ESTIMATES FOR COEFFICIENTS OF L–FUNCTIONS, II

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1. Introduction

As announced in [2] we shall investigate in this paper Dirichlet series

$$\mathcal{A}(s,\chi) = \sum_1^\infty a_n \chi(n) n^{-s}$$

having Euler product with degree 3 and compatible functional equations. We assume that the series converge absolutely for $\mathrm{Re}\, s > 1$ and prove that the series formed by squaring the coefficients also converge absolutely for $\mathrm{Re}\, s > 1$. This result yields immediately a good bound for an individual coefficient, namely that $a_n \ll n^{\frac{1}{2}+\epsilon}$.

In order to get a better bound we shall estimate a sum of $|a_n|^2$ in a short interval. The shorter the interval is, the better the bound for $|a_n|$ follows. However, it is not true that the interval of length $N = 1$ would be the best to choose for the study. In fact the optimal N is predicted by observing the uncertainty principle of harmonic analysis which strongly depends on the type of harmonics being employed. To detect terms of $\mathcal{A} = (a_n)$ in a short interval, traditionally one uses the oscillatory functions $e(nx)$ or n^{it}. In addition to these functions we exploit the additive characters $e_q(an)$ and the multiplicative characters $\chi(n)$ to select terms from an arithmetic progression. Then by special averaging over the moduli we pick up the diagonal $\mathcal{D} = (|a_n|^2)$ of the sequence $\mathcal{A}\mathcal{A} = (a_m \bar{a}_n)$. We regard our approach as an alternative to the circle method as well as to the Rankin–Selberg convolution method for zeta–functions of automorphic forms.

The power of our approach is drawn from the general principle that more characters (independent in the sense of orthogonality) produce a stronger detector. A quantitative analysis of this principle can be made in terms of the size of conductors but we do not

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† Research supported by NSF Grant No. DMS–8902992

dwell to elaborate it here. We just mention that there are $\sim Q^2$ distinct primitive characters of conductor $\sim Q$ while only $\sim T$ independent (asymptotically orthogonal) 'characters' n^{it} with $t \sim T$. There is no particular reason to restrict ourselves to the Dirichlet characters. Probably the Fourier coefficients of cusp forms may serve well as detectors giving plenty of new possibilities.

Acknowledgement. The authors thank Peter Sarnak and the Mathematics Department at Stanford for providing us with excellent conditions to work on this paper in the Summer of 1989. The second author acknowledges the financial support from the organizers of the Amalfi Symposium.

2. Statement of results

Let $\mathcal{A} = (a_n)$ be a sequence of complex numbers such that the series

$$\mathcal{A}(s) = \sum_1^\infty a_n n^{-s}$$

converges absolutely in $\operatorname{Re} s > 1$, where it has the Euler product of type

$$\mathcal{A}(s) = \prod_p (\mathcal{A}_p(p^{-s}))^{-1}$$

with

(1) $$\mathcal{A}_p(X) = 1 + a_1(p)\, X + a_2(p)\, X^2 + a_3(p)\, X^3.$$

Suppose for any primitive character χ to modulus q the series

$$\mathcal{A}(s,\chi) = \sum_1^\infty a_n\, \chi(n)\, n^{-s}$$

has analytic continuation to an entire function and that it satisfies the functional equation

(2) $$\theta(s)\mathcal{A}(s,\chi) = \epsilon_\chi\, \theta(1-s)\mathcal{A}(1-s, \overline{\chi})$$

with $|\epsilon_\chi| = 1$ and

(3) $$\theta(s) = \left(\tfrac{q}{\pi}\right)^{\frac{3}{2}s} \Gamma\left(\tfrac{s}{2} + \kappa_1\right)\Gamma\left(\tfrac{s}{2} + \kappa_2\right)\Gamma\left(\tfrac{s}{2} + \kappa_3\right),$$

where κ_j are complex numbers with $\operatorname{Re} \kappa_j \geq 0$ depending on the parity of χ but not on χ otherwise.

THEOREM 1. For $M > N > M^{3/4}$ we have

$$\sum_{M<n\leq M+N} |a_n|^2 \ll (MN)^{\frac{1}{2}+\epsilon} \tag{4}$$

with any $\epsilon > 0$, the implied constant depending on ϵ.

COROLLARY. If (1) – (3) hold then for all $n \geq 1$ we have

$$a_n \ll n^{\frac{7}{16}+\epsilon} \tag{5}$$

As an application we estimate the Fourier coefficients of a Maass cusp form $u(z)$ for the modular group. Suppose $u(z)$ is an eigenfunction of all the Hecke operators T_n with eigenvalue λ_n. After a suitable normalization the eigenvalues are the Fourier coefficients of $u(z)$. The Dirichlet series $\mathscr{A}(s)$ formed with

$$a_n = \sum_{dk^2=n} \lambda_{d^2} \tag{6}$$

is the symmetric square zeta–function attached to $u(z)$ for which the Shimura theory [10] ensures all the required properties. In this case we have the Euler product with $\mathscr{A}_p(X) = 1-a_pX + a_pX^2-X^3$. By (5) we infer

$$|\lambda_n| \leq n^{\frac{7}{32}} \tau(n). \tag{7}$$

The exponent $7/32$ in (7) is smaller than $1/4$ which is obtainable in various ways with or without Weil's bound for Kloosterman sums (see [7], [4]) and it is larger than $1/5$ which was obtained in [5], [6], [8] by an appeal to the analytic properties of the fourth symmetric power zeta–function attached to the form $u(z)$ (see [9]). We can certainly reduce $7/32$ to $3/14$ along the lines of [2] and this work (see [3]). Had we improved the perturbed large sieve inequality (9) the exponent $1/5$ would have been achieved.

It should be emphasized that we are using the divisibility of the Rankin–Selberg zeta–function by the Riemann zeta–function, so we can work with the symmetric square zeta–function. If we had worked with the Rankin–Selberg zeta–function the presence of one more gamma factor in its functional equation would have spoiled the argument. Since the holomorphy of the symmetric square zeta–function is inherited from the metaplectic

Eisenstein series on $\Gamma_0(4)$, our argument lies entirely within the GL_2 theory. Moreover we do not use Kloosterman sums in this work.

3. The large sieve inequality with perturbation

We need estimates for various character sums. The best possible result of general type is the large sieve inequality (see [1])

$$\sum_{q\leq Q}\ \sum^*_{\chi(\mathrm{mod}\ q)} \Big|\sum_{n\leq N} a_n\chi(n)\Big|^2 \ll (N+Q^2)\sum_{n\leq N}|a_n|^2.$$

In this section we extend the large sieve inequality for perturbed characters $\chi(n)e_q(v(n))$, where $v(x)$ is a real, smooth function on $\mathbb{R}^+$ such that

(8) $\quad 0 < xv'(x) < V$ and $(v'(x))^2 < |v''(x)|V$.

Without loss of generality we assume that $V \geq Q$.

THEOREM 2. For any complex numbers a_n we have

(9) $$\sum_{q\leq Q}\ \sum^*_{\chi(\mathrm{mod}\ q)} \Big|\sum_{n\leq N} a_n\chi(n)e_q(v(n))\Big|^2 \ll (N+Q^{\frac{3}{2}}V^{\frac{1}{2}}\log V)\sum_{n\leq N}|a_n|^2,$$

where the implied constant is absolute.

Proof. By way of Gauss sums the left–hand side is bounded by

$$A = \sum_{q\leq Q}\ \sum^*_{a(\mathrm{mod}\ q)} \Big|\sum_{n\leq N} a_n e_q(an+v(n))\Big|^2.$$

It suffices to estimate the dual expression

$$B = \sum_{n\leq N}\Big|\sum_{q\leq Q}\ \sum^*_{a(\mathrm{mod}\ q)} b_{aq}e_q(an+v(n))\Big|^2.$$

We have

$$B = GN + \sum_{a_1q_2\neq a_2q_1} b_{a_1q_1}\bar{b}_{a_2q_2} S\Big(\frac{a_1}{q_1}-\frac{a_2}{q_2}, \frac{1}{q_1}-\frac{1}{q_2}\Big),$$

where

$$G = \sum_{q\leq Q}\ \sum^*_{a(\mathrm{mod}\ q)} |b_{aq}|^2$$

and

$$S(\alpha,\beta) = \sum_{n\leq N} e(\alpha n+\beta v(n)).$$

By Lemma 4.7 of [11] we have

$$S(\alpha,\beta) = \sum_{|\nu|\leq V} \int_1^N e((\alpha+\nu)x + \beta v(x))dx + O(\log V).$$

Let I be the subinterval of $[1, N]$ in which $|2\beta v'(x)| > |\alpha+\nu|$. By (8) we have $|\beta v''(x)| > |\alpha+\nu|^2(4|\beta|V)^{-1}$ in I. Hence by Lemma 4.4 of [11] the integral over I is bounded in absolute value by $16(|\beta|V)^{\frac{1}{2}}|\alpha+\nu|^{-1}$. In the remaining range we can apply Lemma 4.2 of [11] showing that the relevant integral is bounded in absolute value by $16|\alpha+\nu|^{-1}$. Combining both estimates we obtain

$$\left|\int_1^N e((\alpha+\nu)x + \beta v(x))dx\right| < 16|\alpha+\nu|^{-1}(1+|\beta|V)^{\frac{1}{2}}.$$

Hence, summing over ν we get

$$S(\alpha,\beta) \ll (\|\alpha\|^{-1} + \log V)(1+|\beta|V)^{\frac{1}{2}}.$$

Finally summing over the points a_1/q_1, a_2/q_2 in a familiar way we conclude that

$$B = (N+O(Q^{\frac{3}{2}}V^{\frac{1}{2}}\log V))G$$

from which we infer (9) by the duality principle.

4. The large sieve inequality reversed

Given a sequence $\mathcal{A} = (a_n)$ of complex numbers and $Q \geq 1$ we denote

$$S^*(\mathcal{A},Q) = \sum_{q\leq Q} (qQ)^{-1} \sum_{\chi(\mathrm{mod}\ q)}^{*} \left|\sum_n a_n\chi(n)\right|^2$$

and

$$S(\mathcal{A},Q) = \sum_{q\leq Q} (qQ)^{-1} \sum_{a(\mathrm{mod}\ q)} \left|\sum_n a_n e(n\tfrac{a}{q})\right|^2.$$

subject to the absolute convergence of the innermost series.

LEMMA 1. For any $M \geq 2N \geq 4$ we have

$$\sum_{M<n\leq M+N} |a_n|^2 \leq S(\mathcal{A}f,Q) \tag{10}$$

with some $Q \leq N^{\frac{1}{2}}$, where $\mathcal{A}f = (a_nf(n))$ for some function f which is smooth, supported in a subinterval of $[M-N, M+2N]$ of length $Y = QN^{\frac{1}{2}}$ and whose derivatives satisfy $f^{(j)} \ll Y^{-j}\log Y$ with the implied constant depending on j only.

Proof. For the proof we may assume that $\mathcal{A}$ is supported in a subinterval of $[M-N, M+2N]$ of length $\frac{1}{2}N$. Let us consider a sum of the type

$$A = \sum_r \omega(r) \sum\sum_{m\equiv n(\mathrm{mod}\ r)} a_m \bar{a}_n,$$

where ω is a smooth function supported in $[\frac{1}{2}R, R]$ such that $\omega(r) \ll R^{-1}$, $\sum \omega(r) = 1$ and whose Fourier cosine transform is bounded by

$$\Omega(v) = 2\int_0^\infty \omega(u)\cos(2\pi uv)du \ll e^{-5\sqrt{vR}}.$$

Notice that

$$A = \sum_r \frac{\omega(r)}{r} \sum_{a(\mathrm{mod}\ r)} |\sum_n a_n e(n\tfrac{a}{r})|^2 \ll S(\mathcal{A},R). \tag{11}$$

The terms on the diagonal $m = n$ contribute to A exactly $D = \sum|a_n|^2$ and the remaining terms contribute

$$B = \sum_{1\le s\le S} \sum\sum_{m\equiv n(\mathrm{mod}\ s)} a_m\bar{a}_n\omega(\tfrac{|m-n|}{s}),$$

where $RS = N$. Thus we have $D = A - B$.

It remains to estimate B in terms of $S(\mathcal{A}f, Q)$. To this end we split the outer summation into subintervals of type $(\frac{1}{2}Q, Q]$ with $Q \le S$. Next given Q we make a smooth partition of unity $\sum \rho_k(m) = 1$, say, with ρ_k supported in an interval of length $Y = QR$, whose derivatives satisfy $\rho_k^{(j)} \ll Y^{-j}$ and such that every m is counted two times. The number of terms ρ_k that are needed to cover the support of $\mathcal{A}$ is $\le N(QR)^{-1}$. We obtain

$$B = \sum_{1\le Q\le S} \sum_{|k-\ell|\le 1} B_{k\ell}(Q),$$

where

$$B_{k\ell}(Q) = \sum_{\frac{1}{2}Q<q\le Q} \sum\sum_{m\equiv n\ (\mathrm{mod}\ q)} a_m\bar{a}_n\rho_k(m)\rho_\ell(n)\omega(\tfrac{|m-n|}{q}).$$

To separate m,n in the test function ω we use the Fourier integral

$$\omega(\tfrac{|m-n|}{q}) = \int_{-\infty}^\infty \Omega(v)e(\tfrac{v}{q}(m-n))dv$$

and get

$$B_{k\ell}(Q) = \sum_{\frac{1}{2}Q<q\le Q} \sum_{a(\mathrm{mod}\ q)} \int_{-\infty}^{\infty} \Omega(vq)\left[\sum_m a_m \rho_k(m)e(vm+m\tfrac{a}{q})\right]$$

$$\left[\sum_n \bar{a}_n \rho_\ell(n)e(-vn-n\tfrac{a}{q})\right]dv.$$

By Cauchy's inequality this gives $B_{k\ell}(Q) \ll B_k(Q)+B_\ell(Q)$, where

$$B_k(Q) = \int_0^\infty e^{-3\sqrt{vY}} \sum_{q\le Q} \sum_{a(\mathrm{mod}\ q)} \Big|\sum_m a_m \rho_k(m)e(vm+m\tfrac{a}{q})\Big|^2 dv.$$

Put $f_k(m) = \rho_k(m)e(vm)e^{-\sqrt{vY}}$, so f_k is smooth, supported in an interval of length $Y = QR$ and has derivatives $f_k^{(j)} \ll Y^{-j}$. We get

$$B_k(Q) \ll Q^2Y^{-1}\, S(\mathscr{A}f_k,Q)$$

for some f_k of the above type. Hence summing over k,ℓ,Q we get

$$B \ll NR^{-2}(\log 2S)S(\mathscr{A}f_k,Q)$$

for some $Q \le S = NR^{-1}$ and some f_k. We choose $R = N^{\frac{1}{2}}$ and get

$$|B| \le \tfrac{1}{2}\, S(\mathscr{A}f,Q)$$

for some $Q \le N^{\frac{1}{2}}$ and f of type $c(\log Y)\, f_k$, where c is a large absolute constant. Moreover by (11) we have

$$|A| \le \tfrac{1}{2}\, S(\mathscr{A}f,Q)$$

with $Q = N^{\frac{1}{2}}$, and some f which is smooth, supported in an interval of length N whose derivatives are $f^{(j)} \ll N^{-j}$. Combining these estimates with the relation $D = A - B$ we complete the proof of Lemma 1.

Remark. The arguments used in the proof of Lemma 1 can be refined to give an exact relation (rather than the upper bound) between D and $S(\mathscr{A}f,Q)$.

Since we intend to exploit the functional equations (2) we must replace the additive characters in $S(\mathscr{A}f,Q)$ by the multiplicative ones. We proceed as follows.

$$\begin{aligned} QS(\mathscr{A}f,Q) &= \sum_{q\le Q}\ \sum\sum_{m\equiv n(\mathrm{mod}\ q)} a_m\bar{a}_n f(m)\bar{f}(n) \\ &= \sum_{dr\le Q}\ \sum\sum_{\substack{m\equiv n(\mathrm{mod}\ r)\\ (mn,r)=1}} a_{dm}\bar{a}_{dn} f(dm)\bar{f}(dn) \\ &= \sum_{dr\le Q} \varphi^{-1}(r) \sum_{\psi(\mathrm{mod}\ r)} \Big|\sum_n a_{dn} f(dn)\psi(n)\Big|^2 \\ &= \sum_{dqw\le Q} \varphi^{-1}(qw) \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \Big|\sum_{(n,w)=1} a_{dn} f(dn)\chi(n)\Big|^2 \\ &= \sum_{dqw\le Q} \varphi^{-1}(qw) \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \Big|\sum_{v|w} \mu(v)\chi(v) \sum_n a_{dvn} f(dvn)\chi(n)\Big|^2 \\ &\le \sum_{dquv\le Q} \frac{\tau(uv)}{\varphi(quv)} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \Big|\sum_n a_{dvn} f(dvn)\chi(n)\Big|^2. \end{aligned}$$

Hence

$$S(\mathscr{A}f,Q) \ll (\log Q)^3 \sum_{dq\le Q} (qQ)^{-1} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} \Big|\sum_n a_{dn} f(dn)\chi(n)\Big|^2, \tag{12}$$

where the implied constant is absolute.

Now suppose a_n are the coefficients of $\mathscr{A}(s)$ having Euler product with local factors $\mathscr{A}_p(p^{-s})$ given by (1). Thus

$$\mathscr{A}_p(p^{-s}) \sum_{\nu=0}^{\infty} a_{p^\nu} p^{-\nu s} = 1.$$

Hence for all $\nu \in \mathbb{Z}$ we obtain the recurrence relation

$$a_{p^\nu} + a_1(p)a_{p^{\nu-1}} + a_2(p)a_p a_{p^{\nu-2}} + a_3(p)a_{p^{\nu-2}} = \delta_{0\nu}$$

subject to the convention that $a_{p^\nu} = 0$ for negative ν. In particular we have

$$a_1(p) = -a_p,$$

$$a_2(p) = -a_{p^2} + a_p^2,$$

$$a_3(p) = -a_{p^3} + 2a_p a_{p^2} - a_p^3.$$

Moreover for $d = p^m$ with $m > 0$ we obtain

$$\begin{aligned}\sum_n a_{dn} n^{-s} &= \Big(\sum_{\nu\geq 0} a_{p^{\nu+m}} p^{-\nu s}\Big)\Big(\sum_{(n,p)=1} a_n n^{-s}\Big)\\ &= \Big(\sum_{\nu\geq 0} a_{p^{\nu+m}} p^{-\nu s}\Big)\mathcal{A}_p(s)\mathcal{A}(s)\\ &= (a_{p^m}+b_{p^m}p^{-s}+c_{p^m}p^{-2s})\mathcal{A}(s),\end{aligned}$$

where

$$b_{p^m} = a_{p^{m+1}} + a_1(p)a_{p^m}$$

and

$$c_{p^m} = a_{p^{m+2}} + a_1(p)a_{p^{m+1}} + a_2(p)a_{p^m}.$$

Hence it is clear that a_{dn} factors as follows

(13) $$a_{dn} = \sum_{hm=n} a(d,h)a_m,$$

where $a(d,h)$ is defined for $h\,|\,d^2$ by

$$a(d,h) = \sum_{d_0d_1d_2=d} a_{d_0}b_{d_1}c_{d_2}$$

with d_0,d_1,d_2 mutually coprime such that

$$h = \Big(\prod_{p\,|\,d_1} p\Big)\Big(\prod_{p\,|\,d_2} p\Big)^2.$$

Using the above formulas one can show that

(14) $$a(d,h) \ll d^{\epsilon} \sum_{k\,|\,dh} |a_k|,$$

with any $\epsilon > 0$, the implied constant depending on ϵ only.

By (12)–(14) using Cauchy's inequality we conclude the following

LEMMA 2. We have

(15) $$S(\mathcal{A}f,Q) \ll Q^{\epsilon} \sum_{d\leq Q} d^{-1} \sum_{h\,|\,d^2} |a(d,h)|^2\, S^*(\mathcal{A}F,d^{-1}Q),$$

where $F(y) = f(dhy)$, the implied constant depending on ϵ only.

5. Evaluation of $\mathscr{A}(F,\chi)$

Let $F(y)$ be a smooth function supported on the interval $[X, X+Y]$ with $X > Y > 0$ and whose derivatives satisfy $F^{(j)} \ll Y^{-j}$ for any $j \geq 0$, the implied constant depending on j only. We apply the functional equation (2) to evaluate the character sum

$$\mathscr{A}(F,\chi) = \sum_m a_m \chi(m) F(m).$$

By contour integration we obtain $\mathscr{A}(F,\chi) = \epsilon_\chi \mathscr{A}(G,\overline{\chi})$, where

$$G(\ell) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{F}(s) \frac{\theta(s)}{\theta(1-s)} \ell^{-s} ds$$

with $\sigma \geq \frac{1}{2}$ and

$$\hat{F}(s) = \int F(x) x^{-s} dx = Y X^{-s} R(s),$$

say. Integrating by parts we see that all derivatives of $R(s)$ satisfy

$$R^{(j)}(s) \ll \left(1 + \frac{|s|}{T}\right)^{-\nu} T^{-j} \quad , \text{ where } T = XY^{-1}, \tag{16}$$

for any $\nu \geq 0$, the implied constant depending on ν, σ and j only. Next we put

$$\Lambda(s) = \prod_{j=1}^{3} \Gamma\left(\frac{s}{2} + \kappa_j\right) / \Gamma\left(\frac{1-s}{2} + \kappa_j\right) = \left(\frac{\pi}{q}\right)^{3s-\frac{3}{2}} \frac{\theta(s)}{\theta(1-s)}. \tag{17}$$

In the above notation we obtain $G(\ell) = \left(\frac{\pi}{q}\right)^{\frac{3}{2}} Y J\left(\frac{3}{q}(\ell X)^{\frac{1}{3}}\right)$, where

$$J(z) = \frac{1}{2\pi i} \int_{(\sigma)} R(s) \Lambda(s) \left(\frac{3}{\pi z}\right)^{3s} ds. \tag{18}$$

For any function $R(s)$ having the property (16) one can find two functions $j_+(z)$ and $j_-(z)$ on $\mathbb{R}^+$ such that

$$z\, J(z) = j_+(z) e(z) + j_-(z) e(-z), \tag{19}$$

$$z \frac{d}{dz} j_\pm(z) \ll 1 \tag{20}$$

and

$$j_\pm(z) \ll \left(1 + \frac{z}{T}\right)^{-\nu}, \tag{21}$$

where ν is any positive number, the implied constant depending on $\kappa_1, \kappa_2, \kappa_3$ and ν only.

From the above evaluations we conclude the following

LEMMA 3. We have

$$\mathscr{A}(G,\overline{\chi}) = \frac{1}{3}\left(\frac{\pi^3}{q}\right)^{\frac{1}{2}} Y X^{-\frac{1}{3}} \sum_{\ell,\pm} a_\ell \ell^{-\frac{1}{3}} \overline{\chi}(\ell) e\left(\pm \frac{3}{q}(\ell X)^{\frac{1}{3}}\right) j_\pm\left(\frac{3}{q}(\ell X)^{\frac{1}{3}}\right). \tag{22}$$

6. Estimation of $S^*(\mathscr{A}F,Q)$

Let $F(y)$ be as in the previous section. We have

$$S^*(\mathscr{A}F,Q) = \sum_{q \le Q} (qQ)^{-1} \sum^*_{\chi (\mathrm{mod}\ q)} |\mathscr{A}(F,\chi)|^2.$$

By (20), (21), (22) and partial summation we get

$$S^*(\mathscr{A}F,Q) \ll 1 + YQ^{\epsilon-1}Q_o^{-1} \sum_{q \le Q_o} \sum^*_{\chi (\mathrm{mod}\ q)} \Big| \sum_{\ell \le L_o} a_\ell \ell^{-\frac{1}{2}} \chi(\ell) e\left(\frac{3}{q}(\ell X)^{\frac{1}{3}}\right)\Big|^2$$

with some $Q_o \le Q$ and $L_o \le X^{-1}(Q_oT)^{3+\epsilon}$. Applying Theorem 2 we conclude

LEMMA 4. Let $X > Y > 0$, $Q \ge 1$ and $\epsilon > 0$. We have

$$S^*(\mathscr{A}F,Q) \ll 1 + Q^\epsilon(QT^2 + YT^{\frac{1}{2}}) \sum_{\ell \le L} |a_\ell|^2 \ell^{-1}, \tag{23}$$

where $T = XY^{-1}$ and $L = X^{-1}(QT)^{3+\epsilon}$, the implied constant depending on ϵ.

Remark. It may happen that $L < 1$ in which case the sum is void and the resulting bound is very good, the effect of smoothing.

7. Proof of Theorem 1

Let $1+\eta$, where $0 \le \eta \le 1$, be the abscissa of absolute convergence of the series

$$\mathscr{D}(s) = \sum_1^\infty a_n^2 n^{-s}.$$

By (23) we get

$$S^*(\mathscr{A}F,Q) \ll 1 + (QT^2 + YT^{\frac{1}{2}})L^{\eta+\epsilon}.$$

Hence by (15) we get

$$S(\mathscr{A}f,Q) \ll Q^\epsilon \sum_{d \le Q} d^{-1} \sum_{d | h^2} |a(d,h)|^2 \left[1 + \left(\frac{Q}{d}\left(\frac{M}{Y}\right)^2 + \frac{(MY)^{\frac{1}{2}}}{d\,h}\right)\left(\frac{d\,h}{M}\left(\frac{QM}{d\,Y}\right)^3\right)^{\eta+\epsilon}\right]$$

$$\ll C(M^2N^{-\frac{3}{2}} + M^{\frac{1}{2}}N^{\frac{1}{2}})(M^2N^{-\frac{3}{2}})^{\eta+\epsilon},$$

where

$$C = \sum_{d} \sum_{d|h^2} d^{-2-2\eta} h^{\eta} |a(d,h)|^2 \ll 1$$

by (14). Combining with (10) we get

$$\sum_{M<n\leq M+N} |a_n|^2 \ll (M^2N^{-\frac{3}{2}} + M^{\frac{1}{2}}N^{\frac{1}{2}})(M^2N^{-\frac{3}{2}})^{\eta+\epsilon}. \tag{24}$$

for any $M \geq N \geq 1$. Finally taking in (24) $M = N$ we infer that $\eta = 0$ and then by (24) with $\eta = 0$ we get (4).

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