Extreme values of Artin L-functions and class numbers

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Abstract

Assuming the GRH and Artin conjecture for Artin L-functions, we prove that there exists a totally real number field of any fixed degree (> 1) with an arbitrarily large discriminant whose normal closure has the full symmetric group as Galois group and whose class number is essentially as large as possible. One ingredient is an unconditional construction of totally real fields with small regulators. Another is the existence of Artin L-functions with large special values. Assuming the GRH and Artin conjecture it is shown that there exist an Artin L-functions with arbitrarily large conductor whose value at $s = 1$ is extremal and whose associated Galois representation has a fixed image, which is an arbitrary non-trivial finite irreducible subgroup of $GL(n, \mathbb{C})$ with property Gal_T.

1 Class numbers of number fields.

Let K be a number field whose group of ideal classes has size h, called the class number of K. As K ranges over some natural family, it is interesting to investigate the behavior of h. Unless K is imaginary quadratic, the involvement of an infinite unit group makes certain problems here extremely difficult, even assuming conjectures like the generalized Riemann hypothesis (GRH). An obvious example is Gauss' conjecture that there are infinitely many real quadratic fields with $h = 1$. On the other hand, some rather precise information can be obtained about number fields with large class number.

Consider, for example, the family $\mathcal{K}_n$ where $K \in \mathcal{K}_n$ if K is a totally real number field of degree n whose normal closure has the full symmetric group S_n as its Galois group. By the class number formula for such K

$$h = \frac{d^{1/2}}{2^{n-1}R} L(1, \chi) \tag{1}$$

where $d = \mathrm{disc}(K)$ is the discriminant, R is the regulator and $L(s, \chi) = \zeta_K(s)/\zeta(s)$ is an Artin L-function, $\zeta_K(s)$ being the Dedekind zeta function of

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K. If we assume that $L(s,\chi)$ is entire and satisfies the generalized Riemann hypothesis (GRH), that is all of its non-trivial zeros have real part $\frac{1}{2}$, then a method introduced by Littlewood [9] shows that

$$L(1,\chi) \ll (\log\log d)^{n-1}$$

with an implied constant depending only on n. Remak [16] showed that if K contains no non-trivial subfields (which is true for $K \in \mathcal{K}_n$) then

$$R \gg (\log d)^{n-1}. \tag{2}$$

Thus, under GRH, we have the upper bound

$$h \ll d^{\frac{1}{2}}(\log\log d/\log d)^{n-1}. \tag{3}$$

In this paper I will show, still assuming the GRH, that (3) is best possible, up to the constant.

Theorem 1 *Fix $n \geq 2$ and assume that each $L(s,\chi)$ is entire and satisfies the GRH. Then there is a constant $c > 0$ depending only on n such that there exist $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which*

$$h > c\, d^{\frac{1}{2}}(\log\log d/\log d)^{n-1}.$$

For $n = 2$ this result, which follows easily from Littlewood's paper [9], was made unconditional by Montgomery and Weinberger [12]. The problem of proving Theorem 1 unconditionally for any $n > 2$ remains open.

One issue to be dealt with in the proof is the production of totally real number fields with small regulators. This problem seems to have been first considered successfully by Ankeny, Chowla and Brauer [1]. For $n = 2$ one may take $d = 4t^2 + 1$ square-free, for instance, since then a unit is $2t + d^{\frac{1}{2}}$ and so $R \leq \log d$. After generalizing this construction for any $n \geq 2$ they show, using Brauer's generalization of Siegel's theorem that, for any $\epsilon > 0$, there exist an infinite number of such fields where

$$h > d^{\frac{1}{2}-\epsilon}$$

holds. This is unconditional and they also obtained such a result for fields with any given signature.

In this paper we give a modification of their construction, and ultimately take for K a field obtained by adjoining to $\mathbb{Q}$ a root of

$$f(x,t) = (x-t)(x-2^2t)(x-3^2t)\ldots(x-n^2t) - t \tag{4}$$

for a suitable integral value of t. We show in Proposition 1 of §3 that, for sufficiently large square-free t, these fields are totally real of degree n and satisfy

$$R \ll (\log d)^{n-1}.$$

This is done by explicitly computing a set of $n-1$ multiplicatively independent units. This actually gives a simplification of the construction of [1], where one of the main difficulties is to show that discriminants of the constructed fields are not too small, which is a well known difficulty in explicit Galois theory. With our choice of polynomials this problem disappears since for square-free t all primes dividing t turn out to be completely ramified and hence the discriminant satisfies $\log d \gg \log t$.

Our main goal here is to show that the bound (3) is *sharp*, at the expense of assuming GRH. The idea is to consider (4) as defining an algebraic function field over $\mathbb{Q}(t)$. Following a classical method of Jordan, in Proposition 2 of §4 we compute the Galois group of the polynomial $f(x,t)$ over $\mathbb{Q}(t)$ and find that it is S_n. Then, in Proposition 4 of §5 we show that there are infinitely many square-free integral values t so that the splitting field of this polynomial over $\mathbb{Q}$ has Galois group S_n. This follows from a quantitative form of a result of Hilbert due to S.D. Cohen, which relies on a generalization of the large sieve inequality first applied in this manner by Gallagher [6]. In order to force extreme behavior of the Artin L-values $L(1,\chi)$, it is enough to restrict the t to certain arithmetic progressions. This is seen by considering the function field defined by $f(x,t)$ over $\mathbb{F}_p$ and applying the Riemann hypothesis for curves to its normal closure. We obtain in Proposition 4 in §5 a range of about $\log d$ primes that split completely in K and hence for which $\chi(p) = n-1$. Fortunately, the large size of the modulus required to do this is allowed by the mild restriction that t be square-free. The GRH is needed to approximate $\log L(1,\chi)$ by a sum essentially limited to such primes and hence force it to be large.

In order to make this result unconditional for $n > 2$ one could try to show, as was done in [12] when $n = 2$, that the number of L-functions with zeros violating GRH is small enough. We intend to address this problem in some special cases in another paper. At this point, such a result is available only for zeros very close to $s = 1$ and the best we can get unconditionally in general is that there exist infinitely many $K \in \mathcal{K}_n$ with

$$h \gg d^{\frac{1}{2}} (\log d)^{-n} \tag{5}$$

where the implied constant depends only on n (see (27) below).

2 Families of Artin L-functions.

Since the extremal behavior of arbitrary Artin L-functions at $s = 1$ is of independent interest, especially in view of Stark's conjectures (see [20]), we treat this aspect more generally. Let $L(s,\chi)$ be the Artin L-function associated to a continuous Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{C})$$

where $\bar{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$ and $\chi = \mathrm{trace}(\rho)$. $L(s,\chi)$ is known to be meromorphic, satisfy a standard functional equation, and is conjectured to be entire (Artin Conjecture) unless ρ contains trivial components, in which case it should only have a pole at $s = 1$. If N is the conductor of ρ and we assume that $L(s, \chi)$ is entire and satisfies GRH then again a generalization of Littlewood's proof gives

$$L(1, \chi) \ll (\log \log N)^n, \tag{6}$$

where the implied constant depends only on n (see (25) below).

A natural family of L-functions are those associated to all the Galois representations with the same (finite) image $G \subset GL(n, \mathbb{C})$. It is not known that every such G is the image of a Galois representation over $\mathbb{Q}$, as this includes the inverse Galois problem. In view of the factorization property of L-functions, for or purposes it is enough to assume that G is non-trivial and irreducible.

As is the case above when $G = S_n$, we shall assume that G is isomorphic to the Galois group of a regular extension E of the rational function field $\mathbb{Q}(t)$, where *regular* means that $\bar{\mathbb{Q}} \bigcap E = \mathbb{Q}$. Such G are said to have property Gal_T and include many classes of groups such as abelian, symmetric, and alternating group. In fact, Serre has conjectured (p.35. of [18]) that *every* finite group G has property Gal_T. Given a G with property Gal_T we show that (6) is sharp within this family, still under GRH.

Theorem 2 *Suppose that $G \subset GL(n, \mathbb{C})$ is non-trivial, irreducible and has property* Gal_T*. Assume that every Artin L-function $L(s, \chi)$ whose Galois representation has image G is entire and satisfies GRH. Then there is a constant $c > 0$, depending only on G, so that there exist such L-functions with arbitrarily large conductor N, which satisfy*

$$L(1, \chi) > c(\log \log N)^n.$$

Although we have restricted our attention to large values of $L(1, \chi)$, the same methods apply to small values. One ingredient in the proof of this is an analogue of a result of Bohr and Landau, as refined by Littlewood in [8], which is used to find small values of $\zeta(1 + it)$. Roughly speaking, it states that if y is sufficiently large then there is a positive integer t with $\log t \ll y$ so that p^{it} is uniformly close to -1 for each prime $p \leq y$, the implied constants depending on how good the approximation is required to be.

Let G be a finite subgroup of $GL(n, \mathbb{C})$ with property Gal_T and choose $a(p)$ to be, for each prime p, the trace of an arbitrary element of G. The analogue, given in Proposition 3 of §5, states that if y is sufficiently large then there is a Galois representation whose image is G with $\log N \ll y$ so that $\chi(\sigma_p) = a(p)$ for $1 \ll p \leq y$, where σ_p is any Frobenius element over p.

The other main ingredient is a formula that approximates $\log L(s, \chi)$ by a very short sum over primes:

$$\log L(1, \chi) \sim \sum_{p \leq (\log N)^{\frac{1}{2}}} \chi(p) p^{-1}.$$

This is done in Proposition 5 of §6, assuming the GRH, along the lines of Littlewood's paper [9].

3 Totally real fields with small regulators.

In this section we will produce a convenient family of polynomials that generate totally real number fields whose regulators are as small as the bound (2) allows. Fix $n \geq 2$ and let $a_1 < a_2, \cdots < a_n$ be integers. Define the degree n polynomial

$$f_t(x) = (x - a_1 t)(x - a_2 t) \dots (x - a_n t) - t. \tag{7}$$

Proposition 1 *Let $t \in \mathbb{Z}^+$ be a square-free integer. Then $f_t(x)$ is irreducible. Let K be a number field obtained by adjoining to $\mathbb{Q}$ a root of $f_t(x)$. If t is sufficiently large, then K is totally real of degree n and the regulator R of K satisfies*

$$R \ll_f (\log d)^{n-1}$$

where d is the discriminant of K.

Proof: For a square-free $t > 1$, $f_t(x)$ is an Eisenstein polynomial, hence irreducible, while if $t = 1$ the irreducibility is standard (see p.133. of [14]). Let K be the number field obtained by adjoining to $\mathbb{Q}$ a root α of $f_t(x)$. If $p \,|\, t$ then p ramifies completely in K and does not divide the index of α (see p.60. and p.181. of [13]). Since $\mathrm{disc} f_t(x) = t^{n-1} p(t)$, where $p(t)$ is an integral polynomial of degree $(n-1)^2$, we see that

$$t^{n-1} | d, \tag{8}$$

where d is the discriminant of K.

Since they vary continuously with t, we may label the roots $\alpha^{(1)}, \dots, \alpha^{(n)}$ of $f_t(x) = 0$ so that $\alpha^{(i)}/t \to a_i$ as $t \to \infty$ for each $i = 1, \dots, n$. Since non-real roots come in conjugate pairs, if t is sufficiently large, all of the $\alpha^{(i)}$ will be real.

For $t \in \mathbb{Z}^+$ and $i, j = 1, \dots, n$ let

$$\varepsilon_j^{(i)} = t(\alpha^{(i)} - a_j t)^{-n}.$$

Then, for each i

$$\prod_{j=1}^{n} \varepsilon_j^{(i)} = t^n \prod_{j=1}^{n} (\alpha^{(i)} - a_j t)^{-n} = t^n (f_t(\alpha^{(i)}) + t)^{-n} = 1. \tag{9}$$

Also $(\varepsilon_j^{(i)})^{-1}$ is an algebraic integer since

$$(\alpha^{(i)} - a_j t)^n \equiv \prod_{j=1}^{n} (\alpha^{(i)} - a_j t) \pmod{t}$$

for any i and

$$\prod_{j=1}^{n}(\alpha^{(i)} - a_j t) = f_t(\alpha^{(i)}) + t \equiv 0 \pmod{t}.$$

Thus, for each $1 \leq j \leq n$, we have that $\varepsilon_j^{(i)}$ runs over the conjugates of a unit in the ring of integers $\mathfrak{O}_K$ as $i = 1, \ldots, n$. Furthermore, using that $\alpha^{(i)}/t \to a_i$ as $t \to \infty$ we easily derive that

$$|\varepsilon_j^{(i)}| \sim \begin{cases} c_{i,i}\, t^{(n-1)^2} & \text{if } i = j \\ c_{i,j}\, t^{1-n} & \text{otherwise} \end{cases} \tag{10}$$

as $t \to \infty$, where

$$c_{i,j} = \begin{cases} \prod_{k \neq i} |a_i - a_k|^n \geq 1 & \text{if } i = j \\ |a_i - a_j|^{-n} > 0 & \text{otherwise.} \end{cases}$$

By (9) each row sum of the the $n \times n$ matrix

$$\begin{pmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_n^{(1)}| \\ \vdots & \ddots & \vdots \\ \log|\varepsilon_1^{(n)}| & \cdots & \log|\varepsilon_n^{(n)}| \end{pmatrix} \tag{11}$$

is zero. For t sufficiently large, by (10) each diagonal entry is positive and each off-diagonal entry is negative. By Minkowski's lemma [11], any $(n-1) \times (n-1)$ principal minor is positive. Since each $\varepsilon_j^{(i)}$ is a unit in $\mathfrak{O}_K$ for some fixed i, for t sufficiently large any $n-1$ of them are multiplicatively independent. Thus it follows from (10) and (8) that

$$R \leq A \ll_f (\log t)^{n-1} \ll (\log d)^{n-1}$$

where A is such a minor, finishing the proof of Proposition 1.

4 Classical monodromy.

We now apply the classical theory of monodromy to prove the following result needed in the proof of Theorem 1.

Proposition 2 *For $n \geq 1$ the splitting field of*

$$f(x,t) = (x-t)(x-2^2t)(x-3^2t)\ldots(x-n^2t) - t \tag{12}$$

over $\mathbb{Q}(t)$ is a regular extension with Galois group S_n.

Generally, let E be the splitting field over $\mathbb{Q}(t)$ of a polynomial $f(x,t)$ degree n with coefficients in $\mathbb{Q}[t]$ that is monic in x. Assume that E is regular or, equivalently, that $f(x,t)$ is irreducible over $\mathbb{C}$. Let G be the Galois group of E over $\mathbb{Q}(t)$. The monodromy group is classically defined in the following way. First, compute the zeros (finite ramification points) $t_1,\dots,t_r \in \mathbb{C}$ of the discriminant $D(t)$ of $f(x,t)$, which is a polynomial in t and fix an unramified point t_0. Thus $f(x,t_0)$ has n distinct roots. Letting t start at t_0 and run over a simple loop enclosing t_i induces, by continuity of the roots in t, a permutation σ_i of these roots. The subgroup M of S_n generated by σ_i for $i = 1,\dots n$ is the monodromy group. The point at ∞ is unramified exactly when $\prod_{i=1}^n \sigma_i = 1$. The basic fact, observed already by Jordan and Hermite, is that M is a transitive subgroup of S_n, which is isomorphic to a normal subgroup of G.

The following result was applied by Hilbert [7] in case $f(x,t) = g(x) - t$ to produce infinitely many number fields with Galois group S_n.

Lemma 1 *Suppose that $g(x)$ is a monic integral polynomial of degree $n \geq 1$ such that the zeros $x_1,\dots,x_{n-1}$ of the derivative g' are simple and that $g(x_i) \neq g(x_j)$ for $i \neq j$. Suppose also that m is any positive integer except that $m \neq 2$ if g is a square. Then*

$$f(x,t) = g(x) - t^m,$$

is irreducible over $\mathbb{C}$ and $G = S_n$.

Proof: The irreducibility of $f(x,t)$ is an immediate consequence of Capelli's theorem (see p. 91. of [17]), which implies that for any $g \in \mathbb{C}(x)$, $t^m - g(x)$ is reducible over $\mathbb{C}(x)$ if and only if either for some prime divisor p of m we have $g = h^p$ or $4|m$ and $g = -4h^4$ for some $h \in \mathbb{C}(x)$. Of these possibilities, the only one consistent with the condition that the zeros of g' are simple is that $m = 2$ and $g = h^2$.

One computes that the finite ramification points are solutions t of $t^m = g(x_i)$. Under our assumptions the associated permutations are all transpositions. The proof now follows since a transitive subgroup of S_n generated by transpositions must be S_n (see e.g. p.40 of [18]). □

Proof of Proposition 2: First note that the splitting field over $\mathbb{Q}(t)$ of

$$f(x,t) = \prod_{i=1}^{n}(x - a_i t) - t.$$

for any $a_i \in \mathbb{Q}$ is the same as that of

$$t^n f(t^{-1}x, t^{-1}) = \prod_{i=1}^{n}(x - a_i) - t^{n-1}. \tag{13}$$

The proof is now reduced after (13) to showing that

$$g(x) = \prod_{k=1}^{n}(x - k^2)$$

satisfies the conditions of Lemma 1. This is straightforward since there are clearly $n-1$ distinct real zeros x_i of $g'(x)$ and the values of $|g(x_i)|$ are strictly increasing. □

5 Bohr-Landau for Galois representations.

We now give a general analogue of a classical theorem of Bohr and Landau [3] for Galois representations whose image G has property Gal_T. Recall that this means that G is isomorphic to the Galois group of a regular extension E of the rational function field $\mathbb{Q}(t)$. Then we prove a more specific version needed for the proof of Theorem 1.

Proposition 3 *Let G be a finite subgroup of $GL(n,\mathbb{C})$ for some $n \geq 1$ that has property Gal_T. For each prime p, let $a(p)$ be the trace of an arbitrary element of G. There is a constant $c > 0$ depending only on G such that for any $y > c$ there is a Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ whose image is G, whose conductor N satisfies $\log N \leq cy$ and whose character χ is unramified and satisfies*

$$\chi(\sigma_p) = a(p) \tag{14}$$

for $c \leq p \leq y$, where σ_p is any Frobenius element over p.

Proof: Let $f(x,t)$ be an integral polynomial whose splitting field over $\mathbb{Q}(t)$ is regular and has Galois group isomorphic to G. Let K_t be one of the fields obtained by adjoining to $\mathbb{Q}$ a root of $f(x,t)$ for a specialization $t \in \mathbb{Q}$ of T and let $\hat{K}_t$ be the splitting field of $f(x,t)$. Let C be any conjugacy class of G. As observed by Serre (p.45. in [18]) it follows from Weil's bound that there is a constant $\kappa > 0$ so that for a prime $p \geq \kappa$ there is a $t_C \in \mathbb{Z}$ so that for any $t \equiv t_C (\mathrm{mod}\, p)$, p is unramified in $\hat{K}_t$ and the Frobenius class of p in $\mathrm{Gal}(\hat{K}_t/\mathbb{Q}) \hookrightarrow G$ intersects C. For $y > \kappa$ let

$$q = \prod_{\kappa \leq p \leq y} p \tag{15}$$

and choose C_p so that any element in C_p has trace $a(p)$. Let $t_q \in \mathbb{Z}^+$ be such that $t_q \equiv t_{C_p} (\mathrm{mod}\, p)$ for all p with $\kappa \leq p \leq y$. Consider the set

$$T(y) = \{t \in \mathbb{Z}^+ \mid t \leq q^3, t \equiv t_q (\mathrm{mod}\, q), \mathrm{Gal}(\hat{K}_t/\mathbb{Q}) = G\}. \tag{16}$$

Clearly if $t \in T(y)$ then $\mathrm{Gal}(\hat{K}_t/\mathbb{Q})$ has a faithful Galois representation with image G whose character χ satisfies (14). It follows from a comparison of the polynomial discriminant with the field discriminant that for $t \in T(y)$ we have

$$\log |\mathrm{disc}(\hat{K}_t)| \ll \log q \ll y \tag{17}$$

where in the last inequality we are using (15) and the prime number theorem. It is standard (see p.79. of [10]) that if N is the conductor of χ and if χ is irreducible then N^n divides $\mathrm{disc}(\hat{K}_t)$. Thus we deduce from (17) that in general for $t \in T(y)$ we have $\log N \ll y$.

In order to finish the proof it is enough to show that $T(y)$ is non-empty for $y \geq c$ for some sufficiently large c. This follows immediately from Theorem 2.1 of S.D. Cohen [5], which is a quantitative version of a result of Hilbert, and implies that

$$\#\{t \in \mathbb{Z}^+ | t \leq q^3 \text{ and } \mathrm{Gal}(\hat{K}_t/\mathbb{Q}) \neq G\} \ll q^{3/2} \log q. \tag{18}$$

□

For Theorem 1 we need a more specialized version of this result for the family $\mathcal{K}_n$ of totally real number fields of degree n, each of whose normal closure has the full symmetric group S_n as its Galois group.

Proposition 4 *Fix $n \geq 2$. There is a constant $c > 0$ depending only on n so that there are fields $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which every prime p with $c \leq p \leq \log d$ splits completely in K and such that*

$$R \leq c(\log d)^{n-1},$$

where R is the regulator of K.

Proof: In view of Proposition 2 we apply the same proof as above to the case where $G = S_n$, where $a(p) = n$ for all p and where the polynomial $f(x,t)$ is defined in (12), except that now we restrict our attention to square-free t. In place of $T(y)$ in (16) we consider

$$T^*(y) = \{t \text{ square-free} \mid \delta q^3 \leq t \leq 2\delta q^3, t \equiv t_q (\mathrm{mod}\, q), \mathrm{Gal}(\hat{K}_t/\mathbb{Q}) = G\}$$

where δ is a constant chosen so that if $t \in T^*(y)$, then

$$\log d = \log |\mathrm{disc}(K_t)| \leq y.$$

Since we know from (8) that the primes dividing t are ramified in K_t we have that $(t_q, q) = 1$ and so by [15] that

$$\#\{t \in \mathbb{Z}^+ \text{square-free} \mid \delta q^3 \leq t \leq 2\delta q^3, t \equiv t_q (\mathrm{mod}\, q)\} \gg q^2.$$

Comparison with (18) shows that $T^*(y)$ is not empty for y sufficiently large. It now follows by Proposition 1 that for $t \in T^*(y)$ we have that $K_t \in \mathcal{K}_n$ with the property that all primes with $c \leq p \leq \log d$ split completely in K_t and such that

$$R \ll (\log d)^{n-1}.$$

Finally, $d \to \infty$ as $y \to \infty$ since (8) implies that

$$\log d \gg \log t \gg \log q \gg y$$

for $t \in T^*(y)$, again using (15) and the prime number theorem. □

6 Short sums of group characters.

Let $L(s, \chi)$ be an Artin L-function associated to a Galois representation whose character χ has degree n over $\mathbb{Q}$ and has conductor N. This is given for $\mathrm{Re}(s) > 1$ by the formula

$$\log L(s, \chi) = \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \chi(p^m) p^{-ms} \tag{19}$$

where

$$\chi(p^m) = \frac{1}{|I_p|} \sum_{\iota \in I_p} \chi(\sigma_p^m \iota) \tag{20}$$

where I_p is the inertia group of any of the primes over p. This converges absolutely since we have from (20) the bound

$$|\chi(p^m)| \leq n. \tag{21}$$

Thus $L(s, \chi)$ is holomorphic for $\mathrm{Re}(s) > 1$ and has the uniform bound

$$L(s, \chi) \ll_{n,\delta} 1 \tag{22}$$

for $\mathrm{Re}(s) > \delta \geq 1$. Now $L(s, \chi)$ is known to be meromorphic [4] and to satisfy the functional equation [2]

$$\Lambda(1 - s, \bar{\chi}) = \varepsilon_\chi N^{s - \frac{1}{2}} \Lambda(s, \chi) \tag{23}$$

where $|\varepsilon_\chi| = 1$ and

$$\Lambda(s, \chi) = \pi^{\frac{ns}{2}} \Gamma(s/2)^{\frac{n+\ell}{2}} \Gamma((s+1)/2)^{\frac{n-\ell}{2}} L(s, \chi)$$

where ℓ is the value of χ on complex conjugation.

The following result allows us to approximate $\log L(1, \chi)$ with a very short sum, assuming GRH. For $n = 2$ this was done by Littlewood, who gave a more precise version that yields good constants.

Proposition 5 *Let $L(s, \chi)$ be an entire Artin L-function that satisfies GRH, where χ has degree n and conductor N. Then*

$$\log L(1, \chi) = \sum_{p \leq (\log N)^{\frac{1}{2}}} \chi(p) p^{-1} + O(1)$$

where the constant depends only on n.

Although this result is somewhat standard, we will give details. The next Lemma is classical and proven as Lemma 4 in [9].

Lemma 2 *Suppose that $f(s)$ is holomorphic in $|s-s_0|<r$ and satisfies there*

$$Re(f(s)-f(s_0))\le U.$$

Then there is an absolute constant $A>0$ so that for $|s-s_0|=r_0<r$ we have

$$|f'(s)|<\frac{AUr}{(r-r_0)^2}.$$

Next we give an estimate for $L'(s,\chi)/L(s,\chi)$ in the critical strip.

Lemma 3 *Let $\epsilon>0$ and let $L(s,\chi)$ be an entire Artin L-function that satisfies GRH, where χ has degree n and conductor N. Then for $s=\sigma+it$ we have*

$$|\frac{L'(s,\chi)}{L(s,\chi)}|\ll_\epsilon \log(|t|+2)+\log N$$

for $\frac{1}{2}+\epsilon<\sigma\le 2$.

Proof: First, under our assumptions, the entire function $L(s,\chi)$ is of finite order since it inherits a canonical product of genus 1 from those of the Hecke L-functions in its Brauer decomposition [4], which are well known to be of order 1. By the functional equation (23), the bound (22) and the Phragmén-Lindelöf Theorem we derive that

$$|L(s,\chi)|\ll_n N(|t|+2)^n$$

for $\frac{1}{2}<\sigma\le 2$. It follows easily that for some constant B depending only on n we have the bound

$$\mathrm{Re}(\log L(s_1,\chi))\le B\log(N(|t|+2)) \tag{24}$$

for $|s_1-(2+it)|\le\frac{3}{2}$.

Suppose $\epsilon<1$; since under GRH $f(s)=\log L(s,\chi)$ is holomorphic for $\sigma>\frac{1}{2}$ we have from (24) and Lemma 2 with $s_0=2+it, r=\frac{3}{2}$ and $r_0=\frac{3}{2}-\epsilon$ that

$$|\frac{L'(s_1,\chi)}{L(s_1,\chi)}|\ll\frac{\log(|t|+2)+\log N}{\epsilon^2}$$

for $|s_1-s_0|\le\frac{3}{2}-\epsilon$. Now we take $s_1=\sigma+it$ for $\frac{1}{2}+\epsilon\le\sigma\le\frac{7}{2}-\epsilon$ to finish the proof of Lemma 3.

Lemma 4 *Let $L(s,\chi)$ be an entire Artin L-function that satisfies GRH, where χ has degree n and conductor N. Then, for $\epsilon>0$, $0\le u\le\frac{3}{2}$ and $x>1$, we have*

$$\sum_p \log p\ \chi(p)\,p^{-u}e^{-p/x}+\delta_{u,\epsilon}\frac{L'(u,\chi)}{L(u,\chi)}\ll_{\epsilon,n} x^{\frac{1}{2}-u+\epsilon}\log N+1$$

where $\delta_{u,\epsilon}=1$ if $u-\epsilon>1/2$ and is $\delta_{u,\epsilon}=0$ otherwise.

Proof: We use Mellin's classical formula

$$e^{-y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^{-s}\Gamma(s)ds$$

for $y > 0$. By (19) we get for $0 \leq u \leq \frac{3}{2}$ and $x > 0$ the formula

$$\sum_p \log p \sum_{m=1}^{\infty} \chi(p^m)p^{-mu}e^{-p^m/x} = \frac{-1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'(s+u,\chi)}{L(s+u,\chi)}x^s\Gamma(s)ds.$$

By Lemma 3 and Stirling's formula we can shift the contour to $\mathrm{Re}(s) = 1/2+\epsilon-u$ picking up the residue at $s = 0$ to get

$$\sum_p \log p \sum_{m=1}^{\infty} \chi(p^m)p^{-mu}e^{-p^m/x} + \delta_{u,\epsilon}\frac{L'(u,\chi)}{L(u,\chi)} \ll_{\epsilon,n} x^{\frac{1}{2}-u+\epsilon}\log N$$

Using (21) the sum of the terms with $m > 1$ is $\ll_n 1$ and this finishes the proof of Lemma 4.

Proof of Proposition 5: Integrating over u from 1 to $\frac{3}{2}$ and taking $\epsilon = \frac{1}{8}$ in Lemma 4 gives for $x > 1$

$$\log L(1,\chi) - \sum_p \chi(p)p^{-1}e^{-p/x} \ll_n x^{-\frac{3}{8}}\log N + 1.$$

Hence, letting $x = (\log N)^3$ and $y = x^{\frac{1}{6}}$ we get

$$\log L(1,\chi) - \sum_{p\leq y}\chi(p)p^{-1}$$
$$\ll_n \sum_{p\leq y} p^{-1}(1-e^{-p/x}) + \sum_{y<p\leq x^2} p^{-1}e^{-p/x} + \sum_{x^2<p} p^{-1}e^{-p/x} + 1 \ll 1.$$

finishing the proof of Proposition 5.□

7 Extreme L-values at $s = 1$.

We now use Propositions 3 and 5 to prove Theorem 2 and Propositions 4 and 5 to prove Theorem 1 as well as justify the upper bound (6) and the unconditional result (5).

The upper bound (6): Observe first that the upper bound (6) mentioned before Theorem 2 follows from Proposition 5, giving (under GRH)

$$\log|L(1,\chi)| \leq n \sum_{p\leq(\log N)^{\frac{1}{2}}} p^{-1} + O(1) \leq n\log\log\log N + O(1), \qquad (25)$$

and hence (6).

Proof of Theorem 2: Taking $a(p) = n$ for all primes $p \le y$ in Proposition 3, where y is a sufficiently large parameter, we have that there exists a Galois representation whose image is G, conductor N satisfies

$$\log N \le cy \tag{26}$$

and whose character χ is unramified and satisfies

$$\chi(\sigma_p) = a(p)$$

for $c \le p \le y$, where σ_p is any Frobenius element over p. Thus by Proposition 5 we have that the associated L-function $L(s, \chi)$ satisfies

$$\log L(1, \chi) \ge \sum_{p \le (\log N)^{\frac{1}{2}}} \chi(p) p^{-1} + O(1) \ge n \sum_{c \le p \le (\log N)^{\frac{1}{2}}} p^{-1} + O(1)$$

for y sufficiently large, since then $(\log N)^{\frac{1}{2}} \le y$ follows from (26). Thus for these $L(s, \chi)$ we have

$$\log L(1, \chi) \ge n \log \log \log N + O(1).$$

The fact that such $L(s, \chi)$ have arbitrarily large N follows from Lemma 4 with $u = 0$ and $\epsilon = 1/4$, since otherwise N would be bounded yet taking $x = y^{\frac{1}{2}}$ would give, for x sufficiently large,

$$\sum_{p \le x} \log p \ll x^{\frac{3}{4}},$$

which contradicts the prime number theorem. This can also be seen by applying a uniform version of the Chebotarev theorem.

Proof of Theorem 1: For $K \in \mathcal{K}_n$ we have that

$$\zeta_K(s) = L(s, \chi) \zeta(s)$$

where χ has degree $n - 1$, conductor d and is irreducible. Also, for $p \nmid d$, we have that $\chi(\sigma_p) + 1$ is the number of primes of K over p of degree 1, so

$$\chi(\sigma_p) = n - 1$$

for primes p that split completely in K. Thus by Proposition 4 and Proposition 5 we get, as above, that there exist $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which both

$$\log L(1, \chi) \ge (n - 1) \log \log \log d + O(1)$$

and

$$R \ll_n (\log d)^{n-1}.$$

Theorem 1 now follows from (1).

The unconditional result (5): The result that there are infinitely many $K \in \mathcal{K}_n$ with

$$h \gg d^{\frac{1}{2}} (\log d)^{-n}$$

is a consequence of Proposition 4 and the estimate

$$L(1, \chi) \gg (\log d)^{-1}, \tag{27}$$

which follows from Theorem 1 of [19], since K contains no quadratic subfields in case $n > 2$.

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