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Properties of a Quadratic Fibonacci Recurrence

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Abstract: The terms of A000278, the sequence defined by $h_0 = 0$, $h_1 = 1$, and $h_{n+2} = h_{n+1} + h_n^2$, count the trees in certain recursively defined forests. We show that for *n* large, h_n is approximately $A^{sqrt(2)^n}$ for n even and h_n is approximately $B^{sqrt(2)^n}$ for n odd, with A,B > 1 and A not equal to B, and we give estimates of A and B: A is 1.436 ± .001 and B is 1.452 ± .001. The doubly exponential growth of the sequence is not surprising (see, for example, [AS]) but the dependence of the growth on the parity of the subscript is more interesting. Numerical and analytical investigation of similar sequences suggests a possible generalization of this result to a large class of such recursions.

Asymptotics

We study the growth of

$$h_{n+2} = h_{n+1} + (h_n)^2 \qquad (1)$$

as $n \to \infty$.

The sequence $\{h_n\}$ is bounded by double the square of its translate:

Lemma 1 If $n \geq 1$, then $0 < h_n \leq h_{n+1} \leq 2(h_n)^2$.

Proof This is true for $n = 1$. Since the sequence is increasing, $h_n \leq h_{n+1}$ always, and $h_{n+2} = h_{n+1} + (h_n)^2 \leq (h_{n+1})^2 + (h_{n+1})^2 = 2(h_{n+1})^2$. ■

Rewrite (1) in the following way for $n \geq 1$:

$$h_{n+2} = h_{n+1} + (h_n)^2 = (h_n)^2 \overbrace{\left(1 + \frac{h_{n+1}}{(h_n)^2}\right)}^{\alpha_n}.$$

The preceding lemma shows that $1 \leq \alpha_n \leq 3$ if $n \geq 1$. Then

$$h_{n+2} = (h_{n-2})^4(\alpha_{n-2})^2\alpha_n = (h_{n-4})^8(\alpha_{n-4})^4(\alpha_{n-2})^2\alpha_n = \ldots$$

which gives $h_{2n} = \prod_{j=0}^{n-1}(\alpha_{2n-2j-2})^{2^j}$ if n is a positive integer and α_0 is defined to be 1. More algebra is enlightening, beginning with reversing the product index:

$$h_{2n} = \prod_{j=0}^{n-1}(\alpha_{2j})^{2^{(n-j-1)}} = \exp\left(\sum_{j=0}^{n-1}\log\left((\alpha_{2j})^{2^{(n-j-1)}}\right)\right) = \exp\left(2^n \sum_{j=0}^{n-1}\frac{1}{2^{j+1}}\log\alpha_{2j}\right)$$

$$= \left(\exp\left(\sum_{j=0}^{n-1}\frac{1}{2^{j+1}}\log\alpha_{2j}\right)\right)^{2^n} = \left(\exp\left(\sum_{j=0}^{n-1}\frac{1}{2^{j+1}}\log\alpha_{2j}\right)\right)^{(\sqrt{2})^{2n}}.$$

Therefore if A is defined by requiring

$$\log A = \sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\log\alpha_{2j}\,, \qquad (2)$$

it seems plausible to expect that $h_{2n} \approx A^{(\sqrt{2})^{2n}}$.

A similar analysis for odd integers incorporates the initial condition $h_1 = 1$ and uses the formula $h_{2n+1} = \prod_{j=0}^{n-1}(\alpha_{2n-2j-1})^{2^j}$. This leads to defining B by the equation

$$\log B = \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \log \alpha_{2j+1}\,, \qquad (3)$$

and to the expectation that $h_{2n+1} \approx B^{(\sqrt{2})^{2n+1}}$.

Lemma 2 The series of non-negative constants defined in (2) and (3) converge, and all partial sums of the first N terms of each of them are within $\frac{\log 3}{2^N}$ of the actual sums. By considering partial sums for $N = 15$, we obtain an estimate for A (respectively, B) which is 1.436 (respectively, 1.451) with error less than .001.

Proof Since $0 \leq \log \alpha_n \leq \log 3$ the convergence of the geometric series $\sum \frac{1}{2^j}$ implies the convergence of both series shown. Then infinite tails of both series can be overestimated by $\frac{\log 3}{\text{powers of } 2}$. The numerical results are obtained by direct calculation. ■

Theorem 1 $\lim_{n\to\infty} \frac{h_{2n}}{A^{(\sqrt{2})^{2n}}} = 1$ and $\lim_{n\to\infty} \frac{h_{2n+1}}{B^{(\sqrt{2})^{(2n+1)}}} = 1$

Proof Consider the first limit. Unravel some algebra via

$$\frac{h_{2n}}{A^{(\sqrt{2})^{2n}}} = \frac{\exp\left(2^n \sum_{j=0}^{n-1} \frac{1}{2^{j+1}} \log \alpha_{2j}\right)}{\left(\exp\left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \log \alpha_{2j}\right)\right)^{2^n}} = \exp\left(-\sum_{j=n}^{\infty} 2^{n-j-1} \log \alpha_{2j}\right)$$

to discover that the desired result will follow an estimate which shows that the series $\sum_{j=n}^{\infty} 2^{n-j-1} \log \alpha_{2j} = 2^n \sum_{j=n}^{\infty} 2^{-j-1} \log \alpha_{2j}$ approaches 0 as $n \to \infty$. The estimation needs to be finer than what $0 \leq \log \alpha_{2j} \leq \log 3$ can provide.

First, $0 \leq \log \alpha_j \leq \frac{h_{j+1}}{(h_j)^2}$ because $\log(1+x) \leq x$ for $x \geq 0$. The series (2) and (3) both have positive terms. Since $h_{2j} = \exp\left(2^j \sum_{k=0}^{j-1} \frac{1}{2^{k+1}} \log \alpha_{2k}\right)$, we know by Lemma 2 that for $j \geq 15$, $e^{2^j \log(1.4)} \leq h_{2j} \leq e^{2^j \log A}$. Similarly, for $j \geq 15$, $e^{2^{(j+.5)}\log(1.4)} \leq h_{2j+1} \leq e^{2^{(j+.5)}\log B}$. If $C = \max(A, B)$, $C < 1.46$ and $C^{\sqrt{2}} < 1.46^{1.42} < 1.72$. We have

$$0 \leq \frac{h_{j+1}}{(h_j)^2} \leq \frac{\exp\left(2^{(.5j+.5)}\log C\right)}{[\exp(2^{.5j}\log(1.4))]^2} \leq \frac{\left(C^{\sqrt{2}}\right)^{\sqrt{2}^j}}{\left(1.4^{\sqrt{2}^j}\right)\left(1.4^{\sqrt{2}^j}\right)} < \left(\frac{1.72}{(1.4)^2}\right)^{\sqrt{2}^j} < .9^{\sqrt{2}^j}$$

which allows the series to be estimated easily. If $j \geq 6$, then $\sqrt{2}^j > 1.4j$ and the following estimate is valid:

$$2^n \sum_{j=n}^{\infty} 2^{-j-1} \log \alpha_{2j} < 2^n \sum_{j=n}^{\infty} 2^{-j-1}\left(.9^{\sqrt{2}^j}\right)$$

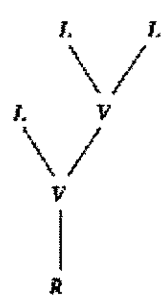
$$< 2^{n-1} \sum_{j=n}^{\infty} 2^{-j}(.9^{1.4j}) = \frac{.5}{1-.5(.9)^{1.4}}(.9^{1.4})^n$$

This overestimate certainly approaches 0 rapidly, so the first limit in the theorem is verified. The second, for odd integers, follows in a similar fashion. ∎

The proof above certainly doesn't use all the information present. In fact, the convergence to the limits is extremely rapid, and very sharp error estimates can be made. A result similar to theorem 1 with similar error estimates can be proved for any non-negative initial conditions. Different initial conditions give rise to different growth constants. The link between the pair of initial conditions and the pair of growth constants for this recurrence has been shown to be a real analytic mapping with further interesting properties. See [GN].

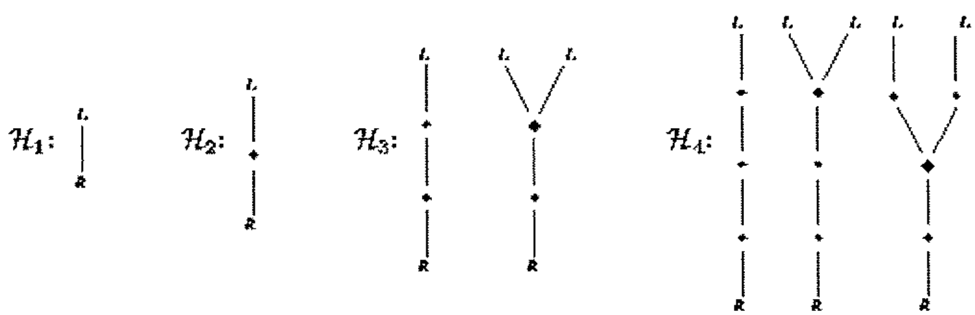
Counting the trees in a forest

A graph is a set of vertices together with a set of edges where the edges connect pairs of distinct vertices. A vertex connected by an edge is called *incident* with that edge. A *tree* here will be a connected graph without cycles, which are closed paths of edges. The number of edges a vertex is incident with is called the *degree* of the vertex. A *rooted tree* has one distinguished vertex with degree 1. The root vertex will be labeled *R*. Any other vertices of degree 1 in a rooted tree are called *leaves* and will be labeled *L*.

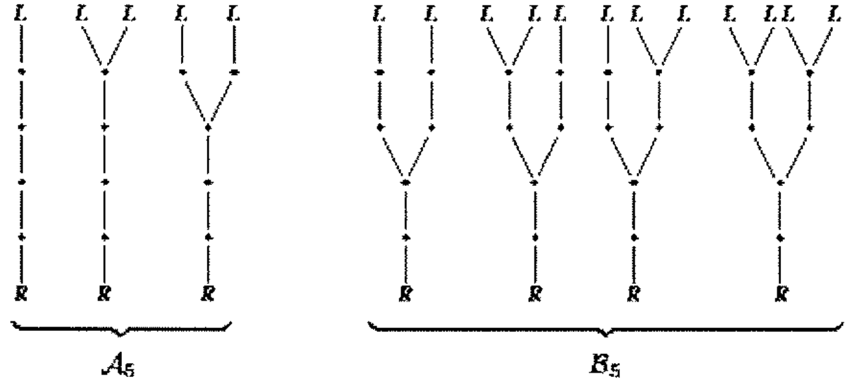


Trees will be drawn here with their roots at the bottom of their pictures. The *level* of a vertex is its distance to the root. The *distance* between two vertices in a tree is the minimum number of edges required to travel from one to the other. A tree with three leaves is displayed. One leaf has level 2 and the two others have level 3. This tree also has two vertices designated V which are neither leaves nor the root. They both have degree 3. One of these has level 1 and the other has level 2.

$\mathcal{H}_n$ will be a set of all rooted trees of a certain type for each integer $n \geq 1$. Every leaf of each tree in $\mathcal{H}_n$ will have level n. Any vertices of degree greater than 1 in trees in $\mathcal{H}_n$ will be one of two types: the diamond (◆) and the circle (•). The diamond will always have degree 3 so the tree must "branch" at a diamond. The distance between two diamonds must always be at least 2, and the level of a diamond must be at least 2. All other vertices of degree greater than 1 will be circles and each circle will have degree 2. There will be no branching at a circle – just a "trunk". Here is a display of some small forests of this species, a rather peculiar sort of binary tree.



The more numerous trees in $\mathcal{H}_5$ can be grouped suggestively.



The forest $\mathcal{H}_5$ divides naturally into a disjoint union of $\mathcal{A}_5$, the trees in $\mathcal{H}_5$ whose level 2 vertex is a circle, and $\mathcal{B}_5$, the trees in $\mathcal{H}_5$ whose level 2 vertex is a diamond. Note that these trees are "oriented": the left and right branches are distinct. More technically, these trees are examples of what are called *simple partially-ordered rooted trees.*

Consider $\mathcal{A}_5$. Delete the root and lowest edge of a tree in this set, and change the lowest circle to a root. The result is an element of $\mathcal{H}_4$. It is not hard to see that this mapping is a bijection.

Now consider $\mathcal{B}_5$. Go to the lowest diamond (which must have level 2) of any tree in $\mathcal{B}_5$. Separate the two branches rising from the diamond, and in each one end the lowest edge by a root. This gives a pair of elements of $\mathcal{H}_3$. This mapping is a bijection of $\mathcal{B}_5$ with $(\mathcal{H}_3)^2$, the set of pairs of trees in $\mathcal{H}_3$.

Thus the number of trees in $\mathcal{H}_5$ is the sum of the number of trees in $\mathcal{H}_5$ and the square of the number of trees in $\mathcal{H}_3$.

Theorem 2 If $n \geq 1$, h_n is the number of trees in $\mathcal{H}_n$.

Proof The correspondence described above extends to $\mathcal{H}_n$ so there is a bijection between $\mathcal{H}_{n+2}$ and $\mathcal{H}_{n+1} \cup (\mathcal{H}_n \times \mathcal{H}_n)$. Then the theorem is true because $\mathcal{H}_1$ and $\mathcal{H}_2$ each contain one tree. ∎

Counting the leaves on the trees in a forest

The pictures above invite the question: how many leaves are there in the forest $\mathcal{H}_n$? Suppose that j_n is the number of leaves in the forest $\mathcal{H}_n$.

Then $j_1 = j_2 = 1$ and the sequence $\{j_n\}$ satisfies the recurrence

$$j_{n+2} = j_{n+1} + 2h_n j_n. \qquad (4)$$

This is fairly clear from the bijection described in Theorem 2, since $j_n h_n$ is the total number of leaves which occur on all the left hand trees, or on all the right hand trees, in pairings from $\mathcal{H}_n \times \mathcal{H}_n$.

Equation (4) may also be obtained by the following argument which may be of independent interest. Define a polynomial $P_n(x)$ by

$$P_n(x) = \sum_{T \in \mathcal{H}_n} x^{\ell(T)},$$

where $\ell(T)$ is the number of leaves on the tree T. Then clearly $P_n(1) = h_n$, $P_n'(1) = j_n$, and the sequence of polynomials $\{P_n\}$ satisfies the original recurrence, (1): $P_{n+2} = P_{n+1} + P_n^2$. These equations imply (4).

Note that (4) is the linearization of the recursion (1) for h: if we alter the initial values for (1) by the infinitesimal perturbations $h_1 \to h_1 + dh_1$ and $h_2 \to h_2 + dh_2$ then the resulting perturbation $h_n \to h_n + dh_n$ satisfies

$$dh_{n+2} = dh_{n+1} + 2h_n dh_n$$

up to higher order terms.

The recursion (4) can be compared to the simpler recursion,

$$J_{n+2} = 2h_n J_n, \qquad (5)$$

which has solution $J_{2m+2} = J_2 2^m \prod_{k=1}^m h_{2k}$ and $J_{2m+3} = J_1 2^m \prod_{k=1}^m h_{2k+1}$. Numerical experiments indicate that if (4) and (5) are given the same initial values, i.e., $J_1 = j_1$ and $J_2 = j_2$, then $\frac{j_{2m}}{J_{2m}}$ and $\frac{j_{2m+1}}{J_{2m+1}}$ converge rapidly to (different) constants. Analytically (from the known asymptotics of h_n) and numerically it appears that $\frac{2^m h_{2m}}{J_{2m}}$ and $\frac{2^m h_{2m+1}}{J_{2m+1}}$ also converge to constants. So apparently $j_{2m+\theta} \approx C_\theta 2^m h_{2m+\theta}$ for $\theta = 0, 1$. Therefore the mean number of leaves per tree in $\mathcal{H}_n$ is asymptotically a constant (which depends on the parity of n) multiple of $(\sqrt{2})^n$.

An approach to more general recurrences with some experimental results

We suggest here one way to analyze sequences defined by polynomial recurrence relations. Begin with a recurrence which can be solved exactly:

> Suppose that $h_{n+2} = h_{n+1}h_n$ with initial condition $(h_0, h_1) = (1, 2)$. An explicit formula is given by $h_n = 2^{(n^{\text{th}} \text{ Fibonacci number})}$. Standard asymptotics for the Fibonacci numbers then imply $h_n \approx K^{\nu^n}$ for n large with $K = 2^{1/\sqrt{5}} \approx 1.363$ and $\nu = \frac{1+\sqrt{5}}{2} \approx 1.618$.

We briefly explain how to find a similar expression for any recurrence determined by one monomial. We assume that

$$h_{n+k} = c\, h_n^{r_0}\, h_{n+1}^{r_1} \cdots h_{n+k-1}^{r_k}, \tag{6}$$

where c is a positive constant and each of the exponents r_j is a nonnegative integer. We further assume that $r_0 > 0$ and that a k-tuple of nonnegative initial values $(h_0, \ldots, h_{k-1})$ is given.

Then the sequence $\{\log h_n\}$ satisfies a linear recurrence with characteristic polynomial $p(x) = x^k - r_{k-1}x^{k-1} - \ldots - r_1 x - r_0$, which can be solved exactly using classical techniques. If $p(1) \neq 0$ ($p(1)$ vanishes only in the uninteresting case $p(x) = x - 1$) then $h_n = C \prod_{i=0}^{k-1} A_i^{\lambda_i^n}$ where $\lambda_0, \ldots, \lambda_{k-1}$ are the roots of p, $C = c^{1/p(1)}$, and $A_0, \ldots, A_{k-1}$ are constants determined by the initial conditions.

Suppose j is the maximum integer so that $p(x) = q(x^j)$ for some polynomial q of degree $m = k/j$. If q has roots $\{\mu_0, \ldots, \mu_{m-1}\}$ then the roots $\{\lambda_0, \ldots, \lambda_{k-1}\}$ of p can be numbered so that $\lambda_{jr+l}^j = \mu_r$ for $l = 0, 1, \ldots, j-1$ and $r = 0, \ldots, m-1$. When $j > 1$, (6) becomes

$$h_{jn+l} = C \prod_{r=0}^{m-1} B_{r,l}^{\mu_r^n}\,,\ l = 0, 1, \ldots, j-1, \tag{7}$$

where $B_{r,l} = \prod_{s=0}^{j-1} A_{jr+s}^{\lambda_{jr+s}^l}$.

The polynomial $p(x)$ has one positive root ν. We let $\nu = \lambda_0 = \mu_0^{1/j}$. The roots $\lambda_0, \ldots, \lambda_{j-1}$ then all have magnitude $\nu(p)$ and make the dominant contribution to the growth of h_n since $\nu > |\lambda_i|$ for $i \geq j$.

This suggests one way to analyze polynomial recurrences with positive coefficients. We suppose that the recurrence is

$$h_{n+k} = \sum_{\alpha} c_\alpha \, h_n^{r_{\alpha,0}} \, h_{n+1}^{r_{\alpha,1}} \cdots h_{n+k-1}^{r_{\alpha,k-1}}, \tag{8}$$

where each term in the finite sum has positive coefficient c_α and all exponents $r_{\alpha,i}$ are nonnegative integers as in (6). Each term has an associated characteristic polynomial, $p_\alpha(x) = x^k - \sum_{i=0}^{k-1} r_{\alpha,i} x^i$, which in turn has a unique positive root ν_α.

If one term indexed by β is *dominant* in the sense that $\nu_\beta > \nu_\alpha$ for $\alpha \neq \beta$, then simulations and some heuristic reasoning suggest that if the initial conditions are chosen large enough so that $h_n \to \infty$, then h_n behaves asymptotically like the exact solution (7) of the recurrence with only this dominant term:

$$\lim_{n\to\infty} h_n \Big/ \left(C \prod_{i=0}^{k-1} A_i^{\lambda_{\beta,i}^n} \right) = 1, \tag{9}$$

where again $C = c^{1/p(1)}$ but now $A_0, \ldots, A_{k-1}$ depend on the remaining terms in the recurrence as well as on the initial condition.

The recurrence (1) analyzed previously has dominant term, h_n^2, with $C = 1$, $k = 2$, $j = 2$, and $\nu = \sqrt{2}$. In this case, $C \prod_{i=0}^{k-1} A_i^{\lambda_{\beta,i}^n}$ is $A_0^{(\sqrt{2})^n} A_1^{(-\sqrt{2})^n}$. When n is even this is $(A_0 A_1)^{(\sqrt{2})^n}$ and when n is odd it is $(A_0/A_1)^{(\sqrt{2})^n}$. Theorem 1 therefore verifies (9) with $A_0 A_1 \approx 1.436$ and $A_0/A_1 \approx 1.451$.

We have no suggestion for the correct asymptotics when (8) has no dominant term, nor have we proved (9) in general. Here is a report of some numerical experiments which also support (9):

Recurrence and initial condition	k, j, ν for the dominant term	Observed asymptotics
$h_{n+2} = h_{n+1} + (h_n)^3,\ (0,1)$	2, 2, $3^{1/2}$	$h_n \approx (K_l)^{3^{n/2}},\ n \equiv l\ (2)$ for $\begin{cases} K_0 \approx 1.144 \\ K_1 \approx 1.166 \end{cases}$
$h_{n+3} = h_{n+2} + (h_n)^2,\ (0,0,1)$	3, 3, $2^{1/3}$	$h_n \approx (K_l)^{2^{n/3}},\ n \equiv l\ (3)$ for $\begin{cases} K_0 \approx 1.454 \\ K_1 \approx 1.438 \\ K_2 \approx 1.442 \end{cases}$
$h_{n+3} = (h_{n+2})^2 + h_n,\ (0,0,1)$	1, 1, 2	$h_n \approx (1.0257)^{2^n}$
$h_{n+2} = (h_{n+1})^2 + (h_n)^2,\ (0,1)$	1, 1, 2	$h_n \approx (1.111)^{2^n}$

Algebraic identities

Suppose $S = \{s_0, s_1, \ldots\}$ is any sequence of integers, and n is a positive integer. Let $S_n \subset Z^n$ be the set of all consecutive n-tuples of elements of S: $(x_1, \ldots, x_n) \in S_n$ exactly when $x_j = s_{k+j}$ for some $k \geq 0$ and all j between 1 and n. Define $I_{S,n}$ to be the ideal of polynomials with integer coefficients in n variables (elements of $\mathbb{Z}[X_1, \ldots, X_n]$) which vanish on S_n. If S is defined as the solution of a recurrence which is polynomial with integer coefficients as discussed above then the recurrence itself produces elements of $I_{S,n}$ for n sufficiently large. When do these elements generate $I_{S,n}$? A specific example may be useful. If F is the sequence of Fibonacci numbers, then $I_{F,3}$ contains $X_1 + X_2 - X_3$, determined by the generating recurrence. $I_{F,3}$ is not principal since it also contains $(X_1X_3 - (X_2)^2)^2 - 1$ (from the classical Cassini identity).

Suppose $H = \{h_0, h_1, \ldots\}$ is the sequence studied in this paper, so $h_{n+2} = h_{n+1} + (h_n)^2$ with $(h_0, h_1) = (0, 1)$. Is $I_{H,3}$ a principal ideal generated by $(X_1)^2 + X_2 - X_3$? Is $I_{H,4}$ generated by $(X_1)^2 + X_2 - X_3$ and $(X_2)^2 + X_3 - X_4$? In other words, does the sequence H satisfy any finite width polynomial identity which is *not* implied by the generating relation? This seems unlikely but we do not know a proof.

References

[AS] A. V. Aho and N. J. A. Sloane, Some doubly exponential sequences, *The Fibonacci Quarterly* 11 (1973), 429-437.

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