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Author(s): W. Duke and H. Iwaniec

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ESTIMATES FOR COEFFICIENTS OF L-FUNCTIONS. IV

By W. Duke and H. Iwaniec

1. Introduction. In this paper we continue to investigate the growth of coefficients of Dirichlet series satisfying standard functional equations along the lines of our previous works [1,2,3]. We shall establish the absolute convergence of the series formed by squaring the coefficients in a half-plane which is wider than that known before in various important cases. When applied to the Rankin-Selberg convolution of the symmetric square L-function our Theorem 1 yields a new estimate for the eighth power-moment of the Hecke eigenvalues λ_n for a Maass cusp form. From this we derive (by interpolation) an estimate for the sixth power-moment of λ_n which in turn is utilized to explore the theory of the symmetric cube L-functions by means of the large sieve inequality with perturbation. Assuming the holomorphy of the twisted symmetric cube L-functions for sufficiently many characters we are able to give a new bound for the individual eigenvalue λ_n, namely

$$(1) \qquad |\lambda_n| \leq n^{\frac{15}{76}}\tau(n),$$

which is sharper than the best known one, namely

$$(2) \qquad |\lambda_p| < p^{\frac{1}{5}} + p^{-\frac{1}{5}}$$

due to F. Shahidi [12] (see also [6], [7], [8]). Shahidi's result is true also for cusp forms over an arbitrary number field (see [12]).

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2. The Convergence of $\mathcal{A}_2$-series. Given a sequence $\mathcal{A} = (a_n)$ of complex numbers such that the series

$$\mathcal{A}(s) = \sum_1^{\infty} a_n n^{-s} \tag{3}$$

converges absolutely in $Re\, s > \sigma_1 = 1$ we wish to know the abscissa σ_2 of absolute convergence of the series

$$\mathcal{A}_2(s) = \sum_1^{\infty} a_n^2 n^{-s}. \tag{4}$$

Trivially, by the absolute convergence of (3) in $Re\, s > 1$ it follows that

$$a_n \ll n^{1+\varepsilon},$$

hence (4) converges absolutely in $Re\, s > 2$, i.e. we have

$$\sigma_2 \leq 2.$$

Clearly this is the best that one can claim in full generality. However, we can do better if the L-functions

$$\mathcal{A}(s, \chi) = \sum_1^{\infty} a_n \chi(n) n^{-s}$$

twisted by Dirichlet characters $\chi \pmod p$ are entire (of finite order) and satisfy standard functional equations

$$\theta(s)\mathcal{A}(s, \chi) = \varepsilon_\chi \theta(1-s)\mathcal{A}(1-s, \bar{\chi}) \tag{5}$$

with $|\varepsilon_\chi| = 1$ and

$$\theta(s) = \left(\frac{p}{\pi}\right)^{\frac{ks}{2}} \prod_{j=1}^{k} \Gamma\left(\frac{s}{2} + \kappa_j\right), \quad Re\, \kappa_j \geq 0. \tag{6}$$

The number k of the gamma factors in $\theta(s)$ is called the degree of $\mathcal{A}(s)$. We need the above properties only for a set of primes of positive density and for each p from the set we allow a few (bounded number of) exceptions in the sense that we claim nothing about $\mathcal{A}(s, \chi)$ for the exceptional characters $\chi \pmod p$. In [1] we exploited the distribution of the sign ε_χ of the functional equations (by means of

Deligne's estimate for generalized Kloosterman sums) to show that

$$a_n \ll n^{\frac{k-1}{k+1}+\varepsilon}. \tag{7}$$

Consequently the abscissa of absolute convergence of $\mathcal{A}_2(s)$ satisfies

$$\sigma_2 \le 2\frac{k}{k+1}. \tag{8}$$

In [2] we considered L-functions for degree $k = 3$ to show that

$$\sigma_2 \le 1 \tag{9}$$

under somewhat extended conditions. This, of course, is the best possible result. In the proof neither the sign ε_χ nor the Kloosterman sums play a role.

Our main result in this paper is the following improvement on (8) for all $k \ge 2$.

THEOREM 1. *Suppose $\mathcal{A}(s)$ has degree $k \ge 2$ and that the functional equations (5) hold true for almost all even characters χ to prime moduli p in a set of positive density (for each p in the set the number of exceptional characters is bounded). Then the series $\mathcal{A}_2(s)$ converges absolutely in $Re\, s > 2\frac{k-1}{k}$, i.e. we have*

$$\sigma_2 \le 2\frac{k-1}{k}. \tag{10}$$

Remark. We exploit only even characters for convenience. Among them only the total number of exceptional characters has to be controlled rather than for each modulus separately.

Let G_p denote the group of even characters $\chi \pmod p$ and let H_p be the subset of exceptional characters. Thus $\mathcal{A}(s, \chi)$ is entire (of finite order) for χ in G_p but not in H_p and it satisfies the functional equation

$$\mathcal{A}(1-s, \chi) = \varepsilon_\chi \Phi(s) \mathcal{A}(s, \bar{\chi}) \tag{11}$$

with $|\varepsilon_\chi| = 1$, where $\Phi(s)$ is holomorphic in $Re\, s = \sigma > \frac{1}{2}$ such that

$$\Phi(s) \ll (p|s|)^{\left(\sigma-\frac{1}{2}\right)k} \tag{12}$$

by Stirling's formula, the implied constant depending on σ, k and κ_j only. We emphasize that $\Phi(s)$ depends on p but not on the characters $\chi \pmod p$.

In what follows it is essential to work with finite sums

$$\mathcal{A}_f(\chi) = \sum_n a_n \chi(n) f(n)$$

instead of the infinite series $\mathcal{A}(s,\chi)$. Here f is a smooth test function supported in $[X, 2X]$ with derivatives bounded by $f^{(j)} \ll X^{-j}$. For $\chi \in G_p \setminus H_p$ we apply contour integration and the functional equation (11) getting

$$\mathcal{A}_f(\chi) = \varepsilon_\chi \mathcal{A}_g(\bar{\chi}), \tag{13}$$

where g is the integral transform of f given by

$$g(y) = \frac{1}{2\pi i} \int_{(\sigma)} F(s)\Phi(s) y^{-s} ds \tag{14}$$

and $F(s)$ is the Mellin integral which satisfies

$$F(s) = \int_0^\infty f(x) x^{-s} dx \ll |s|^{-A} X^{1-\sigma} \tag{15}$$

for any $\sigma > \frac{1}{2}$ with arbitrary $A > 0$, the implied constant depending on σ and A.

In the sequel ε will denote an arbitrarily small positive number, not always the same one in each occurrence. By the absolute convergence of $\mathcal{A}(s)$ for $Re\, s > \sigma_1 = 1$ and by (14) with $\sigma = 1 + \varepsilon$ we obtain

$$\mathcal{A}_f(\chi) \ll X^{1+\varepsilon}$$

and

$$\mathcal{A}_g(\chi) \ll p^{\frac{k}{2}+\varepsilon}$$

respectively. Hence by (13) we get

$$\sum_{\chi \in G_p} (|\mathcal{A}_f(\chi)|^2 - |\mathcal{A}_g(\chi)|^2) \ll |H_p|(p^k + X^2) X^\varepsilon,$$

where $|H_p|$ denotes the cardinality of the exceptional set H_p. The left-hand side is equal to

$$\frac{p-1}{2} \sum\sum_{\substack{m \equiv \pm n \pmod p \\ (p,mn)=1}} a_m \bar{a}_n (f(m) f(n) - g(m)\bar{g}(n)).$$

We sum both sides over primes p in a set $\mathcal{P} \subset [P, 2P]$ getting

$$\sum_m \sum_n a_m \bar{a}_n f(m) f(n) \omega(m,n) \ll \sum_m \sum_n |a_m a_n| h(m) h(n) \omega(m,n) + R(P^k + X^2) X^{\varepsilon},$$

where h is a majorant of $|g|$ which does not depend on p,

$$\omega(m,n) = \sum_{\substack{p \in \mathcal{P}, (p,mn)=1 \\ p \mid (m^2 - n^2)}} \frac{p-1}{2} = \delta_{mn} Q + O(P \log mn),$$

δ_{mn} is the diagonal symbol of Kronecker, $Q = \sum_{p \in \mathcal{P}} \frac{p-1}{2}$ and $R = \sum_{p \in \mathcal{P}} |H_p|$. Hence

$$\sum_n |a_n f(n)|^2 \ll \sum_n |a_n h(n)|^2 + Q^{-1}(P + R)(P^k + X^2) X^{\varepsilon}.$$

We assume that $Q \gg P^{2-\varepsilon}$ and $R \ll P^{1+\varepsilon}$. We choose $P^k = X^2$ getting

$$\sum_n |a_n f(n)|^2 \ll \sum_n |a_n h(n)|^2 + X^{2\frac{k-1}{k} + \varepsilon}.$$

Furthermore, by (12), (14) and (15) we get

$$g(n) \ll P^{(\sigma - \frac{1}{2})k} (nX)^{-\sigma} X = n^{-\sigma} X^{\sigma} = h(n)$$

for any $\sigma > \frac{1}{2}$. We apply this bound with any $\sigma > \frac{1}{2}\sigma_2$ giving

$$\sum_n |a_n f(n)|^2 \ll X^{2\sigma} + X^{2\frac{k-1}{k} + \varepsilon}.$$

Hence (10) follows.

3. Power-Moments of Hecke Eigenvalues. We shall apply Theorem 1 to estimate the sums

$$\Lambda_r(x) = \sum_{n \le x} |\lambda_n|^r,$$

where λ_n is the eigenvalue of the Hecke operator T_n for a Maass cusp form $u(z)$ for the modular group $\Gamma = SL_2(\mathbb{Z})$. The eigenvalues λ_n are multiplicative; precisely we have

$$\lambda_m \lambda_n = \sum_{d \mid (m,n)} \lambda_{mnd^{-2}},$$

so the associated L-function has the Euler product

$$L_1(s) = \sum_1^\infty \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + p^{-2s})^{-1},$$

whose local factors have degree 2. They factor further into linear polynomials

$$1 - \lambda_p p^{-s} + p^{-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})$$

with $\alpha_p + \beta_p = \lambda_p$ and $\alpha_p \beta_p = 1$. The following assertions (known as the Ramanujan conjecture) are equivalent: $\alpha_p = \bar{\beta}_p$, $|\alpha_p| = 1$, $|\lambda_p| \leq 2$, $|\lambda_n| \leq \tau(n)$, $|\lambda_n| \ll n^\varepsilon$ and

$$\Lambda_r(x) \ll x^{1+\varepsilon} \tag{16}$$

for all $r \geq 1$. We know that (16) is true for $r = 1, 2, 3, 4$. The case of $r = 4$ has been settled by F. Shahidi using analytic properties of symmetric power L-functions. The m^{th} symmetric power is defined by the Euler product of degree $m + 1$ as follows

$$L_m(s) = \prod_p \prod_{j=0}^{m} (1 - \alpha_p^{m-j} \beta_p^j p^{-s})^{-1}. \tag{17}$$

For $m = 1, 2, 3, 4, 5$ it is known that the twisted L-functions $L_m(s, \chi)$ have meromorphic continuation to the whole complex s-plane and that they satisfy suitable functional equations (see the surveys in [11] and [12]).

It is conjectured that all of $L_m(s, \chi)$ except for a finite number of primitive characters are in fact entire. This has been proved for $m = 1$ and 2. As regards the symmetric cube $L_3(s, \chi)$, Shahidi showed it may have at most a finite number of poles in the segment $[\frac{1}{2}, 1)$. The case $m = 4$ is even less clear. Shahidi succeeded in showing that the product

$$\mathcal{A}(s, \chi) = L_0(s, \chi) L_2(s, \chi) L_4(s, \chi)$$

satisfies a functional equation of type (5) with $k = 9$. Moreover recent work of J.-L. Waldspurger (see the appendix of [5]) shows that $\mathcal{A}(s, \chi)$ is entire for almost all primitive characters. Finally the work of H. Jacquet and J.A. Shalika [4] implies that the series

$$\mathcal{A}(s) = L_0(s) L_2(s) L_4(s) = \sum_1^\infty a_n n^{-s}$$

converges absolutely for $Re\, s > 1$. Therefore Theorem 1 is applicable giving

$$\sum_{n\leq x} |a_n|^2 \ll x^{\frac{16}{9}+\varepsilon}.$$

If the Ramanujan conjecture fails at a place p then $|a_p| \asymp |\lambda_p|^4$ and we have $a_p \ll 1$ at all other places. Therefore the above estimate yields

$$\sum_{p\leq x} |\lambda_p|^8 \ll x^{\frac{16}{9}+\varepsilon}.$$

Hence by the multiplicativity of λ_n we get

THEOREM 2. *We have*

$$\Lambda_8(x) \ll x^{\frac{16}{9}+\varepsilon}. \tag{18}$$

Remark. One may try to estimate $\Lambda_8(x)$ directly by using (16) for $r = 4$ and estimates for individual eigenvalues λ_ℓ. But then in order to claim (18) one needs a bound

$$\lambda_\ell \ll \ell^{\frac{7}{36}+\varepsilon}$$

which is not yet known. Therefore (18) contains new information.

Applying Cauchy's inequality $\Lambda_6 \leq (\Lambda_4\Lambda_8)^{\frac{1}{2}}$ we obtain

COROLLARY 1. *We have*

$$\Lambda_6(x) \ll x^{\frac{25}{18}+\varepsilon}. \tag{19}$$

In the next section we shall need an estimate for the second moment of the coefficients of the symmetric cube L-function,

$$L_3(s) = \sum_1^\infty b_n n^{-s}. \tag{20}$$

Arguing as before we have $|b_p| \asymp |\lambda_p|^3$ at the places where the Ramanujan conjecture fails and $|b_p| \ll 1$ at all other places. Hence (19) yields

COROLLARY 2. *We have*

$$\sum_{n\leq x} |b_n|^2 \ll x^{\frac{25}{18}+\varepsilon}. \tag{21}$$

Furthermore we shall need an estimate for $|b_n|$ on average over a short interval.

LEMMA. *If* $x^{\frac{5}{6}} < y < x$ *we have*

$$\sum_{x<n<x+y} |b_n| \ll yx^{\varepsilon}. \tag{22}$$

Proof. We have $a_n \geq 0$,

$$\sum_{n\leq x} a_n = cx + O(x^{\frac{4}{5}+\varepsilon}),$$

where c is a positive constant (see [7]) and

$$b_n \ll n^{\varepsilon} \sum_{d|n} a_d.$$

Hence

$$\begin{aligned} \sum_{x<n<x+y} b_n &\ll x^{\varepsilon} \sum_{x<dh<x+y} a_d \\ &< x^{\varepsilon} \sum_{h<\frac{x}{y}} \sum_{\frac{x}{h}<d<\frac{x+y}{h}} a_d + x^{\varepsilon} \sum_{d<2y} a_d \sum_{\frac{x}{d}<h<\frac{x+y}{d}} 1 \\ &\ll x^{\varepsilon} \sum_{h<\frac{x}{y}} \left(\frac{y}{h} + \left(\frac{x}{h}\right)^{\frac{4}{5}} \right) + x^{\varepsilon} \sum_{d<2y} a_d \frac{y}{d} \\ &\ll x^{\varepsilon} \left(y + xy^{-\frac{1}{5}} \right) < 2yx^{\varepsilon}. \end{aligned}$$

4. Estimating Individual Hecke Eigenvalues. We proceed as in [3]. We evaluate the mean-value of the coefficients b_n of $L_3(s)$ over a short arithmetic progression. More precisely, we consider the sum

$$\frac{p-1}{2} \sum_{n\equiv\pm\ell \pmod p} b_n f(n) \tag{23}$$

where f is a smooth test function supported in $[x, x+y]$ with derivatives bounded by $f^{(j)} \ll y^{-j}$, where $x^{\frac{5}{6}} < y < x$. We sum over primes p in a set $\mathcal{P} \subset [P, 2P]$ getting

$$\begin{aligned} \sum_n b_n f(n) \omega(\ell, n) &= \sum_n b_n f(n) (\delta_{\ell n} Q + O(P \log \ell n)) \\ &= b_\ell f(\ell) Q + O(P y x^{\varepsilon}) \end{aligned}$$

by Lemma 1. On the other hand (23) is equal to

$$\sum_{\chi \in G_p} \bar{\chi}(\ell)\mathcal{B}_f(\chi),$$

where

$$\mathcal{B}_f(\chi) = \sum_n b_n \chi(n) f(n) \ll y x^{\varepsilon}.$$

This is a trivial bound (following from Lemma 1) which we use only for exceptional characters $\chi \in H_p$. For all other characters $\chi \in G_p \setminus H_p$ we make the assumption that

(24) $$L_3(s, \chi) \text{ is entire.}$$

In fact Shahidi has proved in [10] (see Corollary 4.2) that the assumption (24) follows from the conjecture that the Hecke-Jacquet-Langlands L-functions do not vanish in $[\frac{1}{2}, 1)$. It is known (without further assumptions) that $L_3(s, \chi)$ satisfies a functional equation of type (5) with $k = 4$ gamma factors. By contour integration and by (5) we get

$$\mathcal{B}_f(\chi) = \varepsilon_\chi \mathcal{B}_g(\bar{\chi}),$$

where g is the corresponding integral transform of f given by (14) and (15). Combining the above results we get

(25) $$b_\ell f(\ell) Q = \sum_{p \in \mathcal{P}} \sum_{\chi \in G_p \setminus H_p} \varepsilon_\chi \bar{\chi}(\ell) \mathcal{B}_g(\bar{\chi}) + O(P y x^{\varepsilon}).$$

We evaluate g as in [2] and [3] by the stationary phase method giving

$$g(m) = p^{-\frac{1}{2}} y (xm)^{-\frac{3}{8}} \, Re \, e_p(v(m)) h(m),$$

where $v(m) = 4(xm)^{\frac{1}{4}}$ and $h(m)$ is a smooth function satisfying

$$h^{(j)}(m) \ll m^{-j} \left(\frac{M}{m}\right)^{\nu}$$

for $j = 0, 1$ and all $\nu \geq 0$ with $M = (PT)^4 x^{-1}$ and $T = xy^{-1}$, the implied constant depending on ν. Hence it follows that $h(m)$ is extremely small for $m > x^{\varepsilon} M$. For $m < x^{\varepsilon} M$, $h(m)$ is bounded. We remove $h(m)$ by partial summation at no cost and obtain

$$\mathcal{B}_g(\chi) \ll P^{-\frac{1}{2}} y x^{-\frac{3}{8}} \left| \sum_{m<M_1} b_m m^{-\frac{3}{8}} \chi(m) e_p(\pm v(m)) \right| + 1,$$

where $M_1 < x^{\varepsilon} M$. Inserting this into (25) we get by Cauchy's inequality that

$$(b_\ell f(\ell) Q)^2 \ll P y^2 x^{-\frac{3}{4}} \sum_{p \in \mathcal{P}} \sum_{\chi \in G_p \backslash H_p} \left| \sum_{m<M_1} b_m m^{-\frac{3}{8}} \chi(m) e_p(\pm v(m)) \right|^2 + P^2 y^2 x^{\varepsilon}.$$

Then by Theorem 2 of [2] (the large sieve inequality with perturbation) we get

$$(b_\ell f(\ell) Q)^2 \ll P y^2 x^{-\frac{3}{4}} (M_1 + P^2 T^{\frac{1}{2}} \log x) \left(\sum_{m<M_1} |b_m|^2 m^{-\frac{3}{4}} \right) + P^2 y^2 x^{\varepsilon}.$$

Next by Corollary 2 we get

$$(b_\ell f(\ell))^2 \ll x^{\varepsilon} P^{-3} y^2 x^{-\frac{3}{4}} (M + P^2 T^{\frac{1}{2}}) M^{\frac{23}{36}} + x^{\varepsilon} P^{-2} y^2.$$

We choose $y = x^{\frac{50}{57}}$ and $P = x^{\frac{65}{228}}$ giving

$$(b_\ell f(\ell))^2 \ll x^{\frac{45}{38}+\varepsilon}.$$

We may assume that $f(\ell) = 1$ by taking $x \sim \ell$, so

$$b_\ell \ll \ell^{\frac{45}{76}+\varepsilon}.$$

Hence

$$|\lambda_\ell|^3 \ll \ell^{\varepsilon} \sum_{d|\ell} |b_d| \ll \ell^{\frac{45}{76}+\varepsilon}$$

and finally

$$\lambda_\ell \ll \ell^{\frac{45}{76}+\varepsilon}, \tag{26}$$

where the implied constant depends on ε and the cusp form $u(z)$ only. This estimate improves itself to give (1) by the multiplicativity property of λ_n.

Added in Proof (July 7, 1993). Since this work was submitted for publication the estimate (1) has been improved by D. Bump, J. Hoffstein, and the authors (*An estimate for the Hecke eigenvalues of Maass forms,* Inter. Math. Res. Notices

No. 4. (1992), in Duke Math. J., **66** p. 75–81). However, the main result of this paper, Theorem 1, remains unsurpassed.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08903 USA

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