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Estimates for coefficients of L-functions. III

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1. Introduction

In this sequence of papers we investigate Dirichlet series

$$\mathcal{A}(s,\chi)=\sum_{1}^{\infty} a_n\chi(n)n^{-s} \tag{1}$$

having Euler products and compatible functional equations with the aim of estimating the coefficients a_n. It was shown in [2], [3] by different techniques that if the analytic continuation and the functional equations hold for sufficiently many characters then a suitable upper bound for a_n is true which is considerably better than that resulting from the absolute convergence of the series. In this installment we combine both techniques to give new results and improve on those of [3] when $\mathcal{A}(s,\chi)$ has an Euler product of degree 3.

Let p be an odd prime. We assume that for every non-principal even character $\chi \pmod p$ the following hold :

(2) $\mathcal{A}(s,\chi)$ converges absolutely in $\Re e s>1$ and has analytic continuation to an entire function in s-plane,

(3) $\mathcal{A}(s,\chi)$ satisfies the functional equation

$$\theta(s)\mathcal{A}(s,\chi)=\varepsilon_\chi\theta(1-s)\mathcal{A}(1-s,\overline{\chi})$$

with $|\varepsilon_\chi|=1$ and

$$\theta(s)=\left(\frac{p}{\pi}\right)^{\alpha s}\prod_j\Gamma(\alpha_j s+\beta_j), \tag{4}$$

where α_j are positive numbers with $\Sigma\alpha_j = \alpha$ and β_j are complex numbers with $\alpha_j + 2\Re e\beta_j > 0$ (α_j, β_j are independent of χ and p),

$$S_p(a) := \sum_{\chi (\operatorname{mod} p)}^{+} \varepsilon_\chi \overline{\chi}(a) \ll p^{\frac{1}{2}}, \tag{5}$$

where $+$ means that the summation ranges over non-principal, even characters.

Remarks : in many cases α is half an integer, $\alpha = \frac{k}{2}$ say, and $\varepsilon_\chi = \tau^k(\chi)p^{\frac{-k}{2}}$, where $\tau(\chi)$ is the Gauss sum

$$\tau(\overline{\chi}) = \sum_{x (\operatorname{mod} p)} \overline{\chi}(x) e_p(x). \tag{6}$$

In such case $S_p(a)$ can be represented in terms of k-dimensional Kloosterman sums. Precisely, we have

$$S_p(a) = (p-1)p^{\frac{-k}{2}} \sum_{x_1 \cdots x_k \equiv a (\operatorname{mod} p)} \cos\Big(\frac{2\pi}{p}(x_1 + \cdots + x_k)\Big) \tag{7}$$

so (5) follows from the Deligne estimate [1].

We shall evaluate the mean-value of a_n over a short interval of an arithmetic progression to modulus p. Let f be a smooth function supported in $[x, x+y]$ with $2 \leq y \leq x$ such that

$$f^{(\nu)} \ll y^{-\nu} \tag{8}$$

for all $\nu \geq 0$, the implied constant depending on ν only. Let $(a,p) = 1$. Put

$$\mathcal{D}_f(p;a) = \sum_{n \equiv \pm a (\operatorname{mod} p)} a_n f(n) - \frac{2}{p-1} \sum_{(n,p)=1} a_n f(n).$$

THEOREM 1. — *Suppose* $(2-5)$ *hold with* $\alpha \geq 1/2$ *and that* f *satisfies* (8). *For any* $a \not\equiv 0 \ (\operatorname{mod} p)$ *we have*

$$\mathcal{D}_f(p;a) \ll x^\varepsilon (pT)^{\alpha - \frac{1}{2}} \tag{9}$$

where $T = xy^{-1}$ *and* ε *is any positive number, the implied constant depending on* ε *and the sequence* $\mathcal{A} = (a_n)$ *only.*

Our main interest is in Euler products of degree $k = 2\alpha = 3$. In this case we have shown in [3] that

$$(10) \qquad \sum_1^\infty a_n^2 n^{-2} \text{ converges absolutely in } \Re e\ s > 1$$

subject to the above conditions for all primitive characters. We shall use (10) to improve on (9) on average with respect to the moduli p. The sign ε_χ of the functional equation does not play a role in [3] nor in the proof of the following.

THEOREM 2. — *Suppose* $(2-4)$ *and* (10) *hold with* $\alpha = 3/2$ *for prime moduli* $p \in \mathcal{P} \subset [1, P]$ *and that* (10) *is true. Suppose* f *satisfies* (8)*. We then have*

$$(11) \qquad \sum_{\substack{p\in\mathcal{P}\\ p\nmid a}} |\mathcal{D}_f(p;a)|^2 \ll x^\varepsilon P(P^2T + PT^2 + (xy)^{\frac{1}{2}}).$$

for any $\varepsilon > 0$*, the implied constant depending on* ε *and* $\mathcal{A}$ *only.*

COROLLARY 1. — *Suppose* $(2-4)$ *and* (10) *hold with* $\alpha = 3/2$ *for prime moduli of positive density. Moreover suppose that* (10) *holds and that*

$$(12) \qquad \sum_{x<n<x+y} |a_n| \ll x^\varepsilon y \quad \text{if} \quad x > y > x^{\frac{3}{5}}.$$

Then for all $n \geq 1$ *we have*

$$(13) \qquad a_n \ll n^{\frac{3}{7}+\varepsilon}.$$

If we apply (13) for the coefficients of the symmetric square zeta-function associated with a Maass cusp form for $SL_2(\mathbb{Z})$ as in [3] we conclude that (see comments in [3] about the best known results).

COROLLARY 2. — *Suppose* $\lambda(n)$ *are eigenvalues of the Hecke operators* T_n *of a cusp form* $u(z)$ *for the modular group. We then have*

$$\lambda(n) \ll n^{\frac{3}{14}+\varepsilon}.$$

2. An application of the functional equations

We have

$$\sum_{\chi (\operatorname{mod} p)}^{+} \chi(a) = \begin{cases} \frac{p-1}{2} & \text{if } a \equiv \pm 1 (\operatorname{mod} p) \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathcal{D}_f(p; a) = \frac{2}{p-1} \sum_{\chi (\operatorname{mod} p)}^{+} \overline{\chi}(a) \mathcal{A}_f(\chi),$$

where

$$\mathcal{A}_f(\chi) = \sum_n a_n \chi(n) f(n).$$

The functional equation (3) yields (through contour integration)

$$\mathcal{A}_f(\chi) = \varepsilon_\chi \mathcal{A}_g(\overline{\chi}),$$

where

$$g(v) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{f}(s) \frac{\theta(s)}{\theta(1-s)} v^{-s} ds, \quad \sigma > 1 \tag{14}$$

and $\hat{f}$ is the Mellin integral,

$$\hat{f}(s) = \int_0^\infty f(u) u^{-s} du.$$

Combining these rélations we obtain

$$\mathcal{D}_f(p; a) = \frac{2}{p-1} \sum_m a_m S_p(am) g(m). \tag{15}$$

3. Evaluation of g

By repeated partial integration ν times we get

$$\hat{f}(s) \ll y x^{-\sigma} \Big(\frac{T}{|s|} \Big)^\nu \tag{16}$$

where $\sigma = \Re e s \geq \frac{1}{2}$, the implied constant depending on σ and ν. Choosing ν sufficiently large we infer that

$$\hat{f}(s) \ll |s|^{-2} x^{-A} \quad \text{if} \quad |s| \geq x^\varepsilon T \tag{17}$$

with any $A > 0$, the implied constant depending on σ, ε and A. Moreover by Stirling's formula we have

$$\frac{\theta(s)}{\theta(1-s)} \ll (p|s|)^{\alpha(2\sigma-1)} \quad \text{if} \quad \sigma \geq \frac{1}{2}. \tag{18}$$

Inserting (18) and (16) with $\nu = 2\alpha\sigma$ into (14) we get

$$g(v) \ll yp^{-\alpha}\left(\frac{M}{v}\right)^{\sigma},$$

where $M = x^{-1}(pT)^{2\alpha}$. Choosing σ sufficiently large we conclude that

$$g(v) \ll v^{-2}x^{-A} \quad \text{if} \quad v \geq x^{\varepsilon}M \tag{19}$$

with any $A > 0$, the implied constant depending on ε and A.

For $v < x^{\varepsilon}M$ we shall evaluate $g(v)$ asymptotically by the stationary phase method. First we move the integration to the line $\sigma = 1/2$, next we truncate the integral at the heights $\pm x^{\varepsilon}T$ controlling the error term by means of (17) and (18) and change the order of integration giving

$$g(v) = \int_0^{\infty} f(u)I(uv)(uv)^{-\frac{1}{2}}\,du + O(x^{-A}) \tag{20}$$

with

$$I(w) = \frac{1}{2\pi}\int_{-x^{\varepsilon}T}^{x^{\varepsilon}T} \frac{\theta(\frac{1}{2}+it)}{\theta(\frac{1}{2}-it)} w^{-it}\,dt. \tag{21}$$

By Stirling's formula [for $s = \sigma + it$ with $t \geq 1$]

$$\Gamma(s) = \sqrt{2\pi}\left(\frac{t}{e}\right)^{it} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}t}\{1 + O(t^{-1})\},$$

we get

$$\frac{\theta(\frac{1}{2}+it)}{\theta(\frac{1}{2}-it)} = \left[\frac{\gamma}{\pi e}|t|p\right]^{2i\alpha t} + O((1+|t|)^{-1}),$$

where $\gamma = \Pi\alpha_j^{\alpha_j/\alpha}$ and the implied constant depends on α_j, β_j. Hence

$$I(w) = \pi^{-1}\Re e \int_0^{x^{\varepsilon}T} e^{ih(t)}\,dt + O(\log x)$$

where $h(t) = 2\alpha t \log\left(\frac{\gamma}{\pi e}tp\right) - t \log w$. The saddle point $t_0 = \frac{\pi}{\gamma p} w^{\frac{1}{2\alpha}}$ lies in the range of integration so we have (see [5])

$$I(w) = (\alpha\gamma p)^{-\frac{1}{2}} \Re e\, \zeta_8 e\left(\frac{-\alpha}{\gamma p} w^{\frac{1}{2\alpha}}\right) + O(\log x). \tag{22}$$

Inserting (22) to (20) we get

$$g(v) = (\alpha\gamma p)^{-\frac{1}{2}} \int_0^\infty f(u) \Re e \zeta_8 e\left(\frac{-\alpha}{\gamma p}(uv)^{\frac{1}{2\alpha}}\right)(uv)^{-\delta} du + O\left[\frac{y \log x}{\sqrt{vx}}\right] \tag{23}$$

for $v < x^\varepsilon M$, where $\delta = \frac{1}{2} - \frac{1}{4\alpha}$.

4. Proof of Theorem 1

By (15), (19) and (23) we obtain

$$\begin{aligned} \mathcal{D}_f(p;a) &= \tfrac{2}{p-1} \textstyle\sum_{m<x^\varepsilon M} a_m S_p(am) g(m) + O(x^{-A}) \\ &= \Re e \int f(u) L_p(u) u^{-\delta} du + O(x^\varepsilon p^{\alpha-\frac{1}{2}} T^{\alpha-1}), \end{aligned} \tag{24}$$

where

$$L_p(u) = c_p \sum_{m<x^\varepsilon M} a_m m^{-\delta} S_p(am) e_p(v(m)) \tag{25}$$

with $c_p = 2(p-1)^{-1}(-i\alpha\gamma p)^{-\frac{1}{2}} \asymp p^{-\frac{3}{2}}$ and $v(m) = -2\alpha\gamma^{-1}(um)^{\frac{1}{2\alpha}}$. By (2) and (5) we get trivially $L_p(u) \ll x^\varepsilon p^{-1} M^{1-\delta}$ whence (9) follows by (24).

5. Proof of Theorem 2

We may assume without loss of generality that $\mathcal{P} \subset \left(\frac{1}{2}P, P\right]$. By (24) and (25) we obtain

$$\sum_{p\in\mathcal{P}, p\nmid a} |\mathcal{D}_f(p,a)|^2 \ll x^\varepsilon |\mathcal{P}| P^2 T + x^{\varepsilon-\frac{2}{3}} y^2 P^{-3} D, \tag{26}$$

where

$$D = \sum_{p\in\mathcal{P}} \Big| \sum_{m<M} a_m m^{-\frac{1}{3}} S_p(am) e_p(v(m)) \Big|^2$$

with $M = x^{\varepsilon-1} P^3 T^3$ and $v(m) = -3\gamma^{-1}(um)^{\frac{1}{3}}$ for some $u \in [x, x+y]$. By Cauchy's inequality and by Theorem 2 of [3] we get

$$\begin{aligned} D &\le \sum_{p\in\mathcal{P}} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, p)} \Big| \sum_{m<M} a_m m^{-\frac{1}{3}} \overline{\chi}(m) e_p(v(m)) \Big|^2 \\ &\ll P(M + P^2 T^{\frac{1}{2}} \log x) \sum_{m<M} |a_m|^2 m^{-\frac{2}{3}}. \end{aligned}$$

Here $\Sigma \ll M^{\frac{1}{3}+\varepsilon}$ by the assumption (10). Inserting these estimates into (26) we obtain (11).

6. Proof of Corollaries

Let $x < \ell < x+y$ and $f(\ell)=1$. We have

$$|a_\ell| = |\mathcal{D}_f(p,\ell)| + O\Big[\sum_{\substack{n\equiv\pm\ell(\bmod p)\\ n\neq\ell}} |a_n f(n)| + \frac{1}{p}\sum_n |a_n f(n)|\Big].$$

Summing over primes p in a set $\mathcal{P}\subset[1,P]$ with $p\nmid\ell$ we get

$$|\mathcal{P}||a_\ell| = \sum_{p\in\mathcal{P}} |\mathcal{D}_f(p,\ell)| + O\Big[x^\varepsilon \sum_{x<n<x+y} |a_n|\Big]$$

$$\ll P(PT^{\frac{1}{2}} + P^{\frac{1}{2}}T + (xy)^{\frac{1}{4}})x^\varepsilon + yx^\varepsilon$$

by (11) and (12) provided $x > y > x^{3/5}$. Suppose $|\mathcal{P}| \asymp P(\log P)^{-1}$. We get

$$a_\ell \ll (PT^{\frac{1}{2}} + P^{\frac{1}{2}}T + (xy)^{\frac{1}{4}} + P^{-1}y)x^\varepsilon.$$

We choose $P = T = x^{2/7}$, so $y = x^{5/7}$ and all four terms are equal giving $a_\ell \ll x^{3/7+\varepsilon}$ which proves (13).

To prove Corollary 2 we apply Corollary 1 for the symmetric square zeta-function whose coefficients are defined by

$$a_n = \sum_{dk^2=n} \lambda(d^2).$$

The hypothesis (12) follows from the asymptotic formula of Selberg (see [4])

$$\sum_{n\le N} \lambda^2(n) = cN + O\Big(N^{\frac{3}{5}+\varepsilon}\Big).$$

For n prime (13) yields $\lambda(n)^2 = \lambda(n^2)+1 = a_n + 1 \ll n^{3/7+\varepsilon}$ whence $\lambda(n) \ll n^{3/14+\varepsilon}$. This result extends to prime powers by a recursive formula and to all composite numbers by the multiplicativity.

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