^n \log^c x$$

as $x \to \infty$. Suppose that n is even. Then, as $T \to \infty$,

$$\sum_{0 < m \leq T} \sum_{\mathscr{P}_m}{}^* \int_{-T}^{T} \left|\phi\left(\tfrac{1}{4}n + it, P_m\right)\right|^2 dt \ll T^n \log^{c+1} T,$$

where the implied constants depend only on $\phi(s)$.

Note that by (2.4) the number of integrals being summed is $\gg_n T^{n-1}$ so that "on average," for $0 < m \leq T$,

$$\int_{-T}^{T} \left|\phi\left(\tfrac{1}{4}n + it, P_m\right)\right|^2 dt \ll T \log^{c+1} T \quad \text{as} \quad T \to \infty.$$

In case $n = 2$, Theorem 2 includes (essentially) Sarnak's result (1.2). Since parts of the proof are similar we shall be brief at times, referring then to [26]. Finally we remark that the condition that n be even is only imposed to simplify the proof of Lemma 1 and is not essential.

Proof: Functions with signature $\langle \Lambda, n, r \rangle$ were first considered by Maass [20] in 1949. In this paper a non-holomorphic, $(n+1)$-dimensional version of Hecke's correspondence is given for such functions. To describe this it is convenient to introduce the Clifford algebra $\mathscr{C}_{n+1}$, which is the associative algebra over $\mathbb{R}$ generated by n elements $i_1, \cdots, i_n$ subject to the relations $i_h i_k = -i_k i_h$ for $k \neq h$, $i_h^2 = -1$, and no others (we will use the notation of [1]). $\mathscr{C}_{n+1}$ is a real vector space of dimension 2^n. The $(n+1)$-dimensional subspace

$$V^{n+1} = \{x = x_0 + x_1 i_1 + \cdots + x_n i_n;\ x_i \in \mathbb{R}\}$$

contains $H^{n+1} = \{x \in V^{n+1};\ x_n > 0\}$, a model for hyperbolic $(n+1)$-space when endowed with the metric $ds^2 = |dx|^2 x_n^{-2}$. After Vahlen [29] (see also [1]) any proper isometry γ of H^{n+1} acts by $\gamma x = (ax+b)(cx+d)^{-1}$ for $a, b, c, d \in \mathscr{C}_n \subset \mathscr{C}_{n+1}$ suitably chosen. The Laplace-Beltrami operator is given by

$$\Delta_{n+1} = x_n^{n+1} \sum_{h=0}^{n} \frac{\partial}{\partial x_h}\left(x_n^{1-n} \frac{\partial}{\partial x_h} \right).$$

Consider Λ as a lattice in V^n and let $\Lambda' \subset V^n \subset V^{n+1}$ be its dual lattice with respect to the inner product $\mathscr{R}e(\bar{x}y)$. Here $\bar{x} = x_0 - x_1 i_1 - \cdots - x_{n-1} i_{n-1}$ if $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$. An analogue of Hecke's triangle group for $n > 1$ is the group of isometries Γ of H^{n+1} generated by $x \to x + \alpha$ for all $\alpha \in \Lambda'$ and $x \to -x^{-1}$ (in $\mathscr{C}_{n+1}$). Say $f \in C^2(H^{n+1})$ is an *automorphic function for* Γ if

$$\text{(2.6)}\quad \begin{aligned} &\text{(i)} \quad \Delta_{n+1} f + \left(r^2 + \frac{n^2}{4} \right) f = 0 \quad \text{for some } r \in \mathbb{R}, \\ &\text{(ii)} \quad f(x) \ll x_n^A \quad \text{as} \quad x_n \to \infty \quad \text{and} \quad f(x) \ll x_N^{-A} \\ &\qquad \text{as} \quad x_n \to 0^+ \text{ uniformly in } x_0, \cdots, x_{n-1}, \\ &\text{(iii)} \quad f(\gamma x) = f(x) \quad \text{for all } \gamma \in \Gamma. \end{aligned}$$

By separation of variables in (2.6)(i) and (ii), (iii) we get the Fourier expansion

$$\text{(2.7)}\qquad f(x) = u(x_n) + \sum_{\beta \in \Lambda}{}' a(\beta) x_n^{n/2} K_{ir}(2\pi|\beta|x_n) e(\mathscr{R}e(\bar{\beta}x)),$$

where

$$\text{(2.8)}\qquad u(x_n) = \begin{cases} a_1 x_n^{n/2+ir} + a_2 x_n^{n/2-ir} & \text{if } r \neq 0, \\ x_n^{n/2}(a_1 + a_2 \log x_n) & \text{if } r = 0, \end{cases}$$

and K_{ir} is the usual Bessel function.

Suppose $a(\beta)$ in (2.7) comes from that in (2.3)(i) and that a_1 and a_2 in (2.8) are determined by the condition that

$$\phi(s,1) - \frac{2a_1}{\left(s - \frac{1}{2}(n+ir)\right) M_r} - \frac{2a_2}{\left(s - \frac{1}{2}(n-ir)\right) M_{-r}}$$

or

$$\text{(2.9)}\qquad \begin{aligned} &\phi(s,1) - \left[\frac{2a_1}{M_0} - \frac{2a_2}{M_0}\left(\frac{\Gamma'(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)} - \log \pi \right) \right] \left(s - \frac{1}{2}n \right)^{-1} \\ &\quad - \frac{a_2}{\left(s - \frac{1}{2}n\right)^2 M_0} \end{aligned}$$

is entire in s, according to whether $r \neq 0$ or $r = 0$, respectively. Here $M_r =$

$\pi^{-n-ir}\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}n+ir)$. Maass shows in [20] that the map $\phi(s)\to f(x)$ so defined gives an isomorphism from the linear space of functions with signature $\langle\Lambda, n, r\rangle$ onto the linear space of automorphic functions for Γ, provided $n>1$.

Define, for $x\in H^{n+1}$,

$$f^*(x)=\begin{cases} f(x)-u(x_n(\gamma x)) & \text{if there is a } \gamma\in\Gamma \text{ s.t. } x_n(\gamma x)>1,\\ f(x) & \text{otherwise.}\end{cases} \tag{2.10}$$

Thus $f^*(x)$ is Γ-invariant. Introduce as in [26] the "conical" coordinates

$$x_n=\rho\cos\theta,\qquad x-i_nx_n=\omega\rho\sin\theta, \tag{2.11}$$

where $\rho>0$, $0\leqq\theta<\frac{1}{2}\pi$, and $\omega\in S^{n-1}\subset V^n$. Consider, for $m\geqq 0$, the integral

$$G(s,\theta,P_m)=\int_0^\infty\int_{S^{n-1}} f^*(\rho,\theta,\omega)\overline{P}_m(\omega)\,d\omega\,\rho^s\frac{d\rho}{\rho}, \tag{2.12}$$

absolutely convergent for $\mathscr{Re}(s)$ sufficiently large. Substituting for f^* its Fourier expansion from (2.7) and (2.10) we get after interchanging the sum and integral that, for $m>0$,

$$\begin{aligned} G(s,\theta,P_m)&=\cos^{n/2}\theta\sum_{\beta\in\Lambda}{}' a(\beta)\int_0^\infty \rho^{n/2+s}K_{ir}(2\pi|\beta|\rho\cos\theta)\\ &\quad\cdot\int_{S^{n-1}}\overline{P}_m(\omega)e(\rho\sin\theta\langle\omega,\beta\rangle)\,d\omega\frac{d\rho}{\rho},\end{aligned} \tag{2.13}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$ and we are using the fact that $\int_{S^{n-1}}\overline{P}_m(\omega)\,d\omega=0$. By [13], p. 25, the inner integral may be evaluated to give

$$\begin{aligned} G(s,\theta,P_m)&=\frac{2\pi i^m\cos^{n/2}\theta}{\Omega_n(\sin\theta)^{n/2-1}}\sum_{\beta\in\Lambda}{}' a(\beta)\overline{P}_m\left(\frac{\beta}{|\beta|}\right)|\beta|^{1-n/2}\\ &\quad\cdot\int_0^\infty\rho^{s+1}K_{ir}(2\pi|\beta|\rho\cos\theta)J_\tau(2\pi|\beta|\rho\sin\theta)\frac{d\rho}{\rho},\end{aligned} \tag{2.14}$$

where $\tau=m+\frac{1}{2}n-1$ as on page 819 and $\Omega_n=2\pi^{n/2}\Gamma(\frac{1}{2}n)^{-1}$ is the area of S^{n-1}. Thus

$$G(s,\theta,P_m)=M(s,\theta)\phi(\tfrac{1}{2}s+\tfrac{1}{4}n,\overline{P}_m), \tag{2.15}$$

where

$$\begin{aligned} M(s,\theta)&=M(s,\theta;m,n,r)\\ &=i^m2^{-s-1}\pi^{-s-n/2}\Gamma(\tfrac{1}{2}n)(\sin\theta)^{1-n/2}(\cos\theta)^{n/2-s-1}H(s,\theta)\end{aligned}$$

with

$$H(s,\theta)=H(s,\theta;\tau,r)=\int_0^\infty\rho^sK_{ir}(\rho)J_\tau(\rho\tan\theta)\,d\rho. \tag{2.15$'$}$$

By Mellin inversion we get from (2.12), for some c,

(2.16)

$$\int_{S^{n-1}} f^*(\rho, \theta, \omega)\bar{P}_m(\omega)\, d\omega = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s,\theta)\phi\left(\tfrac{1}{2}s + \tfrac{1}{4}n, \bar{P}_m\right)\rho^{-s}\, ds.$$

For $m > 0$, by (2.3)(iii), we may move the contour to the left in (2.16) giving

$$\int_{S^{n-1}} f^*(\rho, \theta, \omega)\bar{P}_m(\omega)\, d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} M(2it, \theta)\phi\left(\frac{1}{4}n + it, \bar{P}_m\right)\rho^{-2it}\, dt.$$

By Parseval's relation on $\mathbb{R}^+$ and S^{n-1} we get

(2.17)

$$\sum_{m>0} \sum_{\mathscr{P}_m}^{*} \int_{-\infty}^{\infty} |M(2it,\theta)|^2 \left|\phi\left(\frac{1}{4}n + it, P_m\right)\right|^2 dt \ll \int_0^{\infty}\int_{S^{n-1}} |f^*(\rho,\theta,\omega)|^2\, d\omega \frac{d\rho}{\rho}.$$

Now $M(s, \theta)$ may be written in terms of the hypergeometric function as in [26], p. 172 (see [9], p. 693)

$$M(s,\theta) = \frac{1}{4} i^m \pi^{-s-n/2} (\tan\theta)^m (\cos\theta)^{-s} \cdot \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n + m)} \Gamma(a)\Gamma(b) F\left(a, b; \frac{1}{2}n + m, -\tan^2\theta\right),$$

for $\mathscr{R}e(s) > -m - \frac{1}{2}n$, where

$$a = \tfrac{1}{2}\left(m + \tfrac{1}{2}n + s + ir\right) \quad \text{and} \quad b = \tfrac{1}{2}\left(m + \tfrac{1}{2}n + s - ir\right).$$

If $r \neq 0$, we may proceed as in [26], p. 172 to transform $-\tan^2\theta \to (1 + \tan^2\theta)^{-1} = \cos^2\theta$ and then estimate M from below. Since this initial transformation is not valid if $r = 0$, the case required for Theorem 1, we shall estimate the integral $H(s, \theta)$ in (2.15′) using the method of stationary phase. We remark that here is the only place where we use the assumption in Theorem 2 that n be even.

LEMMA 1. *Let $T = \tan\theta$ and suppose $\tau = xT$ and $t = yT$ for $x, y \in [A, 2A]$ and for some $A > 0$. Then, as $T \to \infty$,*

$$|H(2it, \theta)| \sim T^{-1} K_{ir}\left(\sqrt{x^2 + 4y^2}\right).$$

Here $\tau = \frac{1}{2}n + m - 1$ is assumed integral.

Proof: Use the representation (see [9], p. 952)

$$J_\tau(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{-i\tau v + iz\sin v\}\, dv$$

in (2.15′) and change variables to get

$$H(2it,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} g(u)\exp\{iTh(u,v)\}\, du\, dv,$$

where $g(u) = e^u K_{ir}(e^u)$ and $h(u,v) = ye^u \sin v - xv + 2uy$. There is a single simple stationary point inside the domain of integration at

$$(u_0, v_0) = \left(\tfrac{1}{2}\log(x^2+4y^2), -\tan^{-1}(2y/x)\right), \qquad v_0 \in \left(-\tfrac{1}{2}\pi, 0\right),$$

which is a saddle point since

$$\begin{vmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{vmatrix} = -e^{2u} < 0.$$

Also, on the boundary $v = \pm\pi$ the normals to the level curves $h(u,v) = c$ have u-component $2y \neq 0$ so the level curves do not become tangent to the boundary. By the method of stationary phase for multiple integrals as developed by Jones and Kline [16], p. 19 (see also [3], p. 347) Lemma 1 follows.

Returning to (2.17), by Lemma 1 and (2.15),

(2.18)

$$\sum_{AT \leq m \leq 2AT} \sum_{\mathscr{P}_m}^{*} \int_{AT}^{2AT} \left|\phi\left(\frac{1}{4}n + it, P_m\right)\right|^2 dt \ll_{n,r} T^n \int_0^\infty \int_{S^{n-1}} |f^*|^2\, d\omega \frac{d\rho}{\rho},$$

using the fact that $K_{ir}(z) \gg 1$ for $z \in [\sqrt{5}A, 2\sqrt{5}A]$ with suitable $A > 0$ depending only on r. Thus to complete the proof of Theorem 2 we need to show that, as $\theta \to \frac{1}{2}\pi^-$,

$$\int_0^\infty \int_{S^{n-1}} |f^*(\rho,\theta,\omega)|^2\, d\omega \frac{d\rho}{\rho} \ll \log^{c+1}(\sec\theta). \tag{2.19}$$

Now

$$\int_0^\infty \int_{S^{n-1}} |f^*|^2\, d\omega \frac{d\rho}{\rho} = \left(\int_0^1 + \int_1^\infty\right)\int_{S^{n-1}} |f^*|^2\, d\omega \frac{d\rho}{\rho}$$

$$= \int_1^\infty \int_{S^{n-1}} \left(|f^*(\rho,\theta,\omega)|^2 + |f^*(\rho,\theta,-\omega)|^2\right) d\omega \frac{d\rho}{\rho},$$

in view of the fact that, as $x \to -x^{-1}$, we have $\rho \to \rho^{-1}$, $\theta \to \theta$, and $\omega \to -\omega$, together with the invariance of f^*. By Cauchy's inequality,

$$\int_1^\infty \int_{S^{n-1}} |f^*|^2 \, d\omega \frac{d\rho}{\rho} \ll \int_1^\infty \int_{S^{n-1}} |f(\rho, \theta, \omega) - u(\rho \cos\theta)|^2 \, d\omega \frac{d\rho}{\rho}$$

since

$$\int_1^\infty |u^*(\rho\cos\theta)|^2 \frac{d\rho}{\rho} \ll \cos^n\theta \int_1^{\sec\theta} \rho^{n-1}\, d\rho \ll 1 \quad \text{as} \quad \theta \to \tfrac{1}{2}\pi^-,$$

where $f^*(x) = u^*(x_n) + f(x) - u(x_n)$. Thus, by (2.7), (2.19) follows from

LEMMA 2. *Assuming* (2.5), *as* $\theta \to \frac{1}{2}\pi^-$,

$$\int_1^\infty \int_{S^{n-1}} \left| \sum_{\beta\in\Lambda}{}' a(\beta)(\rho\cos\theta)^{n/2} K_{ir}(2\pi|\beta|\rho\cos\theta) e(\rho\sin\theta\langle\omega, \beta\rangle) \right|^2 d\omega \frac{d\rho}{\rho}$$

$$= A \log^{c+1}(\sec\theta) + O(\log^c(\sec\theta)).$$

Proof: After squaring and interchanging the integral and sums we get

$$\sum_{\alpha,\beta\in\Lambda}{}' a(\alpha)\overline{a(\beta)} \int_1^\infty K_{ir}(2\pi|\alpha|\rho\cos\theta) K_{ir}(2\pi|\beta|\rho\cos\theta)$$

$$\cdot \rho^n \cos^n\theta \int_{S^{n-1}} e(\rho\sin\theta\langle\omega, \alpha-\beta\rangle)\, d\omega \frac{d\rho}{\rho}$$

$$= \cos^n\theta \sum_{\alpha\in\Lambda}{}' |a(\alpha)|^2 \int_1^\infty K_{ir}^2(2\pi|\alpha|\rho\cos\theta)\rho^n \frac{d\rho}{\rho}$$

(2.20)
$$+ \pi^{1-n/2}\Gamma(\tfrac{1}{2}n) \frac{\cos^n\theta}{(\sin\theta)^{n/2-1}} \sum_{\alpha\neq\beta}{}' \frac{a(\alpha)\overline{a(\beta)}}{|\alpha-\beta|^{n/2-1}}$$

$$\cdot \int_1^\infty K_{ir}(2\pi|\alpha|\rho\cos\theta) K_{ir}(2\pi|\beta|\rho\cos\theta)$$

$$\cdot \rho^{n/2} J_{n/2-1}(2\pi\rho\sin\theta|\alpha-\beta|)\, d\rho \quad \text{(using, [13], p. 25 again)}$$

$$= I_1 + I_2.$$

We shall begin with I_2, for which we need the following convolution-type bilinear inequalities:

LEMMA 3. *For* $\{c_i\} \in l^2(\Lambda)$ *and* $A \in \mathbb{R}^+$,

(i) $\sum'_{\alpha \neq \beta \in \Lambda} c_1(\alpha)\bar{c}_2(\beta) J_{n/2}(A|\alpha - \beta|)|\alpha - \beta|^{-n/2} \ll \|c_1\| \|c_2\|$

and

(ii) $\sum'_{\alpha \neq \beta \in \Lambda} c_1(\alpha)\bar{c}_2(\beta)|\alpha - \beta|^{-n-1/2} \ll \|c_1\| c_2\|$.

Proof: This is proved in a similar manner as Lemma 4 in [26], using the fact that, for $B = \{y \in \mathbb{R}^n;\ |y| \leqq A\}$,

$$\hat{\chi}_B(\xi) = (2\pi)^{-n/2} \int_B \exp\{-i\langle y, \xi\rangle\}\, dy = (A/|\xi|)^{n/2} J_{n/2}(A|\xi|).$$

To continue with I_2, integrate by parts in (2.20) to get

$$\begin{aligned} I_2 = {} & \frac{\Gamma(\frac{1}{2}n)\cos^n\theta}{2(\pi \sin\theta)^{n/2}} \sum_{\alpha\neq\beta}' a(\alpha)\overline{a(\beta)} \\ & \cdot\Big\{ K_{ir}(2\pi|\alpha|\rho\cos\theta) K_{ir}(2\pi|\beta|\rho\cos\theta) \\ & \quad \cdot \rho^{n/2} J_{n/2}(2\pi\rho\sin\theta|\alpha-\beta|)|\alpha-\beta|^{-n/2}\Big|_1^\infty \\ & \quad - \int_1^\infty \frac{d}{d\rho}\left[K_{ir}(2\pi|\alpha|\rho\cos\theta) K_{ir}(2\pi|\beta|\rho\cos\theta)\right] \\ & \quad \cdot \rho^{n/2} J_{n/2}(2\pi\rho\sin\theta|\alpha-\beta|)|\alpha-\beta|^{-n/2}\, d\rho \Big\} \\ = {} & I_{21} + I_{22}. \end{aligned}$$

Now

$$\begin{aligned} I_{21} = {} & \frac{\Gamma(\frac{1}{2}n)\cos^n\theta}{2(\pi\sin\theta)^{n/2}} \sum_{\alpha\neq\beta}' a(\alpha)\overline{a(\beta)}\, K_{ir}(2\pi|\alpha|\cos\theta) \\ & \cdot K_{ir}(2\pi|\beta|\cos\theta) J_{n/2}(2\pi\sin\theta|\alpha-\beta|)|\alpha-\beta|^{-n/2} \\ \ll {} & \cos^n\theta \sum_{\alpha\in\Lambda}' |a(\alpha)|^2 K_{ir}^2(2\pi|\alpha|\cos\theta) \end{aligned}$$

by Lemma 3(i), so by (2.5), as $\theta \to \frac{1}{2}\pi^-$, $I_{21} \ll \log^c(\sec\theta)$.

We next integrate by parts each term which results from differentiating

$$\rho^{-ir} K_{ir}(2\pi|\alpha|\rho\cos\theta)\rho^{ir} K_{-ir}(2\pi|\beta|\rho\cos\theta)$$

in I_{22} again, estimating each boundary term that arises by Lemma 3(i) (after embedding in $\mathbb{R}^{n+2}$). We continue this process until we may apply Lemma 3(ii). A typical integral term will then be bounded by

$$A(\cos\theta)^{n+l}\sum_{\alpha\neq\beta}{}' \frac{|a(\alpha)a(\beta)|}{|\alpha-\beta|^{n/2+l-1/2}}|\alpha|^{l_1}|\beta|^{l_2}$$

$$\cdot\int_1^\infty \rho^{(n-1)/2}\left|K_{ir+l_1}(2\pi|\alpha|\rho\cos\theta)K_{-ir+l_2}(2\pi|\beta|\rho\cos\theta)\right|d\rho$$

if one uses the fact that $J_{n/2+l-1}(z)\ll|z|^{-1/2}$ as $z\to\infty$. Here $l=l_1+l_2$ is the number of integrations by parts performed. This is seen to be

$$\ll\cos^n\theta\sum_{\substack{\alpha\neq\beta\\|\alpha|,|\beta|\leq\sec\theta}}\frac{|a(\alpha)a(\beta)|}{|\alpha-\beta|^{n/2+l-1/2}}\log(|\alpha|\cos\theta)\log(|\beta|\cos\theta)$$

using the easily established estimates

$$K_\nu(z)\ll\begin{cases}z^{-|\mathscr{Re}(\nu)|} & \text{if } \nu\neq 0,\\ -\log z & \text{if } \nu=0,\end{cases}\tag{2.21}$$

as $z\to 0^+$ (see [16], p. 108, for the relevant formulae). By Lemma 3(ii) each remaining integral term in I_2 is

$$\ll\cos^n\theta\sum_{|\alpha|\leq\sec\theta}{}'|a(\alpha)|^2\log^2(|\alpha|\cos\theta)$$

$$\ll\log^c(\sec\theta)\quad\text{as}\quad\theta\to\tfrac{1}{2}\pi^-,$$

by (2.5) and summation by parts.

Returning to (2.20) and I_1,

$$I_1=\cos^n\theta\sum_{\alpha\in\Lambda}{}'|a(\alpha)|^2\int_1^\infty K_{ir}^2(2\pi|\alpha|\rho\cos\theta)\rho^n\frac{d\rho}{\rho}$$

$$=\sum_\alpha{}'\frac{|a(\alpha)|^2}{|\alpha|^n}\int_{|\alpha|\cos\theta}^\infty K_{ir}^2(2\pi\rho)\rho^n\frac{d\rho}{\rho}\tag{2.22}$$

$$=\sum_{|\alpha|\leq\sec\theta}{}'|a(\alpha)|^2|\alpha|^{-n}\int_1^\infty K_{ir}^2(2\pi\rho)\rho^n\frac{d\rho}{\rho}$$

$$+O(\log^c(\sec\theta))\quad\text{by}\quad(2.5).$$

Lemma 2 follows after summation by parts in (2.22).

This completes the proof of Theorem 2.

3. The Proof of Theorem 1

In order to deduce Theorem 1 from Theorem 2 we shall first construct an appropriate function $\phi(s)$ with signature $\langle\Lambda,4,0\rangle$. Here $\Lambda=\sqrt{2}\,\mathcal{O}$, where $\mathcal{O}$ is the (maximal) order in the Hamiltonian quaternions over $\mathbb{Q}$ generated over $\mathbb{Z}$ by $\{i,j,k,\frac{1}{2}(1+i+j+k)\}$, thought of as a full lattice in $\mathbb{R}^4$. Let, for $\mathscr{Re}(s)$ large,

$$(3.1)\qquad \phi(s)=2^{-s}\left(\frac{1}{24}\sum_{\alpha\in\mathcal{O}}{}'N(\alpha)^{-s}\right)^2=\frac{1}{24}\sum_{\beta\in\Lambda}{}'d(\beta/\sqrt{2})|\beta|^{-2s},$$

where $N(\alpha)=\alpha\bar{\alpha}$ and $d(\alpha)=\frac{1}{24}\#\{(\delta,\gamma)\in\mathcal{O}^2;\ \delta\gamma=\alpha\}$ is the divisor function for $\mathcal{O}$.

Lemma 4. *$\phi(s)$ has signature $\langle\Lambda,4,0\rangle$.*

Proof: (2.3)(i) follows from (3.1). Next, by a well known variant of Jacobi's theorem on sums of four squares we have

$$\phi(s)=2^{-s}\left(\sum_{n=1}^{\infty}\bar{\sigma}(n)n^{-s}\right)^2,$$

where $\bar{\sigma}(n)=\sum_{d|n,\ d\text{ odd}}d$. Thus

$$(3.2)\qquad \phi(s)=2^{-s}(1-2^{1-s})^2\zeta^2(s)\zeta^2(s-1),$$

showing the first part of (2.3)(ii).

To discuss twists of $\phi(s)$ by spherical harmonics we introduce the unique irreducible unitary $(2l+1)\times(2l+1)$ representation of SU(2) given explicitly as a matrix T_l in [30], p. 115, where in general $l\in\{0,\frac{1}{2},1,\cdots\}$. The entries in $\sqrt{2l+1}\,T_l$ constitute an orthonormal basis of $\mathscr{P}_{2l}$ (Recall the discussion near (2.4) and see p. 160 of [30]). We may identify SU(2) with $S^3\subset V^4$. To show the second part of (2.3)(ii) first notice that, for all of the above l,

$$(3.3)\qquad 2^{-s}\left(\frac{1}{24}\sum_{\alpha\in\mathcal{O}}{}'T_l(\alpha/|\alpha|)N(\alpha)^{-s}\right)^2=\frac{1}{24}\sum_{\beta\in\Lambda}{}'d(\beta/\sqrt{2})T_l(\beta/|\beta|)|\beta|^{-2s}$$

$$=\phi(s,T_l),$$

where in fact $\phi(s,T_l)=0$ unless $l\in\mathbb{Z}$. By [27], p. 69, each entry in $\phi(s,T_l)$ is entire of finite genus if $l>0$ and, furthermore, if $R(s,T_l)=\pi^{-2s}\Gamma^2(s+l)\phi(s,T_l)$, then $R(s,T_l)=(-1)^{2l}R(2-s,T_l')$ for all l. Thus (2.3)(iii) holds as well, proving Lemma 4.

We shall now apply Theorem 2.

LEMMA 5. *$\sum_{0<l\leq T,\, l\in\mathbf{Z}}(2l+1)\int_{-T}^{T}\|\phi(1+it,T_l)\|^2\,dt \ll T^4\log^4 T$ as $T\to\infty$, where $\|M\|^2=\operatorname{tr}(MM^*)$ for any square matrix M. The implied constant is absolute.*

By Lemma 4 and Theorem 2, Lemma 5 follows from

LEMMA 6. *For $d(\alpha)$ defined in (3.1), as $x\to\infty$,*

$$\sum_{\substack{N(\alpha)\leq x\\ \alpha\in\mathcal{O}}}{}' d^2(\alpha) \sim Ax^2\log^3 x.$$

Proof: Lemma 6 follows easily from the identity

$$\tfrac{1}{24}\sum_{\alpha\in\mathcal{O}}{}' d^2(\alpha)\,N(\alpha)^{-s} = \zeta_{\mathcal{O}}^4(s)/\zeta_{\mathcal{O}}(2s), \tag{3.4}$$

where $\zeta_{\mathcal{O}}(s)=\frac{1}{24}\sum'_{\alpha\in\mathcal{O}}N(\alpha)^{-s}$, both for $\mathcal{Re}(s)>2$. The proof of (3.4) is computationally much more involved than its rational prototype, $\sum_{n>0}d^2(n)n^{-s}=\zeta^4(s)/\zeta(2s)$. As before,

$$\zeta_{\mathcal{O}}(s)=\sum_{n>0}\bar{\sigma}(n)n^{-s}=(1-2^{1-s})\zeta(s)\zeta(s-1)$$

so, for $\mathcal{Re}(s)>2$,

$$\zeta_{\mathcal{O}}(s)=\prod_{p\text{ prime}}\frac{1}{24}\left(\sum_{m\geq 0}\sum_{\substack{N(\rho)=p^m\\ \rho\in\mathcal{O}}}N(\rho)^{-s}\right)$$

and

$$\frac{1}{24}\sum_{\alpha\in\mathcal{O}}{}' d(\alpha)\,N(\alpha)^{-s}=\prod_{p}\frac{1}{24}\left(\sum_{m\geq 0}\sum_{N(\rho)=p^m}d(\rho)\,N(\rho)^{-s}\right).$$

It follows from these formulae and an elementary factorization property of integral quaternions (Proposition 1.3, p. 12. in [17]) that

$$\frac{1}{24}\sum_{\alpha\in\mathcal{O}}{}' d^2(\alpha)\,N(\alpha)^{-s}=\prod_{p}\frac{1}{24}\left(\sum_{m\geq 0}\sum_{\substack{N(\rho)=p^m\\ \rho\in\mathcal{O}}}d^2(\rho)\,N(\rho)^{-s}\right);$$

thus we are reduced to showing that

$$(3.5)\qquad \frac{1}{24}\sum_{m\geqq 0}\sum_{\substack{N(\rho)=p^m\\ \rho\in\mathcal{O}}} d^2(\rho)\,p^{-ms} = \begin{cases}(1-2^{-2s})(1-2^{-s})^{-4} & \text{if } p=2,\\ (1+p^{-s})(1-p^{1-2s})(1-p^{-s})^{-3}(1-p^{1-s})^{-4} & \text{if } p>2,\end{cases}$$

for p a rational prime.

For this we need a formula for $d(\rho)$. Let p^n be the largest odd integer dividing ρ. Then it can be shown using the Hurwitz decomposition theorem (see [14] p. 54.) that if $N(\rho)=p^{m+2n}$ then, for $m, n \geqq 0$,

$$(3.6)\qquad d(\rho) = 2(n+1)(1-p)^{-1} + (1-p^{n+1})(1-p)^{-2} \cdot [(1-p)(m+1)-2].$$

It also follows (see [14], p. 56.) that, for odd p,

$$(3.7)\qquad \frac{1}{24}\#\{\rho\in\mathcal{O};\ N(\rho)=p^{2n+m},\ p^n|\rho \text{ maximal}\} = \begin{cases}1 & \text{if } m=0,\\ (p+1)p^{m-1} & \text{if } m>0.\end{cases}$$

For $p=2$, (3.5) follows easily from the identity $\Sigma_{m\geqq 0}(m+1)^2x^m = (1-x^2)(1-x)^{-4}$ since in this case $n=0$ and $d(\rho)=m+1$.

For $p>2$ we write the left-hand side of (3.5) as

$$\frac{1}{24}\sum_{m\geqq 0}\sum_{n\geqq 0}\sum_{\substack{N(\rho)=p^{m+2n}\\ p^n|\rho \text{ maximal}}} d^2(\rho)\,p^{-(2n+m)s}$$

and apply (3.6) and (3.7). A rather long and tedious calculation then yields (3.5).

Define the (Brandt) matrices for $n\in\mathbb{Z}^+$ by

$$B_l(n) = \frac{1}{24}\sum_{\substack{N(\alpha)=n\\ \alpha\in\mathcal{O}}} T_l(\alpha/|\alpha|).$$

Thus, from (3.3),

$$(3.8)\qquad \phi(s,T_l) = 2^{-s}\left(\sum_{n=1}^{\infty} B_l(n)n^{-s}\right)^2.$$

LEMMA 7. (i) *For* $(m, n) = 1$, $B_l(m)B_l(n) = B_l(mn)$ *and*
(ii) $B_l(p^{k+1}) = B_l(p)B_l(p^k) - \chi_0(p)B_l(p^{k-1})$ *for p a prime*, $k \geqq 1$, *and*

$$\chi_0(p) = \begin{cases} 1, & p \neq 2, \\ 0, & p = 2. \end{cases}$$

Proof: This follows from Theorem 2, p. 106, of [7] or it may be proved directly as in the proofs of Theorems 5.4–5.5, p. 299 of [24].

It transpires that $B_l(n)$ form a commuting family of Hermitian (since T_l is unitary) matrices. Thus there is a *unitary* U s.t. $UB_l(n)U^{-1} = D(n)$ for all $n \in \mathbb{Z}^+$, where $D(n)$ is diagonal. By work of Eichler on the "basis problem" (see [21], Corollary 8, p. 696 and [7]), we see that the nonzero entries in $n^l D(n)$ are precisely the n^{th} Fourier coefficients of *all* newforms $f \in \mathcal{N}_k$, where $k = 2l + 2$, each counted exactly once. Of course, their multiplicativity properties follow from Lemma 7.

Thus from Lemma 5 we have

$$\sum_{\substack{8 \leq k \leq T \\ k \text{ even}}} (k-1) \int_{-T}^{T} \sum_{f \in \mathcal{N}_k} \left| L_f(k/2 + it) \right|^4 dt \ll T^4 \log^4 T, \tag{3.9}$$

since $\|\phi(1 + it, T_l)\|^2 = \|U\phi(1 + it, T_l)U^{-1}\|^2$ and, from (3.8),

$$U\phi(s, T_l)U^{-1} = 2^{-s}\left(\sum_{n=1}^{\infty} D(n) n^{-s} \right)^2.$$

Theorem 1 follows easily from (3.9).

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